Math 13 Fall 2004
Calculus of Vector-valued Functions

Example of a function that has different mixed partial derivatives at (0,0)

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Define a scalar-valued function of two variables

\[ f := (x, y) \rightarrow x \ast y \ast (x^2 - y^2) / (x^2 + y^2); \]

\[ f := (x, y) \rightarrow \frac{xy(x^2 - y^2)}{x^2 + y^2} \]

Have a look at its graph

\[ \text{plot3d}(f(x, y), x = -1..1, y = -1..1); \]
Both partial derivatives of $f$ are continuous everywhere, so $f$ is differentiable at $(0, 0)$

```latex
\begin{align*}
  f_x &:= \frac{y(x^4 - y^4 + 4x^2y^2)}{(x^2 + y^2)^2} \\
  f_y &:= \frac{x(x^4 - y^4 - 4x^2y^2)}{(x^2 + y^2)^2}
\end{align*}
```

```latex
> f_x := factor(diff(f(x, y), x)); \\
  f_y := factor(diff(f(x, y), y)); \\
```

```latex
> plot3d(f_x, x = -2..2, y = -2..2); \\
plot3d(f_y, x = -2..2, y = -2..2);
```
Let's explicitly compute the mixed partial derivatives of $f$ at $(0, 0)$

```plaintext
> f_xy_00 := limit(subs(x = h, y = 0, f_x) / h, h = 0);  
f_yx_00 := limit(subs(x = 0, y = h, f_x) / h, h = 0);

f_xy_00 := 0
f_yx_00 := -1
```

They are **different!!!**

Let's plot both mixed partial derivatives of $f$

```plaintext
> plot3d(diff(f(x, y), y, x), x = -1..1, y = -1..1);  
plot3d(diff(f(x, y), x, y), x = -1..1, y = -1..1);
```
They are obviously **discontinuous**!!!

**Remark:** mixed partial derivatives are **the same** away from (0, 0)

```plaintext
> factor(diff(f(x, y), y, x));
> factor(diff(f(x, y), x, y));

\[
\frac{(-y + x) (x + y) (y^4 + 10 x^2 y^2 + x^4)}{(x^2 + y^2)^3}
\]

\[
\frac{(-y + x) (x + y) (y^4 + 10 x^2 y^2 + x^4)}{(x^2 + y^2)^3}
\]

>