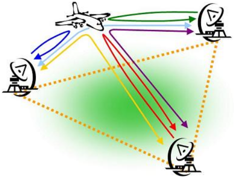
A Multi-frequency Method for the Solution of the Acoustic Inverse Scattering Problem

Carlos Borges & Leslie Greengard

June 29, 2013

Contents

Introduction

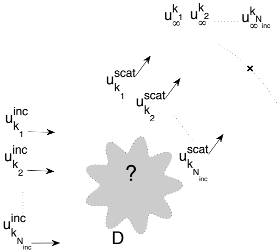


- There is increasing interest in inverse problems in the areas of medical imaging[1], non-destructive testing, sensing, probing, oil and gas prospecting, radar[2] and sonar, among many others.
- In those problems, a set of measured data is given from experimentation and, using this data, the goal is to reconstruct the object or its properties.

[1] Picture from Wikipedia.de
[2] Picture from Wikipedia

Introduction

Problem: *We consider the problem of reconstructing the shape of a sound-soft obstacle from the measured far field pattern from time harmonic plane waves with varying incidence direction and frequencies.*



Introduction

Important points:

- Reconstruction methods (most of the time) depend on the solution of the direct scattering problem.
- To compute the solution of the direct scattering problem, the number of operations increases with the frequency of the incident waves.
- The problem is nonlinear and ill-posed. To deal with the nonlinearity of the problem, we apply damped Newton's method. To deal with the ill-posedness of this problem, we need to apply a regularization method based on the recursive linearization algorithm (RLA) (Chen) and bandlimited approximation for sets of points (Beylkin, Rokhlin).

Direct scattering problem

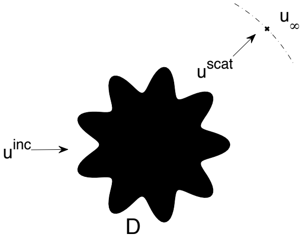


Figure: The direct scattering problem of finding the field scattered by an impenetrable obstacle.

Direct scattering problem

Consider the incident plane wave $u^{\text{inc}}(x) = \exp(ikx \cdot d)$. We seek

$$u(x) = u^{\text{inc}}(x) + u^{\text{scat}}(x),$$

the solution of the Helmholtz equation with Dirichlet condition

$$\begin{aligned} \Delta u(x) + k^2 u(x) &= 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D} \\ u(x) &= 0 \quad \text{on} \quad \partial D, \end{aligned}$$

where $u^{\text{scat}}(x)$ satisfies the Sommerfeld condition

$$\lim_{r \to \infty} r \left(\frac{\partial u^{\text{scat}}}{\partial r} - iku^{\text{scat}} \right) = 0, \qquad r = \|x\|.$$

Direct scattering problem

Theorem

Every radiating solution u to the Helmholtz equation has the asymptotic behavior of an outgoing spherical wave

$$u(x) = \frac{e^{ik|x|}}{|x|^{\frac{1}{2}}} \left\{ u_\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to 0,$$

uniformly in all directions $\hat{x} = x/|x|$, *where the function* u_∞ *defined on the unit disk* Ω *is known as the far field pattern of* u. *We have:*

$$u_\infty(\hat{x}) \quad = \quad \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \left\{ u(y) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y) e^{-ik\hat{x}\cdot y} \right\} \mathrm{d}s(y), \quad \forall \hat{x} \in \Omega.$$

Direct scattering problem – Layer potentials

We define the single layer potential

$$S\varphi(x) := \int_{\partial D} G(x,y)\varphi(y)\ ds(y),$$

and the double layer potential

$$K\varphi(x) := \int_{\partial D} \partial_\nu G(x,y)\varphi(y)\ ds(y),$$

where

$$G(x,y) = \frac{i}{4} H_0^{(1)}(\|x-y\|).$$

Direct scattering problem – Asymptotic of the potentials

We also have the asymptotic operator for the single layer potential

$$(S_\infty\varphi)(\hat{x}) := e^{i\pi/4}/\sqrt{8\pi k}\int_{\partial D} e^{-ik\hat{x}\cdot y}\varphi(y)\;ds(y),$$

and for the double layer potential

$$(K_\infty\varphi)(\hat{x}) := e^{-i\pi/4}\sqrt{\frac{k}{8\pi}}\int_{\partial D} e^{-ik\hat{x}\cdot y}\hat{x}\cdot\nu\;\varphi(y)\;ds(y).$$

Direct scattering problem – First formulation

One way to obtain the far field pattern u_∞ created by waves deflecting off an object D is to first solve

$$(I + K - i\eta S)\varphi = -u^{\text{inc}}$$

for φ, and then use the result to solve

$$u_\infty = (K_\infty + i\eta S_\infty)\varphi.$$

Direct scattering problem – Another formulation

Another way to obtain the far field pattern is to first solve

$$(I + \partial_\nu S - i\gamma S)\frac{\partial u}{\partial \nu} = \frac{\partial u^{\text{inc}}}{\partial \nu} - i\gamma u^{\text{inc}}$$

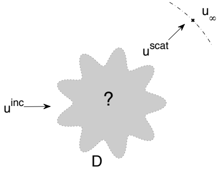
for $\frac{\partial u}{\partial \nu}$, and then we use the result to solve

$$u_\infty = -S_\infty \frac{\partial u}{\partial \nu}.$$

Numerical Implementation

- We solve the system using the Nystrom method [CK98].
- The potential layer operators are implemented using the Alpert quadrature [Alp99].
- We use the trapezoidal rule to implement the far field operators K_∞ and S_∞.
- The inverse method that we are going to use rely on the solution of direct problems. (must be fast)
- It is possible to apply the FMM or fast solvers like the HSS, HODLR, and others. We used the HODLR by Ambikasaran and Darve [CA13].

Inverse problem for a single frequency



Problem 1 *Given N_{ff} measurements $\mathbf{u}_\infty$ obtained from the scattering of u^{inc} by D, reconstruct the shape of D.*

Types of methods

There are several different known methods of solving this inverse problem. In [Kre07], Kress classified them into three different types:

(1) Iterative methods: The inverse problem is interpreted as a nonlinear ill-posed operator equation (Newton's methods (Roger, Kress, Kirsch, many others), Landweber iterations or CG methods);

(2) Decomposition methods: Separate the problem in different problems (Potential Method (Potthast) or Point-Source method);

(3) Sampling methods: Uses an indicator function (Linear Sampling method (Colton, Kirsch), Probe method (Ikehata), Singular Source method (Potthast) or Factorization method (Kirsch)).

Inverse problem for a single frequency

The solution of the direct scattering problem with a fixed incident plane wave u^{inc} defines the operator $F : \partial D \to u_\infty$. Considering the function x represents the shape ∂D, we have the equation:

$$F(x) = u_\infty.$$

- This problem is nonlinear and ill-posed.

Linearization

To deal with the nonlinearity of the problem, we use Newton's method [CK98] with a damping parameter. In this method, given a far field pattern u_∞, the nonlinear equation

$$F(x) = u_\infty$$

is replaced by the linearized equation

$$F(x) + J(x)h = u_\infty,$$

where $J(x)$ is the Fréchet derivative of the operator F.

Linearization

Theorem (Kirsch)

The far field mapping $F : x \to u_\infty$ is Fréchet differentiable from $C^2(\mathbb{R}^2)$ into $L^2(\Omega)$, where Ω is the unit circle. The derivative $J(x)$ is given by

$$J(x)h = v_\infty$$

where v_∞ denotes the far field pattern of the solution v to the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{D}$ satisfying the Sommerfeld radiation condition and the boundary condition

$$v = -\nu \cdot h \quad \frac{\partial u}{\partial \nu} \quad on \quad \partial D.$$

Linearization

The iteration process for the linearization is the following:

- Given an approximation $x^{(i)}$ of the domain, we solve the equation
$$J(x^{(i)})h = u_\infty - F(x^{(i)}). \tag{1}$$
- Obtain the update $x^{(i+1)} = x^{(i)} + \rho h$, where ρ is a damping factor.
- We have for the Fréchet derivative
$$J(x)h = (K_\infty - i\gamma S_\infty)(I + K - i\gamma S)^{-1} h.$$

Regularization

Equation (2) is ill-posed, it is necessary to apply a regularization method. There are several methods available: Tikhonov regularization, truncated SVD, and several others.

We apply a frequency filter $\mathcal{F}_{\mathcal{N}(k)}$ as a right preconditioner. The frequency filter will be a low-pass filter that will filter all the frequencies below the number $\mathcal{N}(k)$. This approach is very similar to the truncated SVD approach.

Solve the equation

$$\mathcal{J}(x^{(i)})p = u_\infty - F(x^{(i)}), \tag{2}$$

where $\mathcal{J}(x^{(i)}) = J(x^{(i)})\mathcal{F}_{\mathcal{N}(k)}$ and $\mathcal{F}_{\mathcal{N}(k)}p = h$. Using h, we update the domain.
After updating the domain, we apply a the algorithm of D. Beylkin and Rokhlin [BR13] to approximate the domain with a bandllimited curve and also reparameterize the curve.

Damped Newton's method

Damped's Newton Method

Given the initial guess $x^{(0)}$, and the far-field pattern u_∞

1. Repeat while the stopping criteria are not reached:

   1.1 Use the domain $x^{(i)}$ and solve for $\frac{\partial \mathbf{u}}{\partial \nu}$ the equation

   $$(I + \partial_\nu S - i\gamma S)\frac{\partial u}{\partial \nu} = \frac{\partial u^{\text{inc}}}{\partial \nu} - i\gamma \mathbf{u}^{\text{inc}}.$$

   1.2 Using $\frac{\partial u}{\partial \nu}$, find the operator $J(x^{(i)})$.

   1.3 Choose the parameter $\mathcal{N}(k)$, obtain the operator $\mathcal{J}(x^{(i)})$ and solve the system

   $$\mathcal{J}(x^{(i)})p = u_\infty - F(x^{(i)})$$

   1.4 Obtain $h = \mathcal{F}_{\mathcal{N}(k)}p$.

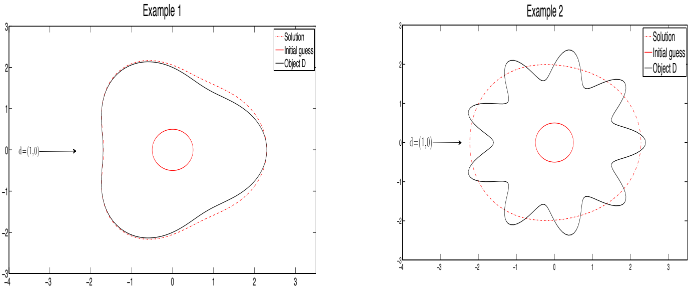
   1.5 Update $x^{(i+1)} = x^{(i)} + \rho h$, re-parameterize and filter the domain $x^{(i+1)}$ for the next step and make $i = i + 1$.

Remarks

We have to consider the following every time that we update the domain:

- We need to check every time that we update the domain if the domain is still valid (not self-intersecting). For this, we can use a naive approach testing if the polygon formed by the points in the curve is not self-intersecting. For a fast algorithm the Bentley and Ottmann sweep algorithm [BO79] can be used.
- If the updated domain is self-intersecting, we adjust the size of ρ.
- For the re-parameterization of the domain we use the algorithm in [BR13] for re-parameterization of limited band curves.
- We also filter the curve during re-parameterization.

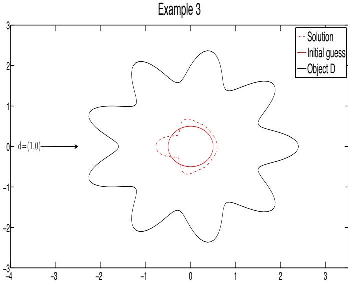
Numerical Results – star-shaped domains



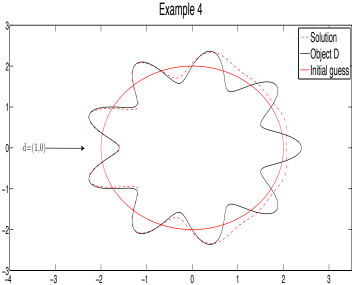
(a) $k = 0.5,\ N_{ff} = 32$ (b) $k = 0.5,\ N_{ff} = 32$

Figure: (a) Pear. (b) 9-gear.

Numerical Results – star-shaped domains



(a) $k = 5$, $N_{ff} = 32$



(b) $k = 5$, $N_{ff} = 32$

Figure: Solution of the inverse scattering problem.

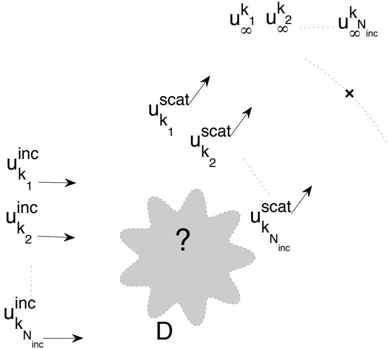
Conclusions – single frequency

Frequency	Initial Guess	Reconstruction
Low	Simple	Fuzzy
High	Closer to object	Sharp

- At low frequencies, the reconstruction problem is uniquely solvable [CS83] , but its stability is poor [SM08]. That means, at low frequencies, it is difficult to reconstruct small details of the obstacle.

- At high frequencies, this inverse problem may not be uniquely solvable but it is more stable [NS12].

Inverse problem for multiple frequencies



Problem 2 Given $N_{\text{total}} = N_{ff} \times N_{\text{inc}}$ far field pattern $u_\infty^{k_1}$, $u_\infty^{k_2}, \ldots,$ $u_\infty^{k_{N_{\text{inc}}}}$, generated by scattering of $u_{k_1}^{\text{inc}}$, $u_{k_2}^{\text{inc}}, \ldots,$ $u_{k_{N_{\text{inc}}}}^{\text{inc}}$, reconstruct the shape of the object D.

Inverse problem for multiple frequencies

- Use the recursive linearization algorithm (RLA) presented in [Che97], and analyzed by [BT10, NS12].
- Choose an initial guess not very close to the object for low frequency measurements and obtain an approximation for the domain.
- The approximation becomes the initial guess at a higher frequency.
- For each frequency, we solve the inverse problem using an iterative method.

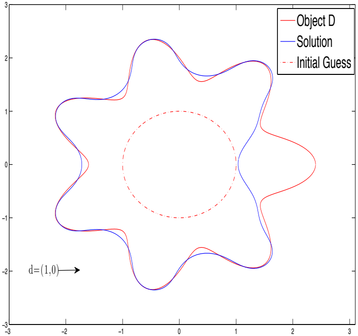
Recursive Linearization Algorithm

Recursive Linearization Algorithm with Damped Newton's method

Given $x^{(0)}$, and $\mathbf{u}_{\infty}^{\mathbf{k_j}}$ for $j = 1, \ldots, N_{\text{inc}}$.

1. For $j = 1, \ldots, N_{\text{inc}}$:
   1.1 Use damped Newton's method with initial guess $x_{k_j}^{(0)}$ and far field pattern $u_{\infty}^{k_j}$. We obtain the result $x_{k_j}^{(i)}$ after i iterations.
   1.2 Make $x_{k_{j+1}}^{(0)} = x_{k_j}^{(i)}$.

Numerical results – star-shaped domain



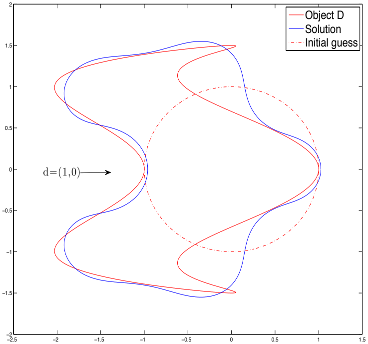
(a) $k_{10} = 5$

Fig3

(b) Video

Figure: RLA for the 7-gear, $k_j = 1 + 0.5j$, $j = 0, \ldots, 13$.

Numerical results – non star-shaped domain



(a) $k_{10} = 5$

Fig3

(b) Video

Figure: RLA for non star shaped figure, $k_j = 0.5j$, $j = 1, \ldots, 10$.

Conclusion – multi-frequency

Improvement

- The technique gives a sharp reconstruction of the object in the illuminated part;

Issues to be solved

- However, our reconstruction is not good in the shadowed part.

Inverse scattering for multiple frequencies and directions

What to do?

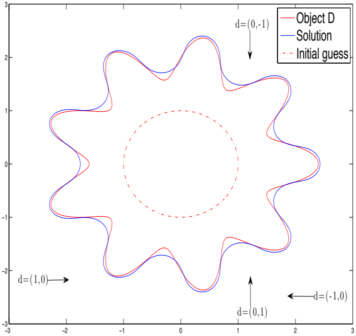
- We can solve the problem using data from multiple directions.

We solve the single frequency inverse problem using data from several directions simultaneously. We would have for each direction d_j an equation

$$\mathcal{J}_{d_j}(x)p = u_{\infty,d_j} - F_{d_j}(x^{(i)}).$$

Solve the equations together for all the directions to obtain the step h.
Remark: This system is better conditioned than the system with one direction.

Numerical results – multiple frequency + multiple directions



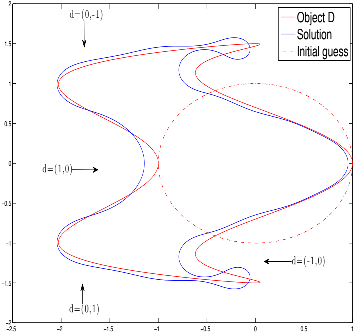
(a) $k_{10} = 5$

Fig3

(b) Video

Figure: RLA for the 9-gear, $k_j = 1 + 0.5j$, $j = 0, \ldots, 19$.

Numerical results – multiple frequency + multiple directions



(a) $k_{10} = 5$

Fig3

(b) Video

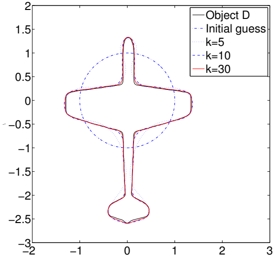
Figure: RLA for non star shaped figure, $k_j = 0.5j$, $j = 1, \ldots, 10$.

Examples using more frequencies

Since we are using a fast solver to accelerate our reconstructions, we can go further and reconstruct objects with more detail.

We have some examples reconstructing shapes of objects similar to an aircraft and a submarine.

Numerical results – multiple frequency + multiple directions



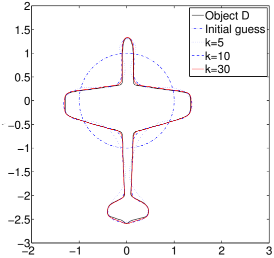
(a) $k_{10} = 5$

FigAircraft

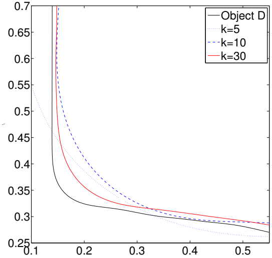
(b) Video

Figure: RLA for aircraft, $k_j = 0.5j$, $j = 1, \ldots, 65$.

Numerical results – multiple frequency + multiple directions



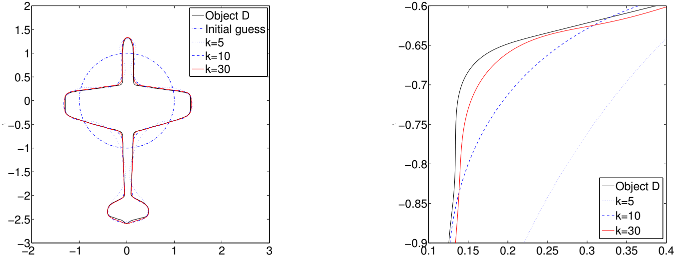
(a) Reconstruction of the aircraft



(b) Top corner of the right wing

Figure: RLA for the aircraft.

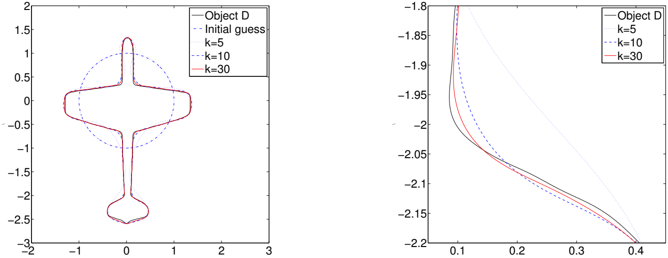
Numerical results – multiple frequency + multiple directions



(a) Reconstruction of the aircraft (b) Bottom corner of the right wing

Figure: RLA for the aircraft.

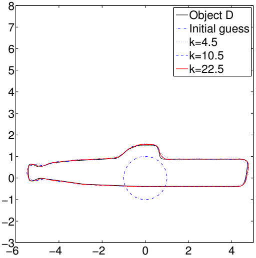
Numerical results – multiple frequency + multiple directions



(a) Reconstruction of the aircraft (b) Top right corner of the tail

Figure: RLA for the aircraft.

Numerical results – multiple frequency + multiple directions



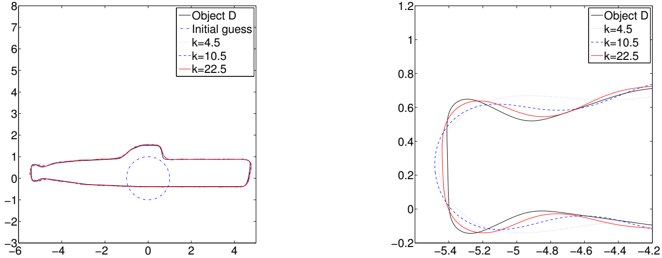
(a) $k_{10} = 5$

Video

(b) Video

Figure: RLA for submarine, $k_j = 0.5j$, $j = 1, \ldots, 65$.

Numerical results – multiple frequency + multiple directions

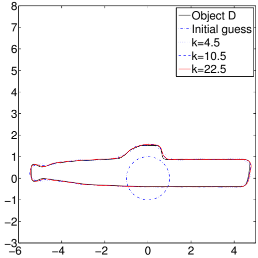


(a) Reconstruction of the submarine

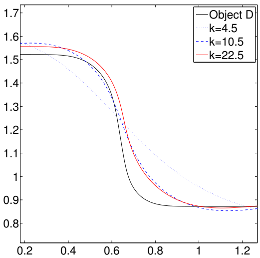
(b) Back part of the submarine

Figure: RLA for submarine.

Numerical results – multiple frequency + multiple directions



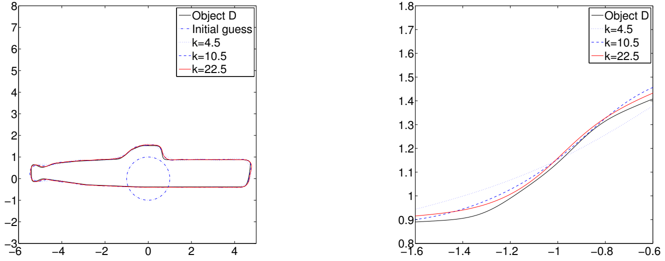
(a) Reconstruction of the submarine



(b) Front part of the tower

Figure: RLA for submarine.

Numerical results – multiple frequency + multiple directions



(a) Reconstruction of the submarine (b) Back part of the tower

Figure: RLA for submarine.

Conclusion

- We are able to obtain accurate reconstructions of the shape of the objects starting from an initial guess that is not very close to the object.
- We presented an alternative to the regularization of the system encoding the frequency information in our preconditioner.
- The inclusion of data generated by incident waves of different directions provides us with data to obtain complete reconstructions of the objects and also improves the conditioning of the problem.

Future Work

- Obtain estimates for better choosing the values of ρ and of the frequencies to be filtered at each step.
- Implement the algorithm for multiple objects.
- Modify for the case of Neumann and Impedance boundary conditions.
- Extend the algorithm for the 3D case.

Thank you!

Works Cited I

- B. Alpert, *Hybrid gauss-trapezoidal quadrature rules*, SIAM J. Sci. Comput. **20** (1999), 1551–1584.
- J. L. Bentley and T. A. Ottmann, *Algorithms for reporting and counting geometric intersections*, IEEE Trans. Comput. **28** (1979), no. 9, 643–647.
- Daniel Beylkin and Vladimir Rokhlin, *Fitting a bandlimited curve to points in a plane*, Yale Computer Science Department – Technical Report (2013).
- G. Bao and F. Triki, *Error Estimates for the Recursive Linearization of Inverse Medium Problems*, Journal of Computational Mathematics (2010).
- E. Darve C. Ambikasaran, *An o(n logn) fast direct solver for partial hierarchically semi-separable matrices*, 2013.

Works Cited II

- Yu Chen, *Inverse scattering via Heisenberg's uncertainty principle*, Inverse Problems **13** (1997), no. 2, 253.
- D. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering theory*, second ed., Springer, 1998.
- D. Colton and B. D. Sleeman, *Uniqueness theorems for the inverse problem of acoustic scattering*, IMA Journal of Applied Mathematics **31** (1983), no. 3, 253–259.
- Andreas Kloeckner, Alexander Barnett, Leslie Greengard, and Michael O'Neil, *Quadrature by expansion: A new method for the evaluation of layer potentials*.
- Rainer Kress, *Uniqueness and numerical methods in inverse obstacle scattering*, Journal of Physics: Conference Series **73** (2007), no. 1, 012003.

Works Cited III

- T. Nguyen and M. Sini, *Inverse acoustic obstacle scattering problems using multifrequency measurements*, Inverse Problems and Imaging **6** (December 2012), no. 4, 749–773.
- E. Sincich and M.Sini, *Local stability for soft obstacles by a single measurement*, Inverse Problems and Imaging **2** (2008), no. 2, 301–315.