1. Let \( n \geq 2 \), and \( x_1, \ldots, x_n \) be indeterminates over a field \( K \). The Vandermonde determinant is given by

\[
V(x_1, \ldots, x_n) = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}
\end{vmatrix}
\]

Show that \( V(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i) \).

**Hint:** You can prove this by induction on \( n \) and by observing that \( V(x_1, \ldots, x_k, z) \) is a polynomial of degree \( k \) in the variable \( z \) with coefficients in the field \( K(x_1, \ldots, x_k) \) having roots \( x_1, \ldots, x_k \).

2. Let \( A \) be an integral domain with quotient field \( K \), and let \( L/K \) be a finite separable extension of fields of degree \( n \). Let \( B \) denote the integral closure of \( A \) in \( L \). Let \( \sigma_1, \ldots, \sigma_n \) denote the distinct embeddings of \( L/K \) into \( \overline{K} \) (an algebraic closure of \( K \)), and let \( \{\alpha_1, \ldots, \alpha_n\} \) be a basis for \( L \) over \( K \). The discriminant of the basis \( \{\alpha_1, \ldots, \alpha_n\} \) is

\[
\Delta_{L/K}(\alpha_1, \ldots, \alpha_n) = \begin{vmatrix}
\sigma_1(\alpha_1) & \cdots & \sigma_n(\alpha_1) \\
\vdots & \ddots & \vdots \\
\sigma_1(\alpha_n) & \cdots & \sigma_n(\alpha_n)
\end{vmatrix}^2
\]

Note that \( \Delta_{L/K}(\alpha_1, \ldots, \alpha_n) \) is independent of the order of the \( \sigma_i \) or \( \alpha_j \) since interchanging two rows or columns of the matrix changes the determinant by \(-1\).

Suppose that \( \{\alpha_1, \ldots, \alpha_n\} \) and \( \{\beta_1, \ldots, \beta_n\} \) are two bases for \( L/K \). Show

(a) If \( \beta_i = \sum_j c_{ij}\alpha_j \), then \( \Delta_{L/K}(\beta_1, \ldots, \beta_n) = \det((c_{ij}))^2 \Delta_{L/K}(\alpha_1, \ldots, \alpha_n) \).

(b) Show that \( \Delta_{L/K}(\alpha_1, \ldots, \alpha_n) \neq 0 \). **Hint:** Using the previous part, show that it is enough to find one basis with nonzero discriminant. Then use the fact that \( L/K \) is a finite separable extension to produce a nice basis.

(c) Show that if \( \{\alpha_1, \ldots, \alpha_n\} \subset B \), and \( A \) is integrally closed, then \( \Delta_{L/K}(\alpha_1, \ldots, \alpha_n) \in A \).

To see this, let \( M \) be the matrix

\[
\begin{pmatrix}
\sigma_1(\alpha_1) & \cdots & \sigma_n(\alpha_1) \\
\vdots & \ddots & \vdots \\
\sigma_1(\alpha_n) & \cdots & \sigma_n(\alpha_n)
\end{pmatrix}
\]

Then show that

\[
\Delta_{L/K}(\alpha_1, \ldots, \alpha_n) = \det(MM^t) = \det((Tr_{L/K}(\alpha_i \alpha_j))).
\]

Now follow your nose.
3. (A generalization of the division algorithm in Euclidean domains) Let $A$ be a com-
mutative ring with identity, and let $f, g \in A[y]$ be polynomials with $\deg(f) = n,$
$\deg(g) = m,$ and $g$ is assumed nonzero. Let $a$ be the leading coefficient of $g$, and let $k = \max(0, n - m + 1).$ Then

(a) Show that there exist polynomials $q, r \in A[y]$ such that $a^k f = gq + r,$ and $r = 0$
or $\deg(r) < \deg(g).$ **Hint:** You might find it useful to proceed by induction of $\deg(f)$.

(b) If $a$ is not a zero divisor, show that $q$ and $r$ are uniquely determined.

(c) If $a$ is a unit, show that there are uniquely determined $q', r' \in A[y]$ such that $f = gq' + r'$ with $r' = 0$ or $\deg(r') < \deg(g).$