Math 11 Fall 2004

Multivariable Calculus
for Two-Term Advanced Placement First-Year Students

Final Exam

Tuesday, December 7, 11:30-2:30
Murdough, Cook Auditorium

Your name (please print): ________________________________

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You are allowed to bring one letter-size sheet of paper with any data you want written on it. You must justify all of your answers in this booklet to receive credit, though because this is a multiple choice exam, justifications can be minimal. However, only the answer you mark on the scantron form will be counted.

You have three hours to work on all 25 problems. Please do all your work in this exam booklet.

The Honor Principle requires that you neither give nor receive any aid on this exam.
Grader’s use only

1-25: Multiple Choice: ______ /150

Total: ______ /150
(1) Let $\mathbf{F}(x, y, z) = (x + \sin y)i + (y - \sin(z))j + zk$. Then $\text{div } F =$
\begin{align*}
a: & \; 0; \\
b: & \; 3; \\
c: & \; i + j + k; \\
d: & \; \cos zi + 0j - \cos yk.
\end{align*}

(2) Let $\mathbf{F}(x, y, z) = (x + \sin y)i + (y - \sin(z))j + zk$. Then $\nabla \times F =$
\begin{align*}
a: & \; 0; \\
b: & \; 3; \\
c: & \; i + j + k; \\
d: & \; \cos zi + 0j - \cos yk.
\end{align*}
(3) Apply the Laplace operator to a function \( h(x, y, z) = 4xy + e^z \). \( \nabla^2 h = \)

a: 4;
b: \( e^z \mathbf{k} \);
c: \( x + y \);
d: \( e^z \).

(4) Find a function \( f(x, y, z) \) such that \( f(0, 0, 0) = 1 \) and \( \nabla f = 2yi + 2xj + e^z \mathbf{k} \).

a: such \( f \) does not exist;
b: \( f(x, y, z) = 2y^2 + 2x^2 + e^z \);
c: \( f(x, y, z) = 2xy + e^z + 1 \);
d: \( f(x, y, z) = 2xy + e^z \).
(5) Does there exist a function \( f(x, y, z) \) with \( \nabla f(x, y, z) = 2xe^{x^2}i + z\sin(y^2)j + z^{2004}k \)?

a: such \( f \) does not exist, since \( \nabla^2 f \neq 0 \);
b: such \( f \) does exist;
c: such \( f \) does not exist, since \( \nabla \times \nabla f \neq 0 \);
d: such \( f \) does not exist, since \( \text{div}(\nabla f) \neq 0 \).

(6) Does there exist a vector field \( G \) with \( F(x, y, z) = \nabla \times G = e^{yz}i + \sin(xz^2)j + z^{2004}k \)?

a: such \( G \) does not exist, since \( \text{curl} F \neq 0 \);
b: such \( G \) does not exist since \( \text{div} F \neq 0 \);
c: such \( G \) does not exist, since \( F(0, 0, 0) \neq 0 \);
d: such \( G \) does exist, since \( \text{div}(F) = 0 \).
(7) Let \( \mathbf{F}(x, y, z) = (x + e^{yz})\mathbf{i} + (y + \sin(x^2z))\mathbf{j} + (z + \cos(x + y))\mathbf{k} \) and let \( S \) be the \textbf{inwards oriented} sphere \( x^2 + y^2 + z^2 = 1 \). Then \( \int \int_S \mathbf{F} \cdot d\mathbf{S} = \)

\begin{itemize}
  \item a: \( 4\pi \);
  \item b: \( \frac{4}{3}\pi \);
  \item c: \(-4\pi\);
  \item d: 1.
\end{itemize}

Recall that in the Divergence Theorem the orientation of the surface is \textbf{outwards} away from the solid.

(8) Let \( F(x, y, z) = x^{2004}\mathbf{i} + \sin(y^2)\mathbf{j} + (x + z)\mathbf{k} \) and let \( C \) be a \textbf{clockwise oriented} circle with parameterization \( \mathbf{r}(t) = \cos(-t)\mathbf{i} + \sin(-t)\mathbf{j} + 0\mathbf{k}, \ t \in [0, 2\pi] \). Then \( \int_C \mathbf{F} \cdot d\mathbf{r} = \)

\begin{itemize}
  \item a: \( \pi \);
  \item b: \(-\pi\);
  \item c: 0;
  \item d: \( \pi^2 \).
\end{itemize}
(9) Let $C$ be the clockwise oriented boundary of the square $\{(x, y) : -1 \leq x \leq 1; -1 \leq y \leq 1\}$ in $\mathbb{R}^2$. Then $\int_C x^3dx + (x + \sin y)dy =$
\begin{align*}
a: & \frac{\pi}{3}; \\
b: & 4; \\
c: & -4; \\
d: & 1.
\end{align*}
Recall that in Green’s Theorem the orientation is counter-clockwise.

(10) Let $\mathbf{F}$ be the force field $\mathbf{F}(x, y, z) = 1\mathbf{i} + 2y\mathbf{j} + 3z^2\mathbf{k}$. The work done by the force $\mathbf{F}$ along a path transporting a particle from the point $(1, 1, 1)$ to the point $(1, 0, 0)$ equals:
\begin{align*}
a: & -2; \\
b: & 2; \\
c: & 1; \\
d: & 0.
\end{align*}
(11) The work \( \int_C \mathbf{F} \cdot d\mathbf{r} \) done by the force \( \mathbf{F}(x, y, z) = x^3 \mathbf{i} + (x + y) \mathbf{j} + \sin z \mathbf{k} \) along the closed circular path \( C \) given by \( \mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + 1 \mathbf{k}, \ t \in [0, 2\pi], \) is equal to:

- a: \(-\pi\);
- b: \(9\pi\);
- c: 0;
- d: 1.

(12) Let \( C \) be a path given by \( \mathbf{r}(t) = \langle 1-t, 2-t, 3-t \rangle, \ 0 \leq t \leq 1. \) Then \( \int_C (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot d\mathbf{r} = \)

- a: 0
- b: \(-1\);
- c: 3;
- d: \(-3\).
(13) Let \( f(x, y, z) = x + y^2 + z^3 \). Then at the point \((0, 1, 1)\) the function \( f \) decreases fastest in the direction of
   \begin{itemize}
   \item a: \(-i - 2j - 3k\);
   \item b: \(-i - j - k\);
   \item c: \(\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle\);
   \item d: \(i + 2j + 3k\).
   \end{itemize}

(14) For a function \( f(x, y) : \mathbb{R}^2 \to \mathbb{R} \) whose gradient is \( \nabla f(x, y) = (x^2 - 1)i + (\sin y)j \), the function \( f \) has
   \begin{itemize}
   \item a: zero critical points;
   \item b: exactly four critical points;
   \item c: exactly two critical points;
   \item d: infinitely many critical points.
   \end{itemize}
(15) \( \lim_{(x,y) \to (0,0)} \frac{3x^2+y^2}{1-\sqrt{3x^2+y^2}+1} \)

\textbf{a}: does not exist since the limits along the paths } C_1(t) = (t, t) \text{ and } C_2(t) = (3t, t) \text{ are different; }

\textbf{b}: does not exist since the limits along the paths } C_1(t) = (t, t^2) \text{ and } C_2(t) = (t, 3t^2) \text{ are different; }

\textbf{c}: equals \(-1\); 

\textbf{d}: equals \(-2\).

(16) \( \lim_{(x,y) \to (0,0)} \frac{y}{x^2+y} \)

\textbf{a}: does not exist since the limits along the paths } C_1(t) = (t, t) \text{ and } C_2(t) = (3t, t) \text{ are different; }

\textbf{b}: does not exist since the limits along the paths } C_1(t) = (t, t^2) \text{ and } C_2(t) = (t, 3t^2) \text{ are different; }

\textbf{c}: equals \(-1\); 

\textbf{d}: equals \(-2\).
(17) Let \( S \) be the **upward oriented** surface that is the part of the cone \( z^2 = x^2 + y^2 \) located between the planes \( z = 1 \) and \( z = 3 \). Let \( \mathbf{F}(x, y, z) = k \) Then \( \int \int_S \mathbf{F} \cdot d\mathbf{S} = \)

- a: \( 8\pi \);
- b: \( -8\pi \);
- c: 0;
- d: 1.

(18) Let \( S \) be the surface that is the part of the cone \( z^2 = x^2 + y^2 \) located between the planes \( z = 1 \) and \( z = 3 \). Then the area of \( S \) is equal to

- a: \( 16\pi \);
- b: \( 12 \);
- c: \( 8\pi \sqrt{2} \);
- d: \( \pi \).
(19) Find the absolute maximum and the absolute minimum of \( f(x, y) = (2x - 1) + y^2 \) on the half disk \( D = \{(x, y) | y \geq 0, x^2 + y^2 \leq 4 \} \). The absolute minimum and the absolute maximum are

a: \(-5\) and \(4\);
b: \(-5\) and \(3\);
c: \(0\) and \(2\);
d: \(-3\) and \(0\).

(20) The function \( f(x, y) = \frac{1}{3}x^3 - x + \frac{1}{2}y^2 - y \) has:

a: no critical points;
b: a local minimum at \((1, 1)\) and a local minimum at \((-1, 1)\);
c: a local minimum at \((1, 1)\) and a saddle point at \((-1, 1)\);
d: local maxima both at \((1, 1)\) and at \((-1, 1)\).
(21) Let \( W(s, t) = f(u(s), v(s, t)) \). Assume that \( u'(1) = 1, \ \frac{\partial u}{\partial s}(1, 3) = 1, \ \frac{\partial v}{\partial t}(1, 3) = 4, \ u(1) = 0, \ v(1, 3) = 4, \ f_u(0, 4) = 1, \) and \( f_v(0, 4) = -2 \). Then \( \frac{\partial W}{\partial s}(1, 3) = \)

\[ \begin{array}{ll}
  \text{a:} & 7.5; \\
  \text{b:} & 0; \\
  \text{c:} & -5; \\
  \text{d:} & -1.
\end{array} \]

(22) The arclength of the curve \( r(t) = (t\sqrt{2}, e^{-7t}, e^{7t}) \), \( 0 \leq t \leq 1 \) is equal to:

\[ \begin{array}{ll}
  \text{a:} & 12; \\
  \text{b:} & e - e^{-1}; \\
  \text{c:} & e^7 - e^{-7}; \\
  \text{d:} & \pi.
\end{array} \]
(23) The area of the triangle with vertices (2, 1, 3), (2, 1, 0), (0, 1, 3) is equal to:
   a: 2;
   b: 3;
   c: 4;
   d: 5.

(24) The particle is moving with the acceleration \( \mathbf{a}(t) = e^t \mathbf{i} \). It has initial velocity \( \mathbf{v}(0) = \mathbf{i} + \mathbf{j} \) and initial position \( \mathbf{r}(0) = \mathbf{i} + \mathbf{k} \). Then \( \mathbf{r}(t) = \)
   a: \( e^t \mathbf{i} + t \mathbf{j} + \mathbf{k} \);
   b: \( \frac{(e^t)^2}{2} \mathbf{i} + t \mathbf{j} + \mathbf{k} \);
   c: \( (e^t + 1) \mathbf{i} + t \mathbf{j} + \mathbf{k} \);
   d: \( e^t \mathbf{i} + \mathbf{k} \).
(25) Let $E$ be a solid bounded from above by the cone $z = \sqrt{x^2 + y^2}$, bounded from below by the plane $z = 0$, and bounded from the sides by the cylinder $x^2 + y^2 = 1$. Then

\[
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\]

a: $\frac{\pi}{4}$;
b: $\frac{\pi}{3}$;
c: $\pi$;
d: $3 + \pi$. 