1. Denote by $\text{Aut}(\mathbb{Z}_n)$ the group of automorphisms of $\mathbb{Z}_n$ (viewing $\mathbb{Z}_n$ as an additive group). Show that $\text{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^\times$ ($\mathbb{Z}_n^\times$ the multiplicative group of the ring $\mathbb{Z}_n$). It may be of use to recall some of the work we did early in the term on homomorphisms with domain $\mathbb{Z}_n$.

2. Semidirect products.

   (a) Suppose that $H_1, H_2$ and $K$ are groups, $\sigma : H_1 \rightarrow H_2$ is an isomorphism, and $\psi : H_2 \rightarrow \text{Aut}(K)$ a homomorphism, so that $\varphi = \psi \circ \sigma : H_1 \rightarrow \text{Aut}(K)$ is also a homomorphism. Show that $K \rtimes_\varphi H_1 \cong K \rtimes_\psi H_2$.

   (b) Suppose that $H$ and $K$ are groups and $\varphi, \psi : H \rightarrow \text{Aut}(K)$ are monomorphisms with the same image in $\text{Aut}(K)$. Show that there exists a $\sigma \in \text{Aut}(H)$ such that $\psi = \varphi \circ \sigma$.

   (c) Suppose that $H$ and $K$ are groups, $\varphi, \psi : H \rightarrow \text{Aut}(K)$ are monomorphisms, and $\text{Aut}(K)$ is finite and cyclic. Show that $\varphi$ and $\psi$ have the same image in $\text{Aut}(K)$.

   (d) Let $p < q$ be primes with $p \mid (q - 1)$. Let $H$ and $K$ be cyclic groups of order $p$ and $q$ respectively. Let $\varphi, \psi : H \rightarrow \text{Aut}(K)$ be nontrivial homomorphisms. Observing that $\text{Aut}(K)$ is cyclic, show that $K \rtimes_\varphi H \cong K \rtimes_\psi H$.

3. Let $p < q$ be primes, and let $G$ be a group of order $pq$. We know from class that $G \cong \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_p$ for some $\varphi : \mathbb{Z}_p \rightarrow \text{Aut}(\mathbb{Z}_q)$. By analyzing all possible $\varphi$, find (up to isomorphism) all groups of order $pq$. If $p = 2$, describe them without using semidirect products.

4. (Proof or counterexample) Consider the validity of a cancellation law for direct products: Let $G, H, K$ be finitely generated abelian groups and suppose that $G \times H \cong G \times K$. Then $H \cong K$. (I asked an unrestricted version of this question on the first homework set)