A fast direct solver for high frequency scattering from a cavity in two dimensions

Jun Lai [1]

Joint work with:
Leslie Greengard (CIMS)
Sivaram Ambikasaran (CIMS)

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[1] Courant Institute of Mathematical Sciences, New York University

Outline

Introduction

Cavity structures exist in many electromagnetic applications

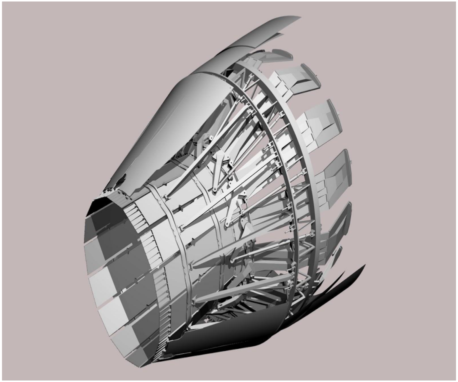


Figure : F16 exhaust nozzle, from: *f16model.blogspot.com*

Geometry

Consider the 2D wave scattering from a cavity embedded in the ground plane

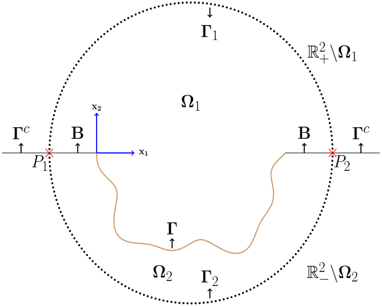


Figure : Geometry of the problem

Related work

- Well-posedness(Ammari-Bao-Wood'02)
- Inverse scattering problem(Bao-Gao-Li'11)
- Stability of the solution(Bao-Yun-Zhou'12,Li-Ma-Sun'12)
- Numerical simulation: bounding and shooting rays(Ling-Chou-Lee'89), Gaussian beam asymptotics(Burkholder'91), finite difference(Bao-Sun'05), finite element(Liu-Jin'00), iterative physical optics(Hemon-Pouliguen'08), mode matching method(Bao-Gao-Lin-Zhang'12),
- Overfilled cavity(Van-Wood'04,Huang-Wood'05)
- Asymptotic of resonances(Bonnetier-Triki'14)
- Optimal design(Bao-Lai'14)

Formulation

- Assumption: permittivity ε_0, permeability μ_0 are constant everywhere
- Plane wave incidence:

$$u^{inc} = e^{i(k_0 \cos\theta x - k_0 \sin\theta y)}$$

- Total field u has the form:

$$u(x,y) = u^s(x,y) + u^{inc} - e^{i(k_0 \cos\theta x + k_0 \sin\theta y)}$$

- Total field u satisfies the equation:

$$\Delta u + k^2 u = 0 \text{ in } \Omega_1 \cup R^2_+$$

where $k = \omega\sqrt{\varepsilon_0\mu_0}$ is the wavenumber.

Helmholtz Equation

The scattered wave satisfies the Helmholtz equation:

$$\begin{cases} \Delta u^s + k^2 u^s & = 0 \text{ in } \Omega_1 \cup R_+^2 \\ u^s & = g \text{ on } \Gamma \cup \Gamma^c \end{cases} \tag{1}$$

along with the Sommerfeld radiation condition (2), where $g = -(u^i + u^r)$, along with the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \tag{2}$$

Approach to find u^s: Integral formulation.

Preliminary

Let

$$\Phi(\mathbf{x},\mathbf{y}) = \frac{i}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|) \tag{3}$$

$$\Phi^H(\mathbf{x},\mathbf{y}) = \frac{i}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|) - \frac{i}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{y}'|) \tag{4}$$

and $\mathcal{S}^H$ and $\mathcal{D}^H$ are the single layer and double layer operators in the half-space:

$$\mathcal{S}^H_{\Gamma_1}\sigma = \int_{\Gamma_1} \Phi^H(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})ds_{\mathbf{y}} \tag{5}$$

$$\mathcal{D}^H_{\Gamma_1}\mu = \int_{\Gamma_1} \frac{\partial \Phi^H(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\mu(\mathbf{y})ds_{\mathbf{y}}. \tag{6}$$

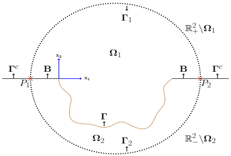
Integral Formulation

Assume Γ_1 is an artificial boundary that covers the cavity Ω_1. A natural formulation to represent the exterior field u^s is:

$$u^s = \mathcal{S}^H_{\Gamma_1}\sigma + \mathcal{D}^H_{\Gamma_1}\mu, \text{ for } \mathbf{x} \in R^2_+ \backslash \Omega_1 \tag{7}$$

The scattered field u^s in Ω_1 is represented by

$$u^s = \mathcal{S}_{\Gamma_1}\sigma + \mathcal{D}_{\Gamma_1}\mu + \mathcal{D}_B\mu + \mathcal{D}_\Gamma\mu \text{ for } \mathbf{x} \in \Omega_1 \tag{8}$$



Integral equation

Combining the boundary condition and jump condition from layer potentials yields

$$\begin{cases} \mu - \mathcal{S}_{\Gamma_2}\sigma - \mathcal{D}_{\Gamma_2}\mu - \mathcal{D}_B\mu - \mathcal{D}_\Gamma\mu = 0, \text{ for } \mathbf{x} \in \Gamma_1 \\ \sigma + \mathcal{N}_{\Gamma_2}\sigma + \mathcal{T}_{\Gamma_2}\mu + \mathcal{T}_B\mu + \mathcal{T}_\Gamma\mu = 0, \text{ for } \mathbf{x} \in \Gamma_1 \\ -\frac{1}{2}\mu + \mathcal{S}_{\Gamma_1}\sigma + \mathcal{D}_{\Gamma_1}\mu + \mathcal{D}_B\mu + \mathcal{D}_\Gamma\mu = g, \text{ for } \mathbf{x} \in \Gamma \cup B \end{cases} \tag{9}$$

i.e.

$$A = \begin{bmatrix} -1/2 & \mathcal{D} & \mathcal{D} & \mathcal{S} \\ \mathcal{D} & -1/2 + \mathcal{D} & \mathcal{D} & \mathcal{S} \\ -\mathcal{D} & -\mathcal{D} & 1 - \mathcal{D}' & \mathcal{S}' \\ \mathcal{T} & \mathcal{T} & \mathcal{T}' & 1 + \mathcal{N}' \end{bmatrix} \begin{bmatrix} \mu_B \\ \mu_\Gamma \\ \mu_{\Gamma_1} \\ \sigma_{\Gamma_1} \end{bmatrix} \tag{10}$$

where

$$\mathcal{N}_\Gamma\sigma = \int_\Gamma \frac{\partial\Phi(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{x})}\sigma(\mathbf{y})ds_\mathbf{y} \tag{11}$$

$$\mathcal{T}_\Gamma\mu = \int_\Gamma \frac{\partial^2\Phi(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{x})\partial n(\mathbf{y})}\mu(\mathbf{y})ds_\mathbf{y} \tag{12}$$

Numerical result

The formulation is not Fredholm equation of second kind. Numerically it is difficult to discretize $\mathcal{T}_B\mu$ to high order.

Table : Results for pot shaped cavity at different wavenumber

k	1	10	20	40	80	160
Error	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-6}	10^{-6}

New formulation

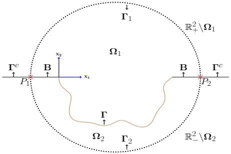
We therefore propose the following formulation:

$$\begin{cases} u^s = \mathcal{S}^H_{\Gamma_1}\sigma + \mathcal{D}^H_{\Gamma_1}\mu + \mathcal{D}_B\mu, \text{ for } \mathbf{x} \in R^2_+ \backslash \Omega_1 \\ u^s = \mathcal{S}^H_{\Gamma_1}\sigma + \mathcal{D}^H_{\Gamma_1}\mu + \mathcal{D}_B\mu + \mathcal{D}_\Gamma\mu, \text{ for } \mathbf{x} \in \Omega_1 \end{cases} \tag{13}$$

Compared to the old formulation:

$$u^s = \begin{cases} \mathcal{S}^H_{\Gamma_1}\sigma + \mathcal{D}^H_{\Gamma_1}\mu, \text{ for } \mathbf{x} \in R^2_+ \backslash \Omega_1 \\ \mathcal{S}_{\Gamma_1}\sigma + \mathcal{D}_{\Gamma_1}\mu + \mathcal{D}_B\mu + \mathcal{D}_\Gamma\mu \text{ for } \mathbf{x} \in \Omega_1 \end{cases} \tag{14}$$

we add some non-physical terms in the new formulation.

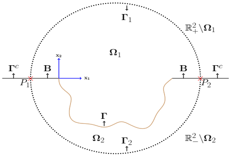


Integral equation

The new formulation leads to the following integral equation:

$$\begin{cases} \mu - \mathcal{D}_\Gamma \mu &= 0, \quad \text{for } \mathbf{x} \in \Gamma_1 \\ \sigma + \mathcal{T}_\Gamma \mu &= 0, \quad \text{for } \mathbf{x} \in \Gamma_1 \\ -\frac{1}{2}\mu + \mathcal{S}^H_{\Gamma_1}\sigma + \mathcal{D}^H_{\Gamma_1}\mu + \mathcal{D}_B\mu + \mathcal{D}_\Gamma\mu &= g, \quad \text{for } \mathbf{x} \in B \cup \Gamma \end{cases}$$

Advantage: the system is still second kind. More importantly, there is no hypersingular term anymore. One can even eliminate μ and σ on Γ_1 by hand and reduce the unknown to μ on $B \cup \Gamma$ only.



Well-posedness

We have the following uniqueness result:

Theorem

Given $k > 0$, the following system

$$\begin{cases} \mu - \mathcal{D}_{\Gamma}\mu &= 0, \quad \text{for } \mathbf{x} \in \Gamma_1 \\ \sigma + \mathcal{T}_{\Gamma}\mu &= 0, \quad \text{for } \mathbf{x} \in \Gamma_1 \\ -\frac{1}{2}\mu + \mathcal{S}^H_{\Gamma_1}\sigma + \mathcal{D}^H_{\Gamma_1}\mu + \mathcal{D}_B\mu + \mathcal{D}_{\Gamma}\mu &= 0, \quad \text{for } \mathbf{x} \in B \cup \Gamma \end{cases}$$

only has a zero solution for $\mu \in C(B) \cup C(\Gamma) \cup C(\Gamma_1)$ and $\sigma \in C(\Gamma_1)$.

from which the existence of the solution follows.

Discretization

In our numerical simulation, we apply Nyström discetization to the integral equation. In other words, we replace the integral

$$\int_{\Gamma} K(\mathbf{x}, \mathbf{y})\sigma(\mathbf{y})dS_{\mathbf{y}} \tag{15}$$

by the quadrature

$$\sum_{i=1}^{N} \sum_{j=1}^{p} \mathcal{K}(\mathbf{x}_{l,m}, \mathbf{y}_{i,j})\sigma(\mathbf{y}_{i,j})w_{l,m,i,j} \tag{16}$$

where $\mathbf{x}_{l,m}$ is the m-th Gauss-Legendre node on panel l, $\mathbf{y}_{i,j}$ is the j-th Gauss-Legendre node on panel i, $w_{l,m,i,j}$ is the quadrature weight and $\mathcal{K}$ is the "quadrature kernel".

Dyadic refinement

In the existence of corners, we apply the dyadic refinement at the end of each segment

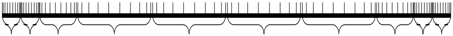


Figure : Graded mesh on a smooth segment

Error analysis: The formal error analysis depends on the regularity of σ and μ. Qualitatively, if ϵ denotes the length of the finest panel, then the error is proportional to $\mathcal{O}(e^{-p} + \epsilon)$.

Fast direct solver

A large dense linear system will be generated from the cavity problem, especially when the frequency is high. Conventional linear solver, either direct or iterative, requires an enormous amount of time. We therefore choose *fast direct solver* to solve the problem. The advantages of fast direct solver are:

- The direct solver is insensitive to multiple reflections inside the cavity, especially at high frequencies and hence is much faster than iterative solver.
- The solver scales exceptionally well with multiple right hand sides.

The linear matrix to be solved

The following is the system that we need to solve:

$$A = \begin{bmatrix} -1/2 & \mathcal{D} & 0 & 0 \\ \mathcal{D} & -1/2 + \mathcal{D} & \mathcal{D}^H & \mathcal{S}^H \\ 0 & -\mathcal{D} & 1 & 0 \\ 0 & \mathcal{T} & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_B \\ \mu_\Gamma \\ \mu_{\Gamma_1} \\ \sigma_{\Gamma_1} \end{bmatrix} \tag{17}$$

Low rank interaction for far field

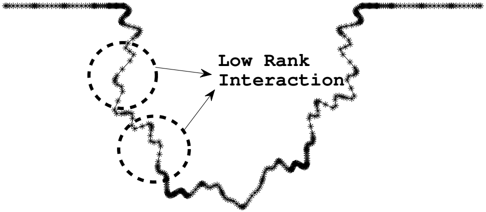


Figure : Low Rank interaction between two different segments

The rank of the far field interaction is roughly on the order of $\log(k)$, which implies the matrix, $A \in \mathbb{R}^{N \times N}$, has a hierarchical off-diagonal low-rank (HODLR) structure.

Inverse of a matrix with low rank perturbation

One can quickly find the inverse of a matrix with low rank perturbation if the inverse of the matrix is known through Sherman-Morrison formula:
If $A \in \mathbb{R}^{n\times n}$ is invertible, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, then

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u} \tag{18}$$

Generalization to rank k perturbation to A is the Sherman-Morrison-Woodbury formula:

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U(I + V^TA^{-1}U)^{-1}V^TA^{-1} \tag{19}$$

HODLR matrix

A 2-level HODLR matrix can be written in the form shown in equation (21).

$$A = \begin{bmatrix} A_1^{(1)} & U_1^{(1)} K_{1,2}^{(1)} {V_2^{(1)}}^T \\ U_2^{(1)} K_{2,1}^{(1)} {V_1^{(1)}}^T & A_2^{(1)} \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} \begin{bmatrix} A_1^{(2)} & U_1^{(2)} K_{1,2}^{(2)} {V_2^{(2)}}^T \\ U_2^{(2)} K_{2,1}^{(2)} {V_1^{(2)}}^T & A_2^{(2)} \end{bmatrix} & U_1^{(1)} K_{1,2}^{(1)} {V_2^{(1)}}^T \\ U_2^{(1)} K_{2,1}^{(1)} {V_1^{(1)}}^T & \begin{bmatrix} A_3^{(2)} & U_3^{(2)} K_{3,4}^{(2)} {V_4^{(2)}}^T \\ U_4^{(2)} K_{4,3}^{(2)} {V_3^{(2)}}^T & A_4^{(2)} \end{bmatrix} \end{bmatrix} \tag{21}$$

The low-rank decomposition of the off-diagonal blocks is obtained using the adaptive cross approximation (ACA) algorithm, which is a minor modification of the partially pivoted LU algorithm.

Multiple level

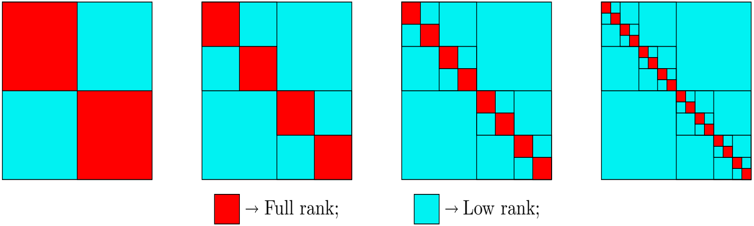


Figure : A hierarchical off-diagonal low-rank matrix at different levels.

factorization

The matrix A is factored as shown in Equation (22).

$$A = A_\kappa A_{\kappa-1} \cdots A_1 A_0 \tag{22}$$

where A_i's are block diagonal matrices with 2^i diagonal blocks and each block is a low-rank perturbation to identity matrix.

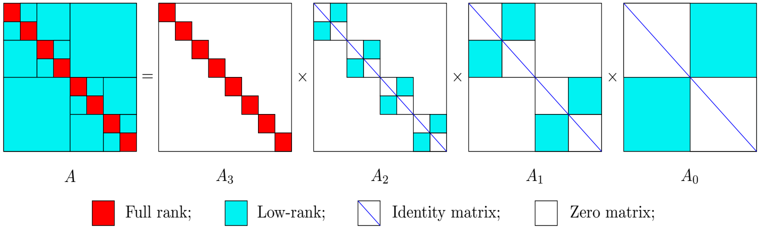


Figure : Factorization of a HODLR matrix at level 3.

Inversion

To invert the matrix after the factorization, one simply needs to apply the Sherman-Morrison-Woodbury formula recursively. The whole cost is on $O(N)$. For multiple right hand sides, the additional cost is negligible.

Numerical experiment

We are trying to mimic the following field in Ω:

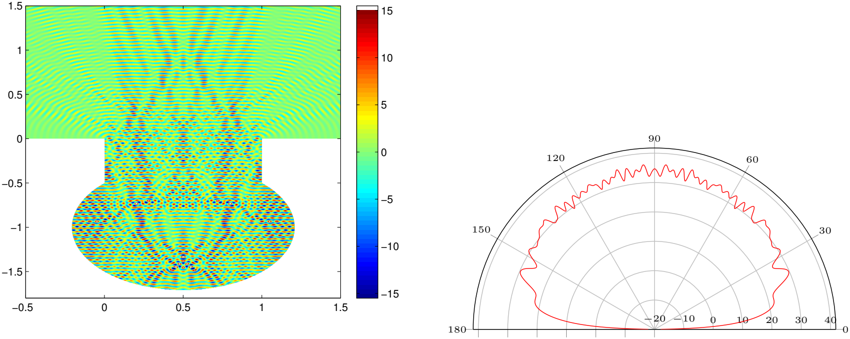
$$u(\mathbf{x}) = \frac{i}{4} H_0^{(1)}(k|(\mathbf{x} - \mathbf{x_0})|) + \frac{i}{4} H_0^{(1)}(k|(\mathbf{x} - \mathbf{x_0'})|), \tag{23}$$

by solving the following equation

$$\begin{cases} -\mu + \mathcal{D}_\Gamma \mu & = u(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Gamma_1 \\ \sigma + \mathcal{T}_\Gamma \mu & = \frac{\partial u(\mathbf{x})}{\partial n}, \quad \text{for } \mathbf{x} \in \Gamma_1 \\ -\frac{1}{2}\mu + \mathcal{S}^H_{\Gamma_1}\sigma + \mathcal{D}^H_{\Gamma_1}\mu + \mathcal{D}_B\mu + \mathcal{D}_\Gamma \mu & = u(\mathbf{x}), \quad \text{for } \mathbf{x} \in B \cup \Gamma \end{cases} \tag{24}$$

where $\mathbf{x_0} = (5, 12)$, $\mathbf{x_0'} = (5, -12)$ and the center of the cavity is at $(0.5, 0)$.

Pot shaped cavity



(a) **(b)**

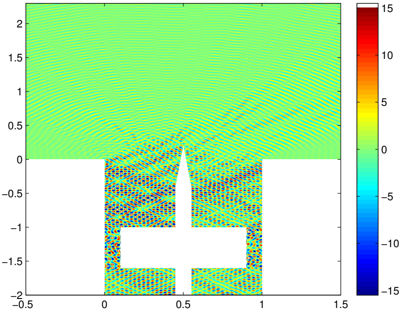
Figure : *Example 1. (a)Real part of the scattered field for a pot shaped cavity from the normal incidence of a plane wave with wavenumber k=160. (b)The backscatter RCS in dB for the pot shaped cavity at k=160*

Pot shaped cavity

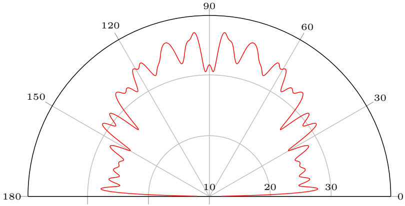
Table : Results for pot shaped cavity at different wavenumber

k	N_{mid}	N_{over}	N_{tot}	$T_{factor}(s)$	$T_{solve}(s)$	E_{error}
1	10	2	1320	0.5	0.01	$7.6 \cdot 10^{-10}$
10	10	10	2200	0.65	0.01	$5 \cdot 10^{-9}$
20	10	20	3300	1.52	0.01	$1.2 \cdot 10^{-9}$
40	10	40	5500	3.14	0.02	$8.5 \cdot 10^{-9}$
80	10	80	9900	9.5	0.05	$4.1 \cdot 10^{-9}$
160	10	160	18700	42.4	0.19	$1.2 \cdot 10^{-9}$
320	10	320	36300	192.8	0.57	$2.1 \cdot 10^{-8}$
640	10	400	45100	581.4	1.43	$5.5 \cdot 10^{-5}$
800	10	400	45100	785.1	1.57	$4.6 \cdot 10^{-6}$

Engine shaped cavity



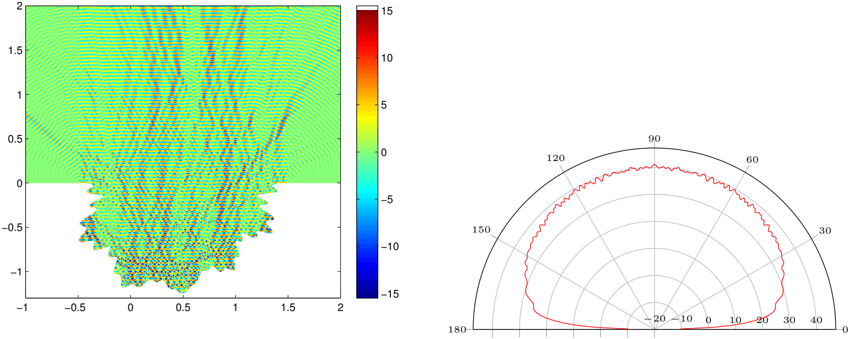
(a)



(b)

Figure : *Example 2. (a)Real part of the scattered field for a engine shaped cavity from the 45^o incidence of a plane wave with wavenumber k=160. (b)The backscatter RCS in dB for the engine shaped cavity at k=160*

Rough bottom cavity



(a) **(b)**

Figure : *Example 3. (a)Real part of the scattered field for a rough bottom cavity from the normal incidence of a plane wave with wavenumber k=160. (b)The backscatter RCS in dB for the rough bottom cavity at k=160*

Conclusion

- Propose an integral formulation for the cavity problem that leads to a high order discretization
- Apply the fast direct solver for the resulted linear system
- The extension to TE case is straightforward
- Future work includes the impedance boundary problem, optimal design problem

Thank you