Math 13 Fall 2004

Calculus of Vector-valued Functions

Example of a function that has different mixed partial derivatives at \((0, 0)\)

Consider a function \(f : \mathbb{R}^2 \to \mathbb{R}\):

\[
f(x, y) = \begin{cases} 
\frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0); \\
0 & \text{if } x = y = 0.
\end{cases}
\]

The partial derivatives of \(f\) are given by

\[
f_x(x, y) = \begin{cases} 
\frac{y(x^4 - y^4 - 4x^2y^2)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0); \\
0 & \text{if } x = y = 0
\end{cases}
\]

and

\[
f_y(x, y) = \begin{cases} 
\frac{x(x^4 - y^4 - 4x^2y^2)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0); \\
0 & \text{if } x = y = 0
\end{cases}
\]

One can check that \(f\) as well as \(f_x\) and \(f_y\) are continuous everywhere in \(\mathbb{R}^2\). Hence, \(f\) is differentiable in \(\mathbb{R}^2\).

We can compute the second order mixed partial derivatives of \(f\) at \((0, 0)\) explicitly:

\[
f_{yx}(0, 0) = \frac{\partial}{\partial x}(f_y)(0, 0) = \lim_{x \to 0} \frac{f_y(x, 0)}{x} = \lim_{x \to 0} \frac{x(x^4 - 0^4 - 4x^20^2)}{(x^2 + 0^2)^2x} = \lim_{x \to 0} \frac{1}{1} = 1;
\]

\[
f_{xy}(0, 0) = \frac{\partial}{\partial y}(f_x)(0, 0) = \lim_{y \to 0} \frac{f_x(0, y)}{y} = \lim_{y \to 0} \frac{y(0^4 - y^4 - 4(0^2y^2))}{(0^2 + y^2)^2y} = \lim_{y \to 0} \frac{-1}{1} = -1.
\]

The mixed partial derivatives \(f_{yx}(0, 0)\) and \(f_{xy}(0, 0)\) of \(f\) at \((0, 0)\) are different

The reason is that they are both not continuous in a neighborhood of \((0, 0)\):

\[
f_{yx}(x, y) = \frac{\partial}{\partial x}(f_y)(x, y) = \frac{(x^2 - y^2)(x^4 + 10x^2y^2 + y^4)}{(x^2 + y^2)^3} = \frac{\partial}{\partial y}(f_x)(x, y) = f_{xy}(x, y).
\]

The limits of both \(f_{yx}\) and \(f_{xy}\) DNE as \((x, y) \to (0, 0)\). If \(x = 0\), they should be \(-1\), but if \(y = 0\), they should be \(1\).

Notice that \(f_{xy}(x, y) = f_{yx}(x, y)\) away from the origin. Indeed, all the derivatives of \(f\) are continuous there.