1. An element $m$ of an $R$-module $M$ is called a torsion element if there exists a nonzero $r \in R$ with $rm = 0$.

(a) If $R$ is an integral domain, show that the torsion elements form a submodule $\text{tor}(M)$ of $M$.

Also, show that $M/\text{tor}(M)$ has no nonzero torsion elements (i.e. it is torsion free).

(b) Show that if $R$ is not an integral domain, then the torsion elements need not form a submodule.

2. An $R$-module is called simple if it is not the zero module and if it has no proper submodule.

(a) Prove that any simple module is isomorphic to $R/M$, where $M$ is a maximal left ideal.

(b) Prove Schur’s Lemma: Let $\varphi: M \rightarrow M'$ be a homomorphism of simple modules. Then either $\varphi$ is zero, or else it is an isomorphism.

(c) Prove that $\text{End}_R(M)$ is a division ring if $M$ is simple.

3. Let $R$ be a ring. Consider the ring $M_n(R)$ of $n \times n$ matrices with entries in $R$.

(a) Show that any two-sided ideal of $M_n(R)$ is of the form $M_n(I)$, all $n \times n$ matrices with entries in $I$, for some two-sided ideal $I$ of $R$.

(b) Conclude that, if $R$ is a simple ring, meaning that it has no nontrivial proper two-sided ideals, then the ring $M_n(R)$ is also simple.

(c) If $R$ is a division ring, is the ring $M_n(R)$ simple?

4. For any index set $T$ and $R$-modules $N, M_t, t \in T$, show that there are group isomorphisms

$$\text{Hom}_R\left(\bigoplus_{t \in T} M_t, N\right) \approx \prod_{t \in T} \text{Hom}_R(M_t, N)$$

and

$$\text{Hom}_R\left(N, \prod_{t \in T} M_t\right) \approx \prod_{t \in T} \text{Hom}_R(N, M_t).$$

5. How many group homomorphisms $\mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/30\mathbb{Z}$ are there?

6. An object $A$ in a category $\mathcal{C}$ is called an initial object if, for every object $X$ in $\mathcal{C}$, there is a unique morphism $A \rightarrow X$. Similarly, an object $Z$ is called a terminal object, if for every object $X$ in $\mathcal{C}$, there is a unique morphism $X \rightarrow Z$.

(a) Prove that initial and terminal objects, if they exist, are unique up to unique isomorphism.

(b) In the category of rings (with $1 \neq 0$ and morphisms preserving $1$), is there an initial object, a terminal object?
(c) Let $A$ and $B$ be objects in a category $C$. Let $\mathcal{D}_{AB}$ be the category with objects all diagrams in $C$ of the form

$$A \rightarrow C \leftarrow B$$

and morphisms all commuting diagrams of the form

$$A \rightarrow C \leftarrow B \quad \xrightarrow{\rho} \quad C'$$

with the obvious notion of composition. What is the initial object in $\mathcal{D}_{AB}$ if it exists?

7. Show that there is a (noncommutative) ring $R$ with $R \approx R \oplus R$, as $R$ modules. Hint: Consider the endomorphism ring of an infinite-dimensional vector space.

8. A retraction of an $R$-module map $i: M' \rightarrow M$ is an $R$-module map $r: M \rightarrow M'$ such that $r \circ i = id_{M'}$. Let

$$0 \rightarrow M' \xrightarrow{i} M \xrightarrow{\pi} M'' \rightarrow 0$$

be a short exact sequence of $R$-modules. If $i$ has a retraction, show that $M \approx M' \times M''$. What is the analogous statement in the category of groups?

9. Give a very short proof of the following standard fact in linear algebra: If $T: V \rightarrow W$ is a linear transformation, then $V \approx \ker T \oplus \text{im} T$.

10. Show that $v = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ extends to a basis $\{v, v_2, \ldots, v_n\}$ of $\mathbb{Z}^n$ if and only if the $a_i$ are coprime, meaning $(a_1) + \cdots + (a_n) = (1)$ as ideals in $\mathbb{Z}$. (Part of this problem can be done quickly using Problem 8.)

11. Let $A = \begin{bmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) If $\varphi: \mathbb{Z}^3 \rightarrow \mathbb{Z}^2$ is the homomorphism whose matrix with respect to the standard bases is $A$, determine the structure of the group $\mathbb{Z}^2 / \text{im} \varphi$ as the direct sum of cyclic groups. Find generators (as few as possible) for this quotient group.

(b) Determine all integer solutions to the system of equations $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

12. Show that if $G$ is a subgroup of the free $\mathbb{Z}$-module $\mathbb{Z}^n$, then there are bases $\{a_1, \ldots, a_k\}$ of $G$ and $\{b_1, \ldots, b_n\}$ of $\mathbb{Z}^n$ such that for each of the basis elements $a_i$ of $G$, there is a $d_i \in \mathbb{Z}$ with $a_i = d_i b_i$.

13. (a) Show that the group of rationals $\mathbb{Q}^+$ under addition is not a free $\mathbb{Z}$-module, even though it’s torsion free.

(b) Show that the torsion $\mathbb{Z}$-module $\mathbb{Q}^+ / \mathbb{Z}^+$ is not an infinite direct sum of cyclic groups.

14. If $G$ is finite abelian group with presentation $0 \rightarrow \mathbb{Z}^n \xrightarrow{\varphi} \mathbb{Z}^n \rightarrow G \rightarrow 0$, show that $|G| = |\text{det}([\varphi])|$, where $[\varphi]$ is the matrix of $\varphi$ with respect to any bases.
15. Let $F$ be a field and $H \leq F^\times$ a finite subgroup of the multiplicative group of units of $F$. Show that $H$ is cyclic. (Hint: Use the characterization of cyclic groups in terms of their exponents.)

16. (a) If $M$ and $N$ are finitely generated torsion modules over a PID $R$, show that

$$\text{Hom}_R(M, N) \approx \bigoplus_p \text{Hom}_R(T_p(M), T_p(N))$$

where the sum is over a finite number of primes $p$ of $R$.

(b) Describe the structure of the abelian group $\text{Hom}_Z(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ as a direct sum of cyclic groups (with as few summands as possible).

17. (a) Let $V$ be a finite-dimensional vector space over any field. If $T^2 = \text{Id}$, can $T$ be diagonalized? If so, what are the possible eigenvalues of $T$?

(b) Same question but possible $T^2 = T$.

(c) $T^2 = 0$.

18. How many $\mathbb{Z}$-bilinear maps are there from $\mathbb{Z} \times \mathbb{Z}$ to $G$, where $G$ is any finite abelian group? Describe them explicitly.

19. (a) Let $I$ and $J$ be two-sided ideals of a ring $R$. Show that $\frac{R}{I} \otimes_R \frac{R}{J} \cong \frac{R}{I+J}$.

(b) Show that $\frac{\mathbb{Z}}{m\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{n\mathbb{Z}}$ is cyclic. What is its order? Describe a generator explicitly.