Green’s, Stokes’s, and Gauss’s Theorems

Let $D$ be a closed bounded region in $\mathbb{R}^2$ such that its boundary $\partial D$ consists of finitely many simple closed curves that are oriented in such a way that $D$ is on the left as one traverses $\partial D$. Let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be a vector field of class $C^1$.

1. (Green’s Theorem) $\oint_{\partial D} M\,dx + N\,dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\,dA$.

2. (Vector form of Green’s Theorem) $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA$.

3. (Divergence Theorem in the plane) If $\mathbf{n}$ is the outward unit normal vector to $D$, then $\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\,ds = \iint_D \nabla \cdot \mathbf{F}\,dA$.

Let $S$ be a bounded, oriented surface in $\mathbb{R}^3$ such that its boundary $\partial S$ consists of finitely many simple closed curves that are oriented consistently with $S$. Let $\mathbf{F}$ be a vector field of class $C^1$.

4. (Stokes’s Theorem) $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$.

Let $W$ be a bounded solid region in $\mathbb{R}^3$ such that its boundary $\partial W$ consists of finitely many closed orientable surfaces that are oriented by unit normals $\mathbf{n}$ pointing away from $W$. Let $\mathbf{F}$ be a vector field of class $C^1$.

5. (Gauss’s Theorem) $\iiint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\partial W} (\mathbf{F} \cdot \mathbf{n})\,dS = \iiint_W \nabla \cdot \mathbf{F}\,dV$.