1. Use the power series method to solve the following initial value problems. Assume \( c \neq 0 \).

(a) \( \frac{df}{dt} = c^2 f \) with \( f(0) = 1 \) and \( \frac{df}{dt}(0) = 0 \).

**Steps**

1. **Assume** \( f(t) = \sum_{n=0}^{\infty} b_n t^n \) & find the \( b_n \).

2. **Plug in** \( \sum_{n=2}^{\infty} n(n-1) b_n t^{n-2} = \frac{c^2}{t} \sum_{n=0}^{\infty} b_n t^n \).

   Or rather \( \frac{c^2}{t} \sum_{n=2}^{\infty} b_n t^{n-2} = \sum_{n=2}^{\infty} (n(n-1) b_n - \frac{c^2}{t} b_{n-2}) t^{n-2} = 0 \).

   Or rather \( \sum_{n=2}^{\infty} \sum_{n=2}^{\infty} (n(n-1) b_n - \frac{c^2}{t} b_{n-2}) t^{n-2} = 0 \).

3. **So** \( b_n = \frac{c^2 b_{n-2}}{n(n-1)} \) for each \( n \geq 2 \).

4. For \( f(0) = 1 = b_0 \) & \( \frac{df}{dt}(0) = 0 = b_1 \).

   \( b_2 = \frac{c^2 b_0}{2!} = \frac{c^2}{2!} \)

   \( b_3 = \frac{c^2 b_1}{3!} = \frac{c^2}{3!} \)

   \( b_4 = \frac{c^2 b_2}{4!} = \frac{c^2}{4!} \)

   \( b_5 = \frac{c^2 b_3}{5!} = \frac{c^2}{5!} \)

   \( \vdots \)

   \( b_{n+1} = 0 \)

(See proof on next page)

5. **Finally** \( f(t) = \sum_{n=0}^{\infty} \frac{c^2}{(2n)!} t^{2n} \) is the need power series.
(b) $\frac{d^2 f}{dx^2} = c^2 f$ with $f(0) = 0$ and $\frac{df}{dx}(0) = a$.

Well from step 3a, we know

$$b_n = \frac{c^2 b_{n-2}}{n(n-1)}$$

Step 0

4. b. $f(0) = 0 = b_0 \quad \frac{df}{dx}(0) = c = b_1$

$$b_2 = \frac{c^2 \cdot b_0}{2 \cdot 1} = \frac{c^2 \cdot 0}{2 \cdot 1} = 0 \quad b_3 = \frac{c^2 \cdot b_1}{3 \cdot 2 \cdot 1} = \frac{c^2}{3 \cdot 2 \cdot 1}$$

$$b_{2n} = 0 \quad b_{2n+1} = \frac{c^{2n+1}}{3 \cdot 2 \cdot 1}$$

5. b. $f_b(t) = \sum_{n=0}^{\infty} \frac{c^{2n+1}}{2n+1} \frac{t^{2n+1}}{(2n+1)!}$

[Series needed for the power series]

Proof: Part 4a. Note $b_0 = 1 = \frac{c^{20}}{0!} \quad b_1 = 0 = \frac{c^2}{1!}$

Assure $b_n = \frac{c^{2n}}{n(n-1)!!}$ Now

$$b_n = \frac{c^{2n} b_{n-2}}{n(n-1)!} = \frac{c^{2n} \cdot c^{2(n-2)}}{n(n-1)!} = \frac{c^{2n}}{2n(2n-1)(2n-3)!}$$

So by induction

$$b_n = \frac{c^{2n}}{(2n)!}$$

Similarly for $b_0$ and $c_1 b_1$. 

Not needed for this class but...
2. (a) Find the radius of convergence of the power series constructed in problems 1a and 1b.

Steps

1. Recall the ratio test
   
   \[ \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = R < 1 \text{ then } \sum_{n=0}^{\infty} b_n \text{ converges} \]
   
   \[ \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = R > 1 \text{ then } \sum_{n=0}^{\infty} b_n \text{ diverges} \]

2a. Well
   
   \[ \lim_{n \to \infty} \left| \frac{\frac{x^n}{(n!)^2}}{\frac{x^n}{(n!)^2}} \right| = \lim_{n \to \infty} \left( \frac{c^n}{(n!)^2} \right) \]
   
   \[ = \left( \frac{c^2}{(n!)^2} \right) \left( \frac{1}{n^2} \right) \]
   
   \[ = \left( \frac{c^2}{n!} \right) \left( \frac{1}{n^2} \right) \]
   
   \[ = 0 < 1 \]

3. So by 1 & 2
   
   \[ \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \text{ converges for all } c \]

   & radius of convergence is \( \infty \)

Similarly for 1b, since

\[ \lim_{n \to \infty} \left( \frac{(ct)^{n+1}}{(n+1)!} \right) = 0 \]

& 3 still holds.
(b) Show the derivative of the function constructed in problem 1a is a constant multiple of the function in constructed in problem 1b.

\[
\frac{d}{dt} (f_a(t)) = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{c^{2n} t^n}{(2n)!}
\]

\[
= \sum_{n=0}^{\infty} \frac{2n c^{2n} t^{n-1}}{(2n-1)!}
\]

\[
= \sum_{n=0}^{\infty} \frac{c^{2(n+1)} t^{n+1}}{(2(n+1)-1)!}
\]

\[
= c \sum_{n=0}^{\infty} \frac{c^{2n} t^{n+1}}{(2n+1)!} = c \left[ f_{b+1}(t) \right]
\]
3. Recall \( \sinh(x) = \frac{e^x - e^{-x}}{2} \) and \( \cosh(x) = \frac{e^x + e^{-x}}{2} \), and let \( \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \) and \( \text{sech}(x) = \frac{1}{\cosh(x)} \).

(a) Using the above formula, show that \( \cosh(x)^2 - \sinh(x)^2 = 1 \) and \( \tanh(x)^2 + \text{sech}(x)^2 = 1 \).

\[
\left( \cosh(x) \right)^2 - \left( \sinh(x) \right)^2 = \left( \frac{e^x + e^{-x}}{2} \right)^2 - \left( \frac{e^x - e^{-x}}{2} \right)^2
\]

foil

\[
= \left( \frac{e^{2x} + 2 + e^{-2x}}{4} \right) - \left( \frac{e^{2x} - 2 + e^{-2x}}{4} \right)
\]

\[
= \frac{2 - (2^2)}{4} = \frac{4}{4} = 1
\]

\[
\cosh^2(x) - \sinh^2(x) = 1
\]

or rather

\[
\left( \frac{\cosh(x)}{\cosh(x)} \right)^2 + \left( \frac{\sinh(x)}{\cosh(x)} \right)^2 = \left( \frac{1}{\cosh(x)} \right)^2
\]

or rather

\[
1 - \left( \tanh(x)^2 \right) = \left( \text{sech}(x) \right)^2
\]

So \( \left( \tanh(x) \right)^2 + \left( \text{sech}(x) \right)^2 = 1 \) as needed.
(b) Show that \( \frac{d}{dx} \sinh(x) = \cosh(x) \) and that \( \frac{d}{dx} \cosh(x) = \sinh(x) \).

\[
\frac{d}{dx} \left( \frac{e^x - e^{-x}}{2} \right) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x)
\]

\[
\frac{d}{dx} (\sinh(x)) = \frac{e^x + e^{-x}}{2}
\]

\[
\frac{d}{dx} (\cosh(x)) = \frac{d}{dx} \left( \frac{e^x + e^{-x}}{2} \right)
\]

\[
= \frac{e^x + e^{-x}}{2} = \sinh(x)
\]
(c) Find a power series expression for \( \sinh(z) \) and \( \cosh(z) \).

\[
\sinh(z) = \frac{e^z - e^{-z}}{2} = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \right)
\]

\[
= \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \right)
\]

\[
\cosh(z) = \frac{e^z + e^{-z}}{2} = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \right)
\]
(d) Let \( \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \) and \( \text{sech}(x) = \frac{1}{\cosh(x)} \). Show \( \frac{d}{dx} \tanh(x) = 1 - \tanh(x)^2 \) and \( \frac{d}{dx} \text{sech}(x) = -\text{sech}(x) \tanh(x) \).

\[
\frac{d}{dx} \left( \tanh(x) \right) = \frac{d}{dx} \left( \frac{\sinh(x)}{\cosh(x)} \right)^{-1} = \frac{\cos(x)}{\cosh(x)} \cdot \left( \frac{\sinh(x)}{\cosh(x)} \right)^{-1} + \sinh(x) \left( -1 \right) \left( \frac{\sinh(x)}{\cosh(x)} \right)^{-2} \cdot \cosh(x)
\]

\[
= 1 - \left( \frac{\sinh(x)}{\cosh(x)} \right)^2 = 1 - \left( \tanh(x) \right)^2
\]

\[
\frac{d}{dx} \left( \text{sech}(x) \right) = \frac{d}{dx} \left( \frac{1}{\cosh(x)} \right)^{-1} = -\left( \frac{1}{\cosh(x)} \right)^{-2} \cdot \frac{d}{dx} \left( \frac{1}{\cosh(x)} \right)
\]

\[
= -\left( \frac{\sinh(x)}{\cosh(x)} \right) \cdot \frac{1}{\cosh(x)} \cdot \left( -1 \right) \left( \frac{\sinh(x)}{\cosh(x)} \right)^{-2} \cdot \cosh(x)
\]

\[
= -\frac{1}{\cosh(x)} \cdot \frac{1}{\cosh(x)} \cdot \left( \frac{\sinh(x)}{\cosh(x)} \right) \cdot \frac{1}{\cosh(x)} \cdot \left( \frac{\sinh(x)}{\cosh(x)} \right)
\]

\[
= -\frac{\sinh(x)}{\cosh(x)} \cdot \frac{1}{\cosh(x)} \cdot \left( \frac{\sinh(x)}{\cosh(x)} \right)
\]

\[
= -\frac{\sinh(x)}{\cosh(x)} \cdot \frac{1}{\cosh(x)} \cdot \frac{\sinh(x)}{\cosh(x)}
\]

\[
= -\frac{\sinh(x) \cdot \sinh(x)}{\cosh(x) \cdot \cosh(x)}
\]

\[
= -\frac{\sinh(x) \cdot \sinh(x)}{\cosh(x) \cdot \cosh(x)}
\]

\[
= -\frac{\sinh(x) \cdot \sinh(x)}{\cosh(x) \cdot \cosh(x)}
\]

\[
= -\frac{\sinh(x) \cdot \sinh(x)}{\cosh(x) \cdot \cosh(x)}
\]
4. (a) Show that $\sinh(ct)$ and $\cosh(ct)$ are solutions to the differential equation $\frac{d^2 f}{dt^2} - c^2 f = 0$ when $c \neq 0$. (Hint you may use problems 1 and 3).

(b) Solve the initial value problem $\frac{d^2 f}{dt^2} - c^2 f = 0$ with $f(0) = A$ and $\frac{df}{dt}(0) = B$ using $\sinh(ct)$ and $\cosh(ct)$.

(a) Well \[ \frac{d^2}{dt^2} (\sinh(ct)) = c \frac{d}{dt} (\cosh(ct)) \quad (\text{by } 3b) \]

\[ = c^2 \sinh(ct) \quad (\text{by } 3b) \]

\[ \text{Similarly } \frac{d^2}{dt^2} (\cosh(ct)) = c \frac{d}{dt} (\sinh(ct)) = c^2 \cosh(ct) \]

(b) We know that any thing in the form \( f(t) = A \cosh(ct) + B \sinh(ct) \) is a solution.

At \( t = 0 \)

\[ A = f(0) = \left( C_0 \cosh(0) + C_1 \sinh(0) \right) \bigg|_{t=0} = C_0 \]

\[ B = \left. \frac{df}{dt}(0) \right|_{t=0} = \left( C_0 \cosh(0) + C_1 \sinh(0) \right) \bigg|_{t=0} = C_0 \]

\[ \Rightarrow f(t) = \frac{B}{c} \sinh(ct) + A \cosh(ct) \] is the solution.
5. Find a general solution to $\frac{d^2f}{dx^2} - e^{2x} = e^{2x}$. (Hint: you may use problem 4).

Steps:

1. Well if we can find a particular solution $f_p(x)$ then any solution is in the form $f(x) = f_p(x) + f_n(x)$ where $f_n(x)$ solves $\frac{d^2f_n}{dx^2} - e^{2x} = 0$. From $f_n(x) = \frac{B}{c} \sinh(cx) + A \cos(cx)$ we only need an $f_p(x)$.

2. The guess method: well we guess $f_p(x) = a_0 e^{2x}$ and write this implies $a_0 \left(4 e^{2x} - c^2 e^{2x}\right) = e^{2x}$

or $a_0 \left(c^2 - 4\right) = 1$. So $c^2 \neq \pm 2$...

\[ f(x) = \frac{1}{4-c^2} e^{2x} + \frac{B}{c} \sinh(cx) + A \cosh(cx) \]

3. With $c = \pm 2$ we guess $a_0 e^{2x}$ and find $a_0 \left(\frac{1}{2} e^{2x} + 2 e^{2x} - c e^{2x}\right) = e^{2x}$

or $a_0 \left((4-c^2) e^{2x} + (10) e^{2x}\right) = e^{2x}$ or $a_0 = \frac{1}{4}$

\[ f(x) = \frac{1}{4} e^{2x} + \frac{B}{c} \sinh(cx) + A \cosh(cx) \]
6. Recall in polar coordinates that \( \hat{r}(\theta) \) parameterizes the unit circle and that \( \frac{d\theta}{dt} = \hat{r} \), and \( \frac{d\hat{r}}{dt} = -\hat{\theta} \). (To solve this problem you may use the results of problem 3.)

(a). Recall the equation of a hyperbola is given by \( x^2 - y^2 = c \) for \( c \neq 0 \).
Explain why \( \hat{r}_h(\theta_h) = \cosh(\theta_h)\hat{r} + \sinh(\theta_h)\hat{\theta} \) parameterizes a hyperbola.

\[
\text{Well by (a)} \quad \left( \cosh(\theta_h) \right)^2 - \left( \sinh(\theta_h) \right)^2 = 1, \quad \text{as needed.}
\]
(b). Let \( \dot{\Theta}_h = \sinh(\Theta_h) \hat{i} + \cosh(\Theta_h) \hat{j} \). Show \( \frac{d}{dt} \dot{\Theta}_h = \dot{\Theta}_h \dot{\Theta}_h \).

Well
\[
\frac{d}{dt} \dot{\Theta}_h = \frac{d}{dt} \left( \left( \cosh(\Theta_h) \right) \hat{i} + \left( \sinh(\Theta_h) \right) \hat{j} \right)
\]
\[
= \frac{d}{dt} \left( \cosh(\Theta_h) \right) \hat{i} + \frac{d}{dt} \left( \sinh(\Theta_h) \right) \hat{j}
\]
by \( \Theta \)
\[
= \left( \sinh(\Theta_h) \right) \dot{\Theta}_h \hat{i} + \sinh(\Theta_h) \dot{\Theta}_h \hat{j}
\]
\[
= \dot{\Theta}_h \dot{\Theta}_h
\]
(c) Can \( \frac{d\theta_n}{dt} \) be expressed as a multiple of \( \hat{r}_n \)? If so derive a formula relating the two, if not explain why not.

\[
\frac{d\theta_n}{dt} = \frac{d}{dt} \left( \sinh(\theta_n) \hat{i} + \cosh(\theta_n) \hat{j} \right)
\]

\[
= \left( \frac{d}{dt} \sinh(\theta_n) \right) \hat{i} + \left( \frac{d}{dt} \cosh(\theta_n) \right) \hat{j}
\]

\[
= \cosh(\theta_n) \hat{\theta}_n \hat{j} + \sinh(\theta_n) \hat{n} \hat{j}
\]

\[
= \hat{\theta}_n \hat{r}_n
\]

So in deed,
(d). Why do the hats in \( \hat{\epsilon}_h \) and \( \hat{\theta}_h \) feel inappropriate?

Well

\[
|\hat{\epsilon}_h| = \left( \cosh(\alpha) \right)^2 + \left( \sinh(\alpha) \right)^2
\]

\[
= \left( \frac{e^x + e^{-x}}{2} \right)^2 + \left( \frac{e^x - e^{-x}}{2} \right)^2
\]

\[
= \frac{e^{2x} + e^{-2x}}{4} = \cosh(2x) \neq 1
\]

except for very special \( x \).

In fact \( e^{x} + e^{-x} \)

\[
eq 1 \quad \text{when}
\]

\[
e^{2x} - 2e^x + 1 = 0
\]

So \( e^x = \frac{2 \pm \sqrt{4 - 4}}{2} = \frac{2 \pm 0}{2} = \pm 1 \)

So \( x = 0 \) is the only such value...
7. How is the area of the parallelogram determined by $\vec{n}_u(\theta_h)$ and $\hat{e}_n(\theta_h)$ from problem 6 changing as $\theta_h$ changes? (Hint: use problem 3.)

Recall the area of this parallelogram is

$$\left| \left( \hat{e}_n(\theta_h) \times \hat{e}_n(\theta_h) \right) \right|$$

$$= \begin{vmatrix}
\hat{e}_n(\theta_h) \\
\hat{e}_n(\theta_h) \\
\hat{e}_n(\theta_h)
\end{vmatrix}$$

$$= \begin{vmatrix}
\cos(\theta_h) & \sinh(\theta_h) & 0 \\
\sin(\theta_h) & \cosh(\theta_h) & 0 \\
0 & 0 & 1
\end{vmatrix}$$

$$= \left( \cosh(\theta_h) \right)^2 - \left( \sinh(\theta_h) \right)^2$$

$$= 1$$

by (3a) so this area does not change as $\theta_h$ varies.
8. (a) Let \( \vec{F} = y^2 \hat{i} + x \hat{j} \) and suppose an object travels along \( \gamma(t) = (t - \tanh(t), \sech(t)) \). Set up (but do not evaluate!) an integral for the work done by this force on this object as time goes from 0 to 2.

\[
W_{\text{fr}} = \int_{\gamma} \vec{F} \cdot d\vec{r}
\]

\[
= \int_{0}^{2} \vec{F}(t - \tanh(t), \sech(t)) \cdot \frac{d}{dt} (t - \tanh(t), \sech(t)) \, dt
\]

\[
= \int_{0}^{2} \left( \sech(t)^2 \frac{d}{dt} (t - \tanh(t)), \left(1 - \tanh(t)^2\right), -\sech(t) \tanh(t) \right) \, dt
\]

(by 3rd)

\[
= \int_{0}^{2} \sech(t) \tanh(t) \left( t + \tanh(t) \right) \, dt
\]
(b) Let \( \vec{F} = y\hat{i} + 2xy\hat{j} \) and suppose an object travels along \( \gamma(t) = (t - \tanh(t), \sech(t)) \). Set up (but do not evaluate!) an integral for the work done by this force on this object as time goes from 0 to 2.

\[
\text{work} = \int_0^2 \left[ (\sech(t) \tanh(t))^2 - 2(\sech(t)) \tanh(t) (t - \tanh(t)) \right] dt
\]
Work = \mathcal{W} = \int_{x_1}^{x_2} F(x) \, dx = \int_{0.1}^{2} \left( \frac{e^x - e^{-x}}{e^x + e^{-x}} \right) \, dx

So work = \int_{0.1}^{2} \left( \frac{e^x - e^{-x}}{e^x + e^{-x}} \right) \, dx

\frac{\text{d}y}{\text{d}x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad \Rightarrow \quad \text{arctanh} \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right) = \ln \left( \frac{e^y + 1}{e^y - 1} \right)

which is constant. So, \ V = -xy^2 + f(y)

\Rightarrow \ \frac{\partial V}{\partial x} = -y^2 = \frac{e^{2x}}{1 + e^{2x}} - \frac{e^{-2x}}{1 + e^{-2x}}

we better find one of the F's is constant

\Rightarrow \ Integrate the work done in either problem 8 or problem 9.

9. Compute the work done in theforce field. (Hint: be sure to think)}
10. Imagine an oxen staring at \((0,0)\) is attached via a taught rope of length one decameter to large stone at \((0,1)\). Suppose the oxen is then driven at a constant speed of one decameter per minute along the \(x\)-axis. Let \(\gamma(t)\) denote the position of the stone at time \(t\). (To solve this problem you may use the results of problem 3.)

(a). Justify in words and/or a picture why the path must satisfy that the line segment starting at \(\gamma(t)\) and ending at \((t,0)\) must have length 1.

\[
\begin{align*}
&\text{(0,1)} \\
&\text{Stone } = \gamma(t) \\
&\text{Oxen } = (t,0)
\end{align*}
\]

The line segment from \(\gamma(t)\) to \((t,0)\) is our taught rope of length 1.
(b). Justify in words and/or a picture why the line starting at \( \gamma(t) \) heading in the direction determined by \( \frac{d}{dt} \) must hit \( x - axis \) at \( (t,0) \).

\[
\text{stone} = \gamma(t) \\
\text{oxen} = (t,0)
\]

Well \( \frac{dy}{dt} \) describes the stone's instantaneous rate & direction of change, which is towards to oxen, since the oxen is pulling the stone.

The oxen is a \((t,0)\) on the \(x-\text{axis}\), as needed.
(e). Show that the path \( \gamma(t) = (t - \text{tanh}(t), \text{sech}(t)) \) satisfies the conditions described in 10a and 10b.

\[
\| \left( t - \text{tanh}(t), \text{sech}(t) \right) - (0,0) \| = \left| \frac{dy}{dt} \right| - (t,0)
\]

\[
\| \left( -\text{tanh}(t), \text{sech}(t) \right) \|
\]

\[
(t\text{anh}(t))^2 + (\text{sech}(t))^2
\]

by \((3d)\)

Similarly

\[
\frac{dy}{dt} = \frac{\text{sech}^2(t) - (1 - (\text{tanh}(t))^2)}{3}\]

\[
= \text{tanh}(t) \left( \text{tanh}(t), -\text{sech}(t) \right)
\]

which has the same direction as \( \frac{dy}{dt} \) from \((21)\) as needed.
(Hard)

EXTRA CREDIT 1) Explain why \( \eta(t) = (t - \tanh(t), \text{sech}(t)) \) is the only path that can satisfy 10a and 10b.

Steps

1. 10(a) implies \( (x-t)^2 + y^2 = 1 \) or \( x = \pm \sqrt{1-y^2} + t \).
   & hence \( x = \frac{-y}{\sqrt{1-y^2}} y + 1 \).

2. 10(b) implies \( \dot{x}, \dot{y} = c (x-t, y) \)
   \( \frac{x}{x-t} = c \frac{y}{y} \) & hence
   \( \dot{y} = \frac{y}{x-t} x = \frac{-y}{\sqrt{1-y^2}} x \).

3. So
   \( \tan \theta = \frac{\dot{y}}{1} = \frac{-y^2}{1-y^2} \).

4. \( \sin^2 \theta = \frac{y^2}{y^2} \).

5. \( \frac{\dot{y}}{\dot{x}} = \frac{y^2}{1-y^2} \).

6. \( \dot{y} = \pm y \sqrt{1-y^2} \).

7. \( \frac{d}{dt} \text{sech}(t) = \frac{1}{\cosh^2(t)} \).

8. \( \frac{d}{dt} \tanh(t) = -\text{sech}^2(t) \).

9. \( \frac{d}{dt} \text{sech}(t) \) satisfies

10. \( \frac{d}{dt} = \frac{d}{dt} \text{sech}(t) \) as needed.
(d). The path \( \gamma(t) \) describes our stone's path. Set up (but do not evaluate) an integral to determine how far the stone has traveled in the first \( t_0 \) minutes.

\[
\int_0^{t_0} \left| \frac{dy}{dt} \right| \, dt = \int_0^{t_0} \left( (\tanh^2 t) - (\sec^2 t) \right) \, dt
\]

\[
= \int_0^{t_0} (\tanh^2 t) \, dt - \int_0^{t_0} (\sec^2 t) \, dt
\]

\[
= \int_0^{t_0} (\tanh^2 t) \, dt - \left[ \tan t \right]_0^{t_0}
\]

\[
= \int_0^{t_0} (\tanh^2 t) \, dt
\]

Incidentally,

\[
= \left. \ln(\cosh t) \right|_0^{t_0} = \ln(\cosh(t_0))
\]
(EXTRA CREDIT 2) Do you expect the stone will have traveled exactly as far, less far, or further than the oxen, and why?

\[
\left| \frac{dy}{dt} \right| = (\tan \theta)^2 = \left( \frac{e^x - e^{-x}}{e^x + e^{-x}} \right)^2 < 1
\]
\[
\text{(since } e^x - e^{-x} < e^x < e^x + e^{-x})
\]

So, \[
\int_0^t \left| \frac{dy}{dt} \right| dt < \int_0^t 1 \ dt = t_0
\]

= distance traveled by oxen.