MAJOR FACTS ABOUT LIMITS AND CONTINUITY

FACT 1. (Uniqueness of limits) If a limit exists, it is unique:

If \( F: X \subset \mathbb{R}^n \to \mathbb{R}^m, \lim_{x \to a} F(x) = L, \) and \( \lim_{x \to a} F(x) = M, \) then \( L = M. \)

FACT 2. (Algebraic properties of limits)

Let \( F, G: X \subset \mathbb{R}^n \to \mathbb{R}^m \) and \( f, g: X \subset \mathbb{R}^n \to \mathbb{R}. \) Let also \( k \in \mathbb{R}. \)

1. If \( \lim_{x \to a} F(x) = L \) and \( \lim_{x \to a} G(x) = M, \) then \( \lim_{x \to a} (F + G)(x) = L + M. \)
2. If \( \lim_{x \to a} F(x) = L, \) then \( \lim_{x \to a} kF(x) = kL. \)
3. If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M, \) then \( \lim_{x \to a} (fg)(x) = LM. \)
4. If \( \lim_{x \to a} f(x) = L, g(x) \neq 0 \) for \( x \in X \) and \( \lim_{x \to a} g(x) = M \neq 0, \) then \( \lim_{x \to a} (f/g)(x) = L/M. \)

FACT 3. Let \( F: X \subset \mathbb{R}^n \to \mathbb{R}^m. \) Then \( \lim_{x \to a} F(x) = L \) if and only if \( \lim_{x \to a} F_i(x) = L_i \) for all \( i = 1, 2, \ldots, m, \) where \( F = (F_1, F_2, \ldots, F_m) \) and \( L = (L_1, L_2, \ldots, L_m). \)

FACT 4. Let \( F, G: X \subset \mathbb{R}^n \to \mathbb{R}^m \) and \( f, g: X \subset \mathbb{R}^n \to \mathbb{R}. \) Let also \( k \in \mathbb{R}. \)

1. If \( F \) and \( G \) are continuous at \( a \in X, \) then \( F + G \) is continuous at \( a. \)
2. If \( F \) is continuous at \( a \in X, \) then \( kF \) is continuous at \( a. \)
3. If \( f \) and \( g \) are continuous at \( a \in X, \) then \( fg \) is continuous at \( a. \)
4. If \( f \) and \( g \) are continuous at \( a \in X \) and \( g \neq 0, \) then \( f/g \) is continuous at \( a. \)
5. \( F: X \subset \mathbb{R}^n \to \mathbb{R}^m \) is continuous at \( a \in X \) if and only if all \( F_i: X \subset \mathbb{R}^n \to \mathbb{R} \) are continuous at \( a, \) where \( F = (F_1, F_2, \ldots, F_m). \)

FACT 5. (Composition of continuous functions) If \( F: X \subset \mathbb{R}^n \to \mathbb{R}^m \) and \( G: Y \subset \mathbb{R}^m \to \mathbb{R}^p \) are continuous and \( \text{Range}(F) \subset Y, \) then \( G \circ F: X \subset \mathbb{R}^n \to \mathbb{R}^p \) is also continuous.
**MAJOR FACTS ABOUT DERIVATIVES**

**Fact 1.** If $F : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in X$, then it is continuous at $a$.

**Fact 2.** Let $F : X \subset \mathbb{R}^n \to \mathbb{R}^m$ such that all $\frac{\partial F_i}{\partial x_j}$ exist and are continuous in a neighborhood of $a \in X$. Then $F$ is differentiable at $a$.

**Fact 3.** $F : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in X$ if and only if all $F_i : X \subset \mathbb{R}^n \to \mathbb{R}$ are differentiable at $a$, where $F = (F_1, F_2, \ldots, F_m)$.

**Fact 4.** *(Linearity of the derivative)*
Let $F, G : X \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in X$ and $k \in \mathbb{R}$. Then

1. $F + G$ is differentiable at $a$ and $D(F + G)(a) = DF(a) + DG(a)$.
2. $kF$ is differentiable at $a$ and $D(kF)(a) = kDF(a)$.

**Fact 5.** Let $f, g : X \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in X$. Then

1. $fg$ is differentiable at $a$ and $D(fg)(a) = g(a)Df(a) + f(a)Dg(a)$.
2. If $g \neq 0$, $f/g$ is differentiable at $a$ and $D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{g(a)^2}$.

**Fact 6.** *(The chain rule)* If $F : X \subset \mathbb{R}^n \to \mathbb{R}^m$ and $G : Y \subset \mathbb{R}^m \to \mathbb{R}^p$ are differentiable at $a$ and $b = F(a)$, respectively, and Range($F$) $\subset Y$, then $G \circ F : X \subset \mathbb{R}^n \to \mathbb{R}^p$ is also differentiable at $a$ and $D(G \circ F)(a) = DG(b)DF(a)$. 