1. Problem #9 on page 59 of the text.  

**ANS:** Start by defining $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$
\varphi(x) = \begin{cases} 
  x + 1 & \text{if } -1 \leq x \leq 0, \\
  1 - x & \text{if } 0 \leq x \leq 1, \\
  0 & \text{otherwise.}
\end{cases}
$$

Thus the graph of $\varphi$ looks like

And we can define

$$f_{n,k}(x) := \varphi(2^n x - k) \quad \text{for } k = 0, 1, \ldots, 2^n.$$

Then restricted to $[0, 1]$ the graphs of $f_{3,3}$, $f_{3,6}$, and $f_{3,8}$, respectively, look like

Note that in order for $f_{n,k}$ to be nonzero, we must have $2^n x - k \in (-1, 1)$ or $2^n x \in (k - 1, k + 1)$. Thus

$$f_{(n,k)}(x) \neq 0 \iff x \in \left(\frac{k}{2^n} - \frac{1}{2^n}, \frac{k}{2^n} + \frac{1}{2^n}\right).$$

Similarly,

$$f_{(n,k)}(x) \geq \frac{1}{2} \iff x \in \left[\frac{k}{2^n} - \frac{1}{2^n+1}, \frac{k}{2^n} + \frac{1}{2^n+1}\right].$$

We will consider $\{f_{(n,k)} : n \in \mathbb{Z}^+ \text{ and } 0 \leq k \leq 2^n\}$ to be the sequence

$$f(1,0), f(1,1), f(2,0), f(2,1), f(2,2), f(2,3), f(2,4), \ldots, f(n,0), \ldots, f(n,2^n), \ldots$$

Since

$$\int_0^1 f_{(n,k)} \, dm \leq \frac{1}{2^n}$$

it follows that

$$\lim_{(n,k) \to \infty} \int_0^1 f_{(n,k)} \, dm = 0.$$ 

Now fix $x \in [0, 1]$. We need to show that

$$\lim_{(n,k) \to \infty} f_{(n,k)}(x) \quad \text{doesn’t exist.}$$

It will suffice to produce infinitely many $(n,k)$ such that $f_{(n,k)}(x) = 0$ as well as infinitely many $(n,k)$ such that $f_{(n,k)}(x) \geq \frac{1}{2}$. But for each $n \in \mathbb{Z}^+$ there is a unique $p_n \in \{0, 1, \ldots, 2^n\}$ such that

$$x \in \left[\frac{p_n}{2^n} - \frac{1}{2^n+1}, \frac{p_n}{2^n} + \frac{1}{2^n+1}\right].$$
2. Problem #4 on page 73 of the text. You may use the result of problem #2 on page 73. On part (c), I found the function defined on \([e, \infty)\) by

\[
f(x) = \frac{1}{x(\log x)^2}
\]

to be interesting.

**ANS.** (a) Suppose \(r < p < s\). If \(|f(x)| \leq 1\), then \(|f(x)|^p \leq |f(x)|^r\). On the other hand, if \(|f(x)| \geq 1\), then \(|f(x)|^p \leq |f(x)|^s\). Therefore for all \(x \in X\),

\[
|f(x)|^p \leq |f(x)|^r + |f(x)|^s.
\]

In particular, if \(r, s \in E\), then \(\varphi(p) \leq \varphi(r) + \varphi(s) < \infty\) and \(p \in E\).

(b) Suppose that \(r, s \in E\). Given \(\lambda \in (0, 1)\) we need to show that

\[
\log \varphi(\lambda s + (1 - \lambda)r) \leq \lambda \log \varphi(s) + (1 - \lambda) \log \varphi(r).
\]

Now let \(p = \frac{1}{\lambda}\) and \(q = \frac{1}{1 - \lambda}\). Then \(\frac{1}{p} + \frac{1}{q} = 1\), and by Hölder:

\[
\varphi(\lambda s + (1 - \lambda)r) = \varphi^{\frac{1}{p} + \frac{1}{q}} = \int_X |f|^\frac{1}{p} |f|^\frac{1}{q} d\mu
\]

\[
\leq \| |f|^{\frac{1}{p}} \|_p \| |f|^{\frac{1}{q}} \|_q
\]

\[
= \varphi(s)^{\frac{1}{p}} \varphi(r)^{\frac{1}{q}}
\]

\[
= \varphi(s)^{\lambda} \varphi(r)^{1 - \lambda}.
\]

Since \(\log\) is monotonic, we obtain (3).

Note that problem 2 in the text proves only that \(\varphi\) is continuous in the interior of \(E\). But suppose \(\{ p_n \}\) is a sequence in \(E\) converging to \(p \in E\). Then \(r := \max\{ p_n, p \}\) and \(s := \min\{ p_n, p \}\) are elements of \(E\) and \(|f|^{p_n} \leq |f|^r + |f|^s \in L^1(\mu)\).

Thus the Lebesgue dominated convergence theorem implies that

\[
\lim_n \varphi(p_n) = \lim_n \int_X |f|^{p_n} d\mu = \int_X |f|^p d\mu = \varphi(p),
\]

which proves that \(\varphi\) is continuous.

(c) It turns out that \(E\) can be any connected subset of \((0, \infty)\): \((a, \infty)\), \([a, \infty)\), \((0, b)\), \((0, b]\), \((a, b)\), and \([a, b]\) for any \(0 \leq a \leq b \leq \infty\). Note this even includes singleton sets.

To prove this, it suffices to consider only \(X = \mathbb{R}\) and Lebesgue measure. It will be convenient to establish some notation and make two observations. Let \(E(f) := \{ p \in (0, \infty) : \int_{\mathbb{R}} |f|^p dm < \infty \}\).

**Observation I:** If \(f, g \geq 0\) then \(0 \leq f \leq f + g \) and \(0 \leq g \leq f + g \) imply that \(f, g \in L^1(\mathbb{R})\) if and only if \(f + g \in L^1(\mathbb{R})\). Thus \(E(f + g) = E(f) \cap E(g)\).

**Observation II:** If \(p \in E(f)\), then \(ap \in E(|f|^\frac{3}{2})\).

**Observation III:** Combining Observations I & II, it will suffice to find functions \(f_1\), \(f_2\), \(f_3\) and \(f_4\) such that \(E(f_1) = (1, \infty)\), \(E(f_2) = (0, 1)\), \(E(f_3) = [1, \infty)\) and \(E(f_4) = (0, 1]\).

But we can let \(\mathbb{I}_A\) be the characteristic function of the set \(A\) and define

\[
f_1(x) = \mathbb{I}_{(0, \infty)}(x) \frac{1}{x} \quad \text{and} \quad f_2(x) = \mathbb{I}_{(0, 1)}(x) \frac{1}{x}.
\]

Then \(\int_{\mathbb{R}} |f_1|^p dm = \int_0^\infty \frac{1}{x^p} dx\) converges if and only if \(p \in (1, \infty)\). Similarly, \(E(f_2) = (0, 1]\).

Now let \(f_3(x) := \mathbb{I}_{(1, \infty)}(x^{-1}(\log x))^{-2}\). Then making the substitution \(u = \log x\) shows that

\[
\int_{\mathbb{R}} f_3 dm = \int_{\mathbb{R}} \frac{1}{x^p (\log x)^2} dx = \int_{1}^{\infty} u^{-2p} e^{(1-p)u} du.
\]
Note that the latter integral converges if and only if $p \in [1, \infty)$ and $E(f_3) = [1, \infty)$ as desired.

Similarly,

$$\int_0^\frac{1}{2} \frac{1}{x^p(\log x)^{2p}} \, dx = \int_{-\infty}^{-1} u^{-2p} e^{(1-p)u} \, du$$

converges only if $p \in (0, 1]$. Thus we can set $f_4(x) := \frac{1}{2} x^{-1}(\log x)^{-2}$.

(d) If $r < p < s$ the there is $\lambda \in (0, 1)$ such that $p = \lambda r + (1 - \lambda)s$. The using the convexity of $\log \varphi$:

$$\log \| f \|_p = \frac{1}{p} \log \varphi(p)$$

$$\leq \frac{1}{\lambda r + (1 - \lambda)s} \left( \lambda \log \varphi(r) + (1 - \lambda) \log \varphi(s) \right)$$

$$= \frac{1}{\lambda r + (1 - \lambda)s} \left( \lambda r \log \| f \|_r + (1 - \lambda)s \log \| f \|_s \right)$$

$$\leq \max \{ \log \| f \|_r, \log \| f \|_s \}.$$ 

Since log is monotone, $\| f \|_p \leq \max\{ \| f \|_r, \| f \|_s \}$.

(e) This was hard. Since the case $f \sim 0$ is trivial, we'll assume $0 < \| f \|_r < \infty$ for some $r$. First, consider the case $\| f \|_\infty = \infty$. Then for any $M > 0$, $A := \{ x : |f(x)| \geq M \}$ has nonzero measure. Since $\| f \|_r < \infty$, $\mu(A) < \infty$. Thus $\| f \|_p \geq M \mu(A)^{\frac{1}{r}}$. Since $\mu(A)^{\frac{1}{r}} \to 1$ as $p \to \infty$, we have $\lim\sup_p \| f \|_p \geq M$ for all $M > 0$. Thus $\lim\sup_p \| f \|_p = \infty = \| f \|_\infty$.

Now assume $\| f \|_\infty < \infty$. Since the $L^p$-norms are homogenous, we can replace $f$ by $\| f \|_\infty^\lambda f$, and assume from here on that $\| f \|_\infty = 1$. Then $p > r$ implies

$$\| f \|_p = \int_X |f|^p \, d\mu$$

$$\leq \| |f|^p \|_1 \| f \|^p_\infty$$

$$\leq \| f \|^p_1 - 1.$$

Thus $\| f \|_p \leq \| f \|^p_1/p$, and

$$\lim\sup_p \| f \|_p \leq \lim_p \| f \|_r^p = 1.$$  \hspace{1cm} (4)

Now fix $\epsilon > 0$ and note that $\| f \|_\infty = 1$ and $\| f \|_r < \infty$ imply that $A := \{ x : |f(x)| \geq 1 - \epsilon \}$ has finite nonzero measure. Thus $\| f \|_p \geq (1 - \epsilon)\mu(A)^{\frac{1}{r}}$. Thus

$$\lim\inf_p \| f \|_p \geq \lim(p) (1 - \epsilon)\mu(A)^{\frac{1}{r}} = 1 - \epsilon.$$  \hspace{1cm} (5)

Since (5) holds for all $\epsilon > 0$, $\lim\inf \| f \|_p \geq 1$. Combining with (4) gives the result.

3. Problem #7 on page 73 of the text. The examples $L^p([0, 1], m)$, and $\ell^p$ will be important here. Note that $\ell^p$ is the set of complex sequences $(a_i)$ such that $\sum_{i=1}^\infty |a_i|^p < \infty$. Thus, $\ell^p = L^p(\mathbb{Z}^+, \nu)$ where $\nu$ is counting measure.

**ANS:** In order to give a fairly general answer to this question, I'll start with a few comments. Throughout, $(X, \mathcal{M}, \mu)$ will be a measure space and $1 < r < s < \infty$.

**Comment 1:** If $\mu(X) < \infty$, then $L^s(\mu) \subset L^r(\mu)$.

**Proof.** Suppose that $f \in L^s(\mu)$. Since $\frac{1}{r}$ and $\frac{1}{s}$ are conjugate exponents, Hölder implies

$$\int_X |f|^r \, d\mu \leq \left( \int_X |f|^s \, d\mu \right)^{\frac{r}{s}} \left( \int_X 1^{\frac{s}{r'}} \, d\mu \right)^{\frac{r'}{s}} \leq \| f \|_s \mu(X)^{\frac{s}{r'}} < \infty$$

Thus $f \in L^r(\mu)$ and the claim is proved.  \hfill $\square$
I suggest the following strategy:

since

Proof. Again, some work is required to see that there are pairwise disjoint sets \( \{ A_n \}_{n=1}^\infty \) such that \( 0 < \mu(A_n) < \infty \) and \( \mu(A_n) \searrow 0 \). We can pass to a subsequence if necessary, relabel, and assume that \( \mu(A_n) \leq \left( \frac{1}{n} \right)^{\frac{1}{2+\varepsilon}} \). Set \( a_n = \mu(A_n)^{-\frac{1}{2}} \) and let \( f := \sum_n a_n 1_{A_n} \). Since the \( A_n \) are disjoint,

\[
\|f\|^r_n = \sum_n a_n^r \mu(A_n) = \sum_n \mu(A_n)^{1-\frac{r}{2}} \\
\leq \sum_n \left( \frac{1}{n} \right)^{\frac{2r}{2+\varepsilon}} \\
= \sum_n \left( \frac{1}{n} \right)^2 < \infty.
\]

Thus \( f \in L^r(\mu) \). But

\[
\|f\|_s^r = \sum_n a_n^r \mu(A_n) = \sum_n 1 = \infty.
\]

Thus \( f \notin L^s(\mu) \).

Comment 3: Suppose that \( X \) contains subsets of arbitrary large finite measure. Then \( L^s(\mu) \not\subset L^r(\mu) \).

Proof. Again, some work is required to see that there are pairwise disjoint sets \( \{ A_n \} \) such that \( \mu(A_n) \geq 1 \) for all \( n \). Now let \( a_n := \left( \frac{1}{n} \right)^{\frac{1}{2+\varepsilon}} \mu(A_n)^{-\frac{1}{2}} \), and \( f := \sum_n a_n 1_{A_n} \). Then

\[
\|f\|^r_n = \sum_n a_n^r \mu(A_n) = \sum_n \left( \frac{1}{n} \right)^{\frac{r}{2+\varepsilon}} < \infty,
\]

since \( \frac{1}{2+\varepsilon} > 1 \). Therefore \( f \in L^s(\mu) \), but

\[
\|f\|_s^r = \sum_n a_n^r \mu(A_n) = \sum_n \left( \frac{1}{n} \right)^{\frac{r}{2+\varepsilon}} \mu(A_n)^{1-\frac{r}{2}} \geq \sum_n \left( \frac{1}{n} \right)^{\frac{r+2}{2+\varepsilon}} = \infty,
\]

since \( 1 - \frac{r}{2} > 0 \) and \( \frac{r+2}{2+\varepsilon} < 1 \). Thus \( f \notin L^r(\mu) \).

A set \( A \in \mathcal{M} \) is called a atom if \( \mu(A) > 0 \) and the only proper measurable subsets of \( A \) are null sets.

Comment 4: Suppose that \( X \) doesn’t contain sets of arbitrary small positive measure. Then \( L^s(\mu) \subset L^r(\mu) \).

Proof. By assumption, there is \( \varepsilon > 0 \) such that \( \mu(A) > 0 \) implies \( \mu(A) \geq \varepsilon \). Then it is possible to prove that \( X = \bigcup_j A_j \) where each \( A_j \) is an atom in \( X \). Of course, we have \( \mu(A_j) \geq \varepsilon \) for all \( j \). Also, if \( f \) is measurable, then \( f = \sum_j a_j 1_{A_j} \).

Given all this — some of which is nontrivial to prove — it follows that \( \|f\|^r_n = \sum_j |a_j|^r \mu(A_j) \). If \( f \in L^r(\mu) \), then we must have \( \sum_j |a_j|^r \mu(A_j) < \infty \) and eventually \( |a_j|^r \mu(A_j) < \varepsilon \). Since \( \mu(A_j) \geq \varepsilon \) this means eventually \( |a_j| < 1 \). Thus eventually, \( |a_j|^r \mu(A_j) \leq |a_j|^r \mu(A_j) \) and \( \sum_j |a_j|^r \mu(A_j) \) converges by the comparison theorem. Thus \( f \in L^s(\mu) \) as required.

Now for specific examples. Using Comments 1 and 2 we see that \( L^s([0,1]) \subset L^r([0,1]) \) and \( L^r([0,1]) \not\subset L^s([0,1]) \).

Using Comments 3 and 2, we see that \( L^s(\mathbb{R}) \not\subset L^r(\mathbb{R}) \) and \( L^r(\mathbb{R}) \not\subset L^s(\mathbb{R}) \).

Using Comments 4 and 3, we see that \( \ell^r \subset \ell^s \) and \( \ell^s \nsubseteq \ell^r \).

4. Let \((X, \mathcal{M}, \mu)\) be a measure space. Show that simple functions are dense in \( L^p(X, \mathcal{M}, \mu) \) for \( 1 \leq p \leq \infty \). I suggest the following strategy:

(a) Observe that it suffices to consider nonnegative \( f \).

(b) For \( 1 \leq p < \infty \), apply Theorem 1.17 in the text.
(c) For \( p = \infty \), observe that if \( f \) is bounded then the \( s_i \) constructed in class (or in 1.17 in the text) not only satisfy \( s_i \not\to f \), but \( \|s_i - f\|_{\infty} < 2^{-i} \) provided \( \|f\|_{\infty} \leq 1 \). (More simply said, the convergence is uniform.)

**ANS:** Suppose that \( f \in L^p(\mu) \), \( 1 \leq p \leq \infty \). Then \( f = f_1 - f_2 + i(f_3 - f_4) \) with each \( f_i \geq 0 \). Since \( f_i^p \leq \|f\|^p \) for \( 1 \leq p < \infty \), we have each \( f_i \in L^p(\mu) \) as well. Therefore, we need only consider the case where \( f \geq 0 \). Then we can choose simple functions \( s_i \) such that \( s_i \not\to f \).

First consider the case \( 1 \leq p < \infty \). Notice that \( |f - s_i|^p \leq f^p \), and that the latter is in \( L^1(\mu) \). Since \( |f - s_i|^p \to 0 \) (pointwise), the LDCT implies that

\[
\int_X |f - s_i|^p \, d\mu \to 0, \tag{6}
\]

and \( s_i \to f \) in \( L^p(\mu) \) as desired.

If \( p = \infty \), we can assume that \( |f| \leq N \) (everywhere). By construction, \( |f(x) - s_i(x)| \leq 2^{-i} \) provided \( i \geq N \)! That is, \( \|f - s_i\|_{\infty} < 2^{-i} \) whenever \( i \geq N \). That is, \( s_i \to f \) in \( L^\infty(\mu) \).

5. Let \( \mu, \nu, \) and \( \lambda \) be \( \sigma \)-finite measures on \((X, M)\). We’ll denote the Radon-Nikodym derivative of \( \nu \) by \( \mu \) by \( \frac{d\nu}{d\mu} \).

(a) Show that if \( \nu \ll \mu \) and \( g : X \to [0, \infty] \) is measurable, then \( \int_X g \, d\nu = \int_X g \frac{d\nu}{d\mu} \, d\mu \). (As observed in class, this is a Corollary of an old Theorem.) Conclude that \( f \in L^1(\nu) \) if and only if \( f \frac{d\nu}{d\mu} \in L^1(\mu) \), and that the same formula holds.

(b) Suppose that \( \nu \ll \mu \ll \lambda \). Show that \( \frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \). Of course, “\( = \)” means “equal almost everywhere \( [\lambda] \).”

(c) Suppose that \( \mu \ll \nu \) and \( \nu \ll \mu \) (we say the \( \mu \) and \( \nu \) are equivalent and write \( \nu \approx \mu \)). Show that

\[
\frac{d\mu}{d\nu} = \left[ \frac{d\nu}{d\mu} \right]^{-1}.
\]

Again “\( = \)” means “equal almost everywhere \( [\mu] \) (or \([\nu]\))”.

**ANS:** We proved in class that if \( f : X \to [0, \infty] \) was measurable, then \( \nu(E) = \int_E f \, d\mu \) defined a measure on \((X, M)\) and that \( \int_E g \, d\nu = \int_E g f \, d\mu \) for all measurable functions \( g : X \to [0, \infty] \). The first assertion follows merely by replacing \( f \) by \( \frac{d\nu}{d\mu} \). On the other hand, if \( f : X \to \mathbb{C} \) is measurable, then we’ve just argued that

\[
\int_X |f| \, d\nu = \int_X |f| \frac{d\nu}{d\mu} \, d\mu. \tag{7}
\]

Since the LHS is finite exactly when the RHS is, we have \( f \in L^1(\nu) \) if and only if \( f \frac{d\nu}{d\mu} \in L^1(\mu) \). Furthermore if \( f = f_1 - f_2 + i(f_3 - f_4) \) with \( f_i \geq 0 \), then \( \int_E f \, d\nu = \int_E f \frac{d\nu}{d\mu} \, d\mu \) by the above. It follows that \( f f \, d\nu = f f \frac{d\nu}{d\mu} \, d\mu \) for all \( f \in L^1(\nu) \) as desired. This takes care of part (a).

Parts (b) and (c) follow by the uniqueness part of the R-N Theorem. For example in part (b), we have

\[
\nu(E) = \int_E \frac{d\nu}{d\mu} \, d\mu, \quad \text{by R-N,}
\]

\[
= \int_E \frac{d\nu}{d\lambda} \, d\lambda, \quad \text{by the first part.}
\]

Thus, \( \frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda} \) (a.e.) by the R-N Theorem. Part (c) now follows from part (b) and the fact that \( \frac{d\mu}{d\lambda} = 1 \) (a.e.).

6. Let \( \nu \) be a complex measure on \((X, M)\).
(a) Show that there is a measure \( \mu \) and a measurable function \( \varphi : X \to \mathbb{C} \) so that \( |\varphi| = 1 \), and such that for all \( E \in \mathcal{M} \),

\[
\nu(E) = \int_E \varphi \, d\mu.
\]  

(Hint: write \( \nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4) \) for measures \( \nu_i \). Put \( \mu_0 = \nu_1 + \nu_2 + \nu_3 + \nu_4 \). Then \( \mu_0 \) will satisfy (†) provided we don’t require \( |\varphi| = 1 \). Now use the “useful lemma” from lecture.)

(b) Show that the measure \( \mu \) above is unique, and that \( \varphi \) is determined almost everywhere \( [\mu] \). (Hint: if \( \mu' \) and \( \varphi' \) also satisfy (†), then show that \( \mu' \ll \mu \), and that \( \frac{d\mu'}{d\mu} \) is 1 a.e. Also note that if \( \varphi' \) is unimodular and \( E \in \mathcal{M} \), then \( E = \bigcup_{i=1}^4 E_i \) where \( E_1 = \{ x \in E : \text{Re} \varphi' > 0 \} \), \( E_2 = \{ x \in E : \text{Re} \varphi' < 0 \} \), \( E_3 = \{ x \in E : \text{Im} \varphi' > 0 \} \), and \( E_4 = \{ x \in E : \text{Im} \varphi' < 0 \} \).

Comment: the measure \( \mu \) in question 6 is called the total variation of \( \nu \), and the usual notation is \( |\nu| \). It is defined by different methods in your text: see chapter 6. One can prove facts like \( |\nu|(E) \geq |\nu(E)| \), although one doesn’t always have \( |\nu|(E) = |\nu(E)| \); this also proves that even classical notation can be unfortunate.

ANS. Let \( \nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4) \) be a decomposition of \( \nu \) into finite (positive) measures as in class. Define \( \mu_0 = \nu_1 + \nu_2 + \nu_3 + \nu_4 \). Clearly \( \nu_i \ll \mu_0 \) so there are non-negative functions \( h_i \) so that \( \nu_i(E) = \int_E h_i \, d\mu_0 \). Since each \( \nu_i \) is finite, we can assume that each \( h_i \in L^1(\mu_0) \) and hence that each \( h_i \) is finite-valued. Therefore we can define \( h = h_1 - h_2 + i(h_3 - h_4) \) and observe that

\[
\nu(E) = \int_E h \, d\mu_0.
\]  

By our lemma, we can write \( h = \varphi |h| \) with \( \varphi \) unimodular and everything in sight measurable. Then we can define \( \mu \) by \( \mu(E) = \int_E |h| \, d\mu_0 \) and then \( \nu(E) = \int_E \varphi |h| \, d\mu_0 \) as desired. This proves part (a).

To establish part (b), suppose that \( \nu(F) = \int_F \varphi |h| \, d\mu_0 \) for all \( F \in \mathcal{M} \) for another measure \( \mu' \) and unimodular function \( \varphi' \). Suppose that \( \mu'(E) = 0 \). We first need to show that \( \mu'(E_i) = 0 \). Let \( E_i = \{ x \in E : \text{Re} \varphi' > 0 \} \), \( E_3 = \{ x \in E : \text{Im} \varphi' > 0 \} \), and \( E_4 = \{ x \in E : \text{Im} \varphi' < 0 \} \). Since \( 1 = |\varphi'|^2 + (\text{Re} \varphi')^2 + (\text{Im} \varphi')^2 \), we must have \( E = \bigcup_{i=1}^4 E_i \). On the other hand, \( E_i \subseteq E \) implies that \( \mu(E_i) = 0 \) for all \( i \). But then \( \nu(E_i) = 0 \), and

\[
\int_{E_i} \varphi' \, d\mu' = 0
\]

for all \( i \). But then \( 0 = \text{Re} \left( \int_{E_i} \varphi' \, d\mu' \right) = \int_{E_i} \text{Re} \varphi' \, d\mu' \), and we must have \( \mu'(E_i) = 0 \) for \( i = 1, 2 \). Similarly, \( \int_{E_3} \text{Im} \varphi' \, d\mu' = 0 \) and \( \mu'(E_3) = 0 \) for \( i = 3, 4 \). Therefore \( \mu'(E_i) = 0 \) and \( \mu' \ll \mu \). But in that case for all \( E \in \mathcal{M} \),

\[
\nu(E) = \int_E \varphi' \, d\mu' = \int_E \varphi' \, \frac{d\mu'}{d\mu} \, d\mu.
\]

Since this holds for all \( E \), we must have \( \varphi = \varphi' \frac{d\mu'}{d\mu} \) (a.e.). Since \( \varphi \) and \( \varphi' \) are unimodular and \( \frac{d\mu'}{d\mu} \) is nonnegative, we must have \( \frac{d\mu'}{d\mu} = 1 \) (a.e.). Thus, \( \varphi = \varphi' \) (a.e.) and \( \mu = \mu' \).

7. If \( \nu \) and \( \lambda \) are complex measures on the same measurable space we define \( \nu \perp \lambda \) and \( \nu \ll \lambda \) if the corresponding relations hold for \( |\nu| \) and \( |\lambda| \).

(a) Suppose that \( \lambda \) is a positive measure. Show that \( \nu \ll \lambda \) if and only if \( \lambda(E) = 0 \) implies \( \nu(E) = 0 \). (You’ll want to use Equation (†).)

(b) Show that if \( \nu \perp \lambda \) and \( \nu \ll \lambda \), then \( \nu \ll \lambda \).

(c) Prove the uniqueness assertion in the Lebesgue Decomposition theorem. (Hint: start with the case where \( \mu \) and \( \nu \) are finite—so that for example, \( \mu - \nu \) is a complex measure. Then use \( \sigma \)-finiteness.)

\(^1\)Note that if \( f(x) > 0 \) for all \( x \in A \) and \( \int_A f \, d\mu = 0 \), then \( \mu(A) = 0 \). To see this note that \( A = \bigcup \{ x \in A : f(x) > 0 \} \).
ANS: In part (a), the \((\implies)\) direction is trivial: \(\lambda(E) = 0\) implies that \(\nu(E) = 0\) (since \(\nu \ll \lambda\) by hypothesis) which implies \(\nu(E) = 0\) by problem #6. Notice that for the other direction, we cannot use the (complex version of the) R-N Theorem because we need to know \(|\Phi|\) to invoke it. However, by problem #6, we do have \(\nu(E) = \int_E \varphi \, d\nu\) with \(\varphi\) unimodular. Then if \(\lambda(E) = 0\), define \(E_i, i=1,2,3,4\), as in the uniqueness proof in problem #6; i.e., \(E_1 = \{ x \in E : \Re\varphi > 0 \}\), \ldots Then as before, \(E = E_1 \cup \cdots \cup E_4\) and \(\lambda(E_i) = 0\) for all \(i\). Therefore by hypothesis, \(\nu(E_i) = 0\) for all \(i\). Because the definition of the \(E_i\), it follows that \(|\nu(E)| = 0\) for all \(i\); in particular, \(|\nu(E) = 0\) as desired.

In part (b), let \(A\) and \(B\) be disjoint measurable sets such that \(E = A \cup B\) and such that \(|\nu(B) = |\lambda(A) = 0\). But \(|\nu \ll |\lambda\) implies that \(|\nu(A) = 0\) as well. But then if \(E\) is any measurable set, \(|\nu(E) = |\nu(E \cap A) + \nu(E \cap B)| \leq |\nu(A) + |\nu(B) = 0\). That is, \(\nu = 0\).

Part (c): Suppose that \(\nu\) and \(\mu\) are \(\sigma\)-finite measures, and that \(\nu = \nu_s + \nu_a = \nu'_s + \nu'_a\) are two decompositions such that \(\nu_s \perp \mu\), \(\nu'_s \perp \mu\), \(\nu_a \ll \mu\), and \(\nu'_a \ll \mu\). We want to show that \(\nu_s = \nu'_s\) and \(\nu_a = \nu'_a\).

To begin with, assume that both \(\nu\) and \(\mu\) are finite. Then \(\nu_s - \nu'_s\) and \(\nu'_a - \nu_a\) are bonafide complex measures and \(\nu_s - \nu'_s \perp \lambda\) while \(\nu'_a - \nu_a\) \(\ll \lambda\). Since \(\nu_s - \nu'_s = \nu'_a - \nu_a\), we have \(\nu_s - \nu'_s = \nu'_a - \nu_a = 0\) by part (b); this takes care of the finite case.

In general, we can assume that \(X = \bigcup_{n=1}^\infty X_n\) with both \(\nu(X_n) < \infty\) and \(\mu(X_n) < \infty\). We can also require that \(X_n \subseteq X_{n+1}\). Using the above, we have \(\nu_n(E) = \nu'_n(E)\) for all \(E \subseteq X_n\). If \(E\) is any measurable set, \(\nu_n(E) = \lim_n \nu_n(E \cap X_n) = \mu_n \nu'_n(E \cap X_n) = \nu'_n(E)\); that is, \(\nu_n = \nu'_n\). A similar argument shows that \(\nu_s = \nu'_s\).

8. Show that the \(\sigma\)-finite hypothesis is necessary in the Radon-Nikodym theorem. (Hint: let \(\nu\) be Lebesgue measure on \([0, 1]\) and let \(\mu\) be counting measure (restricted to the Lebesgue measurable sets in \([0, 1]\)).)

ANS: Since \(\mu(E) = 0\) if and only if \(E = \emptyset\), we clearly have \(\nu \ll \mu\). But if \(\nu(E) = \int_E h \, d\mu\) for all \(E \in \mathcal{M}\), then for each \(x \in [0, 1]\) we would have \(0 = \nu(\{ x \}) = h(x)\). Thus, \(h\) would have to be identically zero. Then \(1 = \nu([0, 1]) = \int_{[0, 1]} h \, d\mu = 0\). This is silly and provides the required counter-example.