

# PRACTICAL PRETENDERS

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ABSTRACT. Following Srinivasan, an integer  $n \geq 1$  is called *practical* if every natural number in  $[1, n]$  can be written as a sum of distinct divisors of  $n$ . This motivates us to define  $f(n)$  as the largest integer with the property that all of  $1, 2, 3, \dots, f(n)$  can be written as a sum of distinct divisors of  $n$ . (Thus,  $n$  is practical precisely when  $f(n) \geq n$ .) We think of  $f(n)$  as measuring the “practicality” of  $n$ ; large values of  $f$  correspond to numbers  $n$  which we term *practical pretenders*. Our first theorem describes the distribution of these impostors: Uniformly for  $4 \leq y \leq x$ ,

$$\#\{n \leq x : f(n) \geq y\} \asymp \frac{x}{\log y}.$$

This generalizes Saias’s result that the count of practical numbers in  $[1, x]$  is  $\asymp \frac{x}{\log x}$ .

Next, we investigate the maximal order of  $f$  when restricted to non-practical inputs. Strengthening a theorem of Hausman and Shapiro, we show that every  $n > 3$  for which

$$f(n) \geq \sqrt{e^{\gamma n} \log \log n}$$

is a practical number.

Finally, we study the range of  $f$ . Call a number  $m$  belonging to the range of  $f$  an *additive endpoint*. We show that for each fixed  $A > 0$  and  $\epsilon > 0$ , the number of additive endpoints in  $[1, x]$  is eventually smaller than  $x/(\log x)^A$  but larger than  $x^{1-\epsilon}$ .

## 1. INTRODUCTION

In 1948, Srinivasan [15] initiated the study of *practical numbers*, natural numbers  $n$  with the property that each of  $1, 2, 3, \dots, n - 1$  admits an expression as a sum of distinct divisors of  $n$ . For example, every power of 2 is practical (since every natural number admits a binary expansion), but there are many unrelated examples, such as  $n = 6$  or  $n = 150$ . Srinivasan posed two problems: Classify all practical numbers and say something interesting about their distribution.

The first of these tasks was carried to completion by Stewart [16] in 1954. The same classification was discovered independently, and almost concurrently, by Sierpiński [14]. Given a natural number  $n$ , write its canonical prime factorization in the form

$$(1.1) \quad n := p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}, \quad \text{where } p_1 < p_2 < \cdots < p_r.$$

Put  $n_0 = 1$ , and for  $1 \leq j \leq r$ , put  $n_j := \prod_{i=1}^j p_i^{e_i}$ . Then  $n$  is practical if and only if

$$(1.2) \quad p_{j+1} \leq \sigma(n_j) + 1 \quad \text{for all } 0 \leq j < r.$$

Below, we refer to this as the *Stewart–Sierpiński classification* of practical numbers. This criterion implies, in particular, that all practical numbers  $n > 1$  are even. Stewart and Sierpiński also showed that if all of the inequalities (1.2) hold, then not only are all integers in  $[1, n - 1]$  expressible as a sum of distinct divisors of  $n$ , but the same holds for all integers in the longer interval  $[1, \sigma(n)]$ . Note that  $[1, \sigma(n)]$  is the largest interval one could hope to represent, since the sum of all distinct divisors of  $n$  is  $\sigma(n)$ .

The distribution of practical numbers has proved more recalcitrant. Let  $PR(x)$  denote the count of practical numbers not exceeding  $x$ . Already in 1950, Erdős [2] claimed he

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could show that the practical numbers have asymptotic density zero, i.e., that  $PR(x) = o(x)$  as  $x \rightarrow \infty$ , but he gave no details. In 1984, Hausman and Shapiro [6] made the more precise assertion that  $PR(x) \leq x/(\log x)^{\beta+o(1)}$ , with  $\beta = \frac{1}{2}(1 - 1/\log 2)^2 \approx 0.0979\dots$ . Their proof has an error (specifically, [6, Lemma 3.2] is incorrect); one should replace  $\beta$  with the smaller exponent  $1 - \frac{1+\log \log 2}{\log 2} \approx 0.0860713$ . Much sharper results on  $PR(x)$  were soon established by Tenenbaum [17, 19], who proved that  $PR(x) = \frac{x}{\log x}(\log \log x)^{O(1)}$ . By a refinement of Tenenbaum's methods, Saias [13] established in 1997 what is still the sharpest known result: There are absolute constants  $c_1$  and  $c_2$  with

$$(1.3) \quad c_1 \frac{x}{\log x} \leq PR(x) \leq c_2 \frac{x}{\log x} \quad \text{for all } x \geq 2.$$

On the basis of the numerical data, Margenstern [9] has conjectured that  $\frac{PR(x)}{x/\log x}$  tends to a limit  $\approx 1.341$ .

In this paper, we are concerned with what we term *near-practical numbers* or *practical pretenders*. Define  $f(n)$  as the largest integer with the property that all of the numbers  $1, 2, 3, \dots, f(n)$  can be written as a sum of distinct divisors of  $n$ . By definition,  $n$  is practical precisely when  $f(n) \geq n - 1$ . We define a *near-practical* number as one for which  $f(n)$  is ‘‘large’’. This definition is purposely vague; its nebulous nature suggests that we investigate the behavior of the two-parameter function

$$N(x, y) := \#\{n \leq x : f(n) \geq y\}$$

for all  $x$  and  $y$ . Our first result gives the order of magnitude of the near-practical numbers for essentially all interesting choices of  $x$  and  $y$ .

**Theorem 1.1.** *There are absolute positive constants  $c_3$  and  $c_4$  so that for  $4 \leq y \leq x$ , we have*

$$c_3 \frac{x}{\log y} \leq N(x, y) \leq c_4 \frac{x}{\log y}.$$

**Remark.** To see why the technical restriction  $y \geq 4$  is necessary, note that  $N(x, x) = 0$  for all  $3 < x < 4$ .

Theorem 1.1 has the following easy corollary, proved in §3.

**Corollary 1.2.** *For each  $m$ , the set of natural numbers  $n$  with  $f(n) = m$  possesses an asymptotic density, say  $\rho_m$ . The constant  $\rho_m$  is positive whenever there is at least one  $n$  with  $f(n) = m$ . Moreover,*

$$\sum_{m=1}^{\infty} \rho_m = 1.$$

We call a natural number  $m$  for which  $\rho_m$  is nonvanishing (equivalently, an  $m$  in the image of  $f$ ) an *additive endpoint*. Thus, Corollary 1.2 shows that  $\rho_m$  is the probability mass function for additive endpoints. The first several additive endpoints are

$$1, 3, 7, 12, 15, 28, 31, 39, 42, 56, 60, 63, 73, 90, 91, 96, 100, 104, 108, 112, 120, \dots$$

Just from this limited data, one might conjecture that  $\rho_m$  is usually zero, i.e., zero apart from a set of  $m$  of vanishing asymptotic density. This guess is confirmed, in a much sharper form, in our next theorem.

**Theorem 1.3.** *For each fixed  $A > 0$  and all  $x \geq 3$ , the number of integers in  $[1, x]$  which occur as additive endpoints is  $\ll_A x/(\log x)^A$ . In the opposite direction, the number of additive endpoints up to  $x$  exceeds*

$$x / \exp(c_5(\log \log x)^3)$$

for all large  $x$ , for some absolute constant  $c_5 > 0$ .

Above, we noted Stewart's result that if  $f(n) \geq n - 1$ , then  $f(n) = \sigma(n)$ . In this statement, a weak lower estimate on  $f(n)$  implies that  $f(n)$  is as large as possible. Hausman and Shapiro [6] proposed investigating the extent of this curious phenomenon. More specifically, they asked for the slowest-growing monotone function  $g(n)$  for which  $f(n) \geq g(n)$  implies (at least for  $n$  large) that  $n$  is practical. Set

$$HS(n) := \sqrt{e^\gamma n \log \log n},$$

where  $\gamma$  is the Euler–Mascheroni constant. The next proposition appears as [6, Theorems 2.1, 2.2].

**Proposition 1.4.** *Let  $\epsilon > 0$ . Apart from finitely many exceptional  $n$ , all solutions to  $f(n) \geq (1 + \epsilon)HS(n)$  are practical. On the other hand, there are infinitely many non-practical  $n$  with  $f(n) \geq (1 - \epsilon)HS(n)$ .*

Our final theorem removes the factor  $1 + \epsilon$  from the first half of Proposition 1.4.

**Theorem 1.5.** *If  $n > 3$  and  $f(n) \geq HS(n)$ , then  $n$  is practical.*

**Notation.** We use the Landau–Bachmann  $o$  and  $O$  symbols, as well as Vinogradov's  $\ll$  notation, with their usual meanings; subscripts indicate dependence of implied constants. We write  $\omega(n) := \sum_{p|n} 1$  for the number of distinct prime factors of  $n$  and  $\Omega(n) := \sum_{p^k|n} 1$  for the number of prime factors of  $n$  counted with multiplicity;  $\Omega(n; y) := \sum_{p^k|n, p \leq y} 1$  denotes the number of prime divisors of  $n$  not exceeding  $y$ , again counted with multiplicity. The number of divisors of  $n$  is denoted  $d(n)$ . We use  $P^-(m)$  for the smallest prime factor of  $m$ , with the convention that  $P^-(1)$  is infinite. Absolute positive constants are denoted by  $c_1, c_2, c_3$ , etc., and have the same meaning each time they appear.

## 2. PROOFS OF THEOREM 1.1 AND COROLLARY 1.2

We begin by recording some useful lemmas. Our first gives a formula for  $f(n)$  in terms of the prime factorization of  $n$ .

We assume that the factorization of  $n$  has been given in the form (1.1). We define  $n_0 := 1$  and  $n_j := \prod_{1 \leq i \leq j} p_i^{e_i}$ . Let  $0 \leq j < r$  be the first index for which  $p_{j+1} > \sigma(n_j) + 1$ , putting  $j = r$  if no such index exists (i.e., if  $n$  is practical). Then  $n_j$  is a practical number, by the Stewart–Sierpiński classification, and we call  $n_j$  the *practical component* of  $n$ .

**Lemma 2.1.** *We have  $f(n) = \sigma(n_j)$ , where  $n_j$  is the practical component of  $n$ .*

*Proof.* Since  $n_j$  is practical,  $f(n) \geq f(n_j) = \sigma(n_j)$ . On the other hand,  $\sigma(n_j) + 1$  is not representable as a sum of proper divisors of  $n$ . Indeed, if  $d$  is a divisor of  $n$  involved in an additive representation of  $\sigma(n_j) + 1$ , then  $d \leq \sigma(n_j) + 1 < p_{j+1} < p_{j+2} < \dots < p_r$ . It follows that the only primes dividing  $d$  are  $p_1, \dots, p_j$ , so that  $d$  is a divisor of  $n_j$ . But the largest number which can be formed as a sum of distinct divisors of  $n_j$  is  $\sigma(n_j)$ , which is smaller than  $\sigma(n_j) + 1$ . So  $\sigma(n_j) + 1$  is not representable as a sum of distinct divisors of  $n$ , and hence  $f(n) = \sigma(n_j)$ , as claimed.  $\square$

The following lemma was observed by Margenstern [9, Corollaire 1] to follow from the Stewart–Sierpiński classification.

**Lemma 2.2.** *If  $n$  is practical and  $m \leq \sigma(n) + 1$ , then  $mn$  is practical.*

We now employ Lemma 2.2 to show that reasonably short intervals contain a positive proportion of practical numbers.

**Lemma 2.3.** *Let  $\epsilon > 0$ . For  $x > x_0(\epsilon)$ , the number of practical numbers in  $((1 - \epsilon)x, x]$  is  $\gg_\epsilon x / \log x$ .*

*Proof.* We can assume that  $0 < \epsilon < 1$ . With  $c_1$  and  $c_2$  as defined in (1.3), we set  $r := \lceil 2c_2/c_1 \rceil$  and  $s := \lceil 1/\epsilon \rceil$ . From (1.3), we have that for large  $x$  (depending on  $\epsilon$ ), the number of practical numbers in the interval  $(x/rs, x/s]$  is

$$\geq c_1 \frac{x/s}{\log(x/s)} - c_2 \frac{x/rs}{\log(x/rs)} > \frac{c_1}{3s} \frac{x}{\log x} \geq \frac{c_1}{6} \epsilon \frac{x}{\log x}.$$

By the pigeonhole principle, one of the intervals  $(\frac{x}{s+1}, \frac{x}{s}]$ ,  $(\frac{x}{s+2}, \frac{x}{s+1}]$ ,  $\dots$ ,  $(\frac{x}{rs}, \frac{x}{rs-1}]$  contains  $> \frac{c_1}{6rs} \epsilon x / \log x \gg \epsilon^2 x / \log x$  practical numbers. Suppose this interval is  $(\frac{x}{j+1}, \frac{x}{j}]$ , where  $s \leq j < rs$ , and let  $n$  be a practical number contained within. If  $x > (rs)^2$ , then  $j < \frac{x}{j+1} < n$ , and so  $jn$  is practical by Lemma 2.2. (Note that the lower bound on  $x$  assumed here depends only on  $\epsilon$ .) Letting  $n$  run through the practical numbers in  $(\frac{x}{j+1}, \frac{x}{j}]$ , we obtain  $\gg \epsilon^2 x / \log x$  practical numbers  $jn \in (x \frac{j}{j+1}, x]$ . But  $(x \frac{j}{j+1}, x] \subset ((1 - \epsilon)x, x]$ , by our choice of  $s$ . This proves Lemma 2.3. Moreover, we have shown that the implied constant in the lemma statement may be taken proportional to  $\epsilon^2$ .  $\square$

The next result, due to Hausman and Shapiro [6, Theorem 4.1], shows that substantially shorter intervals than those considered in Lemma 2.3 always contain at least one practical number.

**Lemma 2.4.** *For all real  $x \geq 1$ , there is a practical number  $x < n < x + 2x^{1/2}$ .*

Let  $\Phi(x, y)$  denote the number of natural numbers  $n \leq x$  divisible by no primes  $\leq y$ . The following lemma is a consequence of Brun's sieve. Variants can be found, e.g., as [3, Theorem 1, p. 201] or [18, Theorem 3, p. 400].

**Lemma 2.5.** *Uniformly for  $2 \leq y \leq x$ , we have  $\Phi(x, y) \ll x / \log y$ . If we assume also that  $x > c_6 y$  for a suitable large absolute constant  $c_6$ , then  $\Phi(x, y) \gg x / \log y$ .*

We now prove Theorem 1.1, treating the upper and lower estimates separately.

*Proof of the upper bound in Theorem 1.1.* Suppose that  $n \leq x$  and  $f(n) \geq y$ . By the upper bound in (1.3), we may restrict our attention to non-practical  $n$ . Let  $d$  be the practical component of  $n$  and write  $n = dq$ . By Lemma 2.1,  $\sigma(d) = f(n)$ . In particular, since we are assuming that  $f(n) \geq y \geq 4$ , we must have that  $d > 1$ . Moreover, since  $n$  is not practical,  $d < n$ . Thus,  $q > 1$  and

$$P^-(q) > \sigma(d) + 1 > d.$$

Hence,

$$d^2 < d \cdot P^-(q) \leq dq = n \leq x,$$

and so  $d \leq \sqrt{x}$ .

Given  $d$ , the number of possibilities for  $n$  is bounded above by the number of  $q \leq x/d$  with  $P^-(q) > d$ . Since  $2 \leq d \leq x/d$ , we may apply Lemma 2.5 to find that the number of possibilities for  $q$  is  $\ll \frac{x}{d \log d}$ . Since  $\sigma(d) = f(n) \geq y$  and (crudely)  $\sigma(d) < d^2$ , it follows that  $d > \sqrt{y}$ . Hence, using partial summation and (1.3), we see that the number

of possibilities for  $n$  is

$$\begin{aligned} \ll x \sum_{\substack{\sqrt{y} < d \leq \sqrt{x} \\ d \text{ practical}}} \frac{1}{d \log d} &\leq x \frac{PR(\sqrt{x})}{\sqrt{x} \log \sqrt{x}} + x \int_{\sqrt{y}}^{\sqrt{x}} PR(t) \frac{1 + \log t}{(t \log t)^2} dt \\ &\ll \frac{x}{(\log x)^2} + x \int_{\sqrt{y}}^{\sqrt{x}} \frac{dt}{t(\log t)^2} \ll \frac{x}{(\log x)^2} + \frac{x}{\log y} \ll \frac{x}{\log y}. \quad \square \end{aligned}$$

*Proof of the lower bound in Theorem 1.1.* The proof is suggested by that offered for the upper bound, but some care is necessary to ensure uniformity throughout the stated range of  $x$  and  $y$ .

First, we treat the range when  $x^{1/10} \leq y \leq x$ . In this domain, we use the trivial lower bound  $N(x, y) \geq N(x, x)$ . We estimate the right-hand side from below by counting practical numbers  $n$  belonging to the interval  $[\frac{x+1}{2}, x]$ . Note that for such  $n$ , we have  $f(n) = \sigma(n) \geq 2n - 1 \geq x$  (using for the first inequality that  $n - 1$  is a sum of proper divisors of  $n$ ), and so  $n$  is indeed counted by  $N(x, x)$ .

If  $6 \leq x \leq 11$ , then  $n = 6$  is a practical number in  $[\frac{x+1}{2}, x]$ . Similarly, if  $4 \leq x \leq 6$ , then  $n = 4$  works. Finally, if  $x \geq 11$ , then Lemma 2.4 gives a practical number  $n$  with

$$\frac{x+1}{2} < n < \frac{x+1}{2} + 2\sqrt{\frac{x+1}{2}} \leq x.$$

Hence, we always have  $N(x, x) \geq 1$ . (Recall that we only consider  $x \geq 4$ .) Moreover, by Lemma 2.3, there are  $\gg x/\log x$  practical numbers in  $[\frac{x+1}{2}, x]$  once  $x$  is large. It follows that  $N(x, x) \gg x/\log x$  for all  $x \geq 4$ . So if  $x^{1/10} \leq y \leq x$ , then

$$N(x, y) \geq N(x, x) \gg x/\log x \gg x/\log y,$$

which gives the lower bound of the theorem in this case.

Now suppose that  $4 \leq y \leq x^{1/10}$ . We consider numbers of the form  $n = dq \leq x$ , where  $d$  is a practical number in  $(y, y^3]$  and where  $P^-(q) > y^6$ . For any such  $n$ , we have  $f(n) \geq f(d) \geq d > y$ . Moreover, each  $n$  constructed in this way arises exactly once, since  $q$  is determined as the largest divisor of  $n$  supported on primes  $> y^6$ . Given  $d$ , the number of corresponding  $q$  is  $\Phi(x/d, y^6)$ . If  $x$  is large, then

$$\frac{x/d}{y^6} \geq \frac{x}{y^9} \geq x^{1/10} > c_6,$$

and so Lemma 2.5 gives

$$(2.1) \quad \Phi(x/d, y^6) \gg \frac{x}{d \log y}.$$

On the other hand, (2.1) is trivial for bounded  $x$ , since 1 is always counted by  $\Phi(x/d, y^6)$ . Thus, (2.1) holds in any case. Hence, the number of  $n$  constructed in this way is

$$\gg \frac{x}{\log y} \sum_{\substack{y < d \leq y^3 \\ d \text{ practical}}} \frac{1}{d}.$$

That the sum appearing here is  $\gg 1$  for large  $y$  follows from partial summation and the lower bound in (1.3). For bounded  $y$ , the sum is also  $\gg 1$ , since Lemma 2.4 guarantees that there is at least one practical number between  $y$  and  $y^3$ . (Certainly  $y + 2y^{1/2} < 3y < y^3$  when  $y \geq 4$ .) This completes the proof of the lower bound.  $\square$

*Proof of Corollary 1.2.* One can detect whether or not  $f(n) = m$  given just the list of divisors of  $n$  not exceeding  $m + 1$ . Thus, whether or not  $f(n) = m$  depends only on the residue class of  $n$  modulo  $(m + 1)!$ . This gives the first two assertions of the corollary. For the third, notice that  $1 - \sum_{n=1}^N \rho_m$  represents the density of the set of  $n$  with  $f(n) > N$ , which is  $\ll 1/\log N$  by Theorem 1.1. Letting  $N \rightarrow \infty$  completes the proof.  $\square$

### 3. PROOF OF THEOREM 1.3

We divide the proof of Theorem 1.3 into two parts.

**3.1. The upper bound in Theorem 1.3.** Central to the proof of both halves of Theorem 1.3 is the observation, immediate from Lemma 2.1, that  $m$  belongs to the range of  $f$  precisely when  $m = \sigma(n)$  for some practical number  $n$ . Thus, we are really asking in Theorem 1.3 for estimates on the range of  $\sigma$  restricted to practical inputs.

**Lemma 3.1.** *Let  $A \geq 30$ . Suppose that  $x \geq 3$ . If  $n$  is a practical number with  $x^{3/4} < n \leq x$ , then either*

$$(3.1) \quad \Omega(n) > 2A \log \log x$$

or

$$(3.2) \quad \omega(n) > \frac{1}{2 \log A} \log \log x.$$

*Proof.* Since  $n$  is practical, every integer in  $[1, n]$  can be written as a subset-sum of divisors of  $n$ . Thus,  $2^{d(n)} \geq n$ , so we can use the hypothesis that  $n > x^{3/4}$  to show

$$d(n) \geq \frac{\log n}{\log 2} > \frac{3/4}{\log 2} \log x > \log x.$$

Suppose that  $n = \prod_{i=1}^l p_i^{e_i}$  is the factorization of  $n$  into primes, where  $l = \omega(n)$ . Since  $d(n) = \prod_{i=1}^l (e_i + 1) > \log x$ , the inequality between the arithmetic and geometric means gives that

$$(3.3) \quad \frac{1}{l} \left( \sum_{i=1}^l (e_i + 1) \right)^l \geq \prod_{i=1}^l (e_i + 1) > \log x.$$

Now assume that (3.1) fails. Then  $\sum_{i=1}^l (e_i + 1) \leq 2 \sum_{i=1}^l e_i \leq 4A \log \log x$ , and (3.3) gives  $(\frac{4A \log \log x}{l})^l > \log x$ . Writing  $l = \lambda \log \log x$ , we deduce that

$$\left( \frac{4A}{\lambda} \right)^{\lambda \log \log x} > \log x, \quad \text{and so} \quad \lambda \log \frac{4A}{\lambda} > 1.$$

This latter inequality, along with the condition  $A \geq 30$ , implies that  $\lambda > \frac{1}{2 \log A}$  (by a short exercise in calculus). Since  $\omega(n) = \lambda \log \log x$ , we have (3.2).  $\square$

The next lemma, which belongs to the study of the anatomy of integers, bounds from above the number of  $n$  with an abnormally large number of small prime factors.

**Lemma 3.2.** *Let  $x, y \geq 2$ , and let  $k \geq 1$ . The number of  $n \leq x$  with  $\Omega(n; y) \geq k$  is  $\ll \frac{k}{2^k} x \log y$ .*

**Remark.** As a special case (when  $y = x$ ), the number of  $n \leq x$  with  $\Omega(n) \geq k$  is  $\ll \frac{k}{2^k} x \log x$ .

*Proof.* The proof is almost identical to that suggested in Exercise 05 of [4, p. 12], details of which can be found in [8, Lemmas 12, 13]. Thus, we only sketch it. Let  $v := 2 - 1/k$ . Let  $g$  be the arithmetic function determined through the convolution identity  $v^{\Omega(n;y)} = \sum_{d|n} g(d)$ . Then  $g$  is multiplicative. For  $e \geq 1$ , we have  $g(p^e) = v^e - v^{e-1}$  if  $p \leq y$ , and  $g(p^e) = 0$  if  $p > y$ . Hence,

$$\begin{aligned} \sum_{n \leq x} v^{\Omega(n;y)} &= \sum_{d \leq x} g(d) \left\lfloor \frac{x}{d} \right\rfloor \leq x \sum_{d \leq x} \frac{g(d)}{d} \\ &\leq x \prod_{p \leq y} \left( 1 + \frac{v-1}{p} + \frac{v^2-v}{p^2} + \dots \right) = \frac{x}{2-v} \prod_{3 \leq p \leq y} \left( 1 + \frac{v-1}{p-v} \right). \end{aligned}$$

Now  $2 - v = 1/k$ , and the rightmost product is

$$\leq \exp \left( \sum_{3 \leq p \leq y} \frac{v-1}{p-v} \right) \leq \exp \left( \sum_{3 \leq p \leq y} \frac{1}{p-2} \right) \leq \exp \left( \sum_{p \leq y} \frac{1}{p} + O(1) \right) \ll \log y.$$

Collecting our estimates, we have shown that

$$\sum_{n \leq x} v^{\Omega(n;y)} \ll kx \log y.$$

But each term with  $\Omega(n; y) \geq k$  makes a contribution to the left-hand side that is  $\geq v^k \geq (2 - 1/k)^k = 2^k (1 - \frac{1}{2k})^k \gg 2^k$ . Thus, the number of such terms is  $\ll \frac{k}{2^k} x \log y$ .  $\square$

The next lemma can be viewed as a partial shifted-primes analogue of the Hardy–Ramanujan inequalities. A proof can be found in the text of Prachar [11, Lemma 7.1, p. 166] (cf. Erdős [1]). There a slightly stronger assertion is shown for shifted primes  $p - 1$ ; only trivial changes are required to replace  $p - 1$  with  $p + 1$ .

**Lemma 3.3.** *Let  $t \geq 3$ , and let  $k \geq 1$ . The number of primes  $p \leq t$  with  $\omega(p + 1) = k$  is*

$$\ll \frac{t}{(\log t)^2} \left( \frac{(\log \log t + c_7)^{k+2}}{(k-1)!} + 1 \right).$$

*Proof of the upper bound in Theorem 1.3.* It is enough to prove the result for large values of  $A$ . Suppose that  $m \leq x$  is an additive endpoint, and write  $m = \sigma(n)$  with  $n$  practical. Put  $Z := 2A \log \log x$ . The number of values of  $m$  corresponding to an integer  $n \leq x^{3/4}$  or an  $n$  with  $\Omega(n) > Z$  is, by Lemma 3.2,

$$\ll x^{3/4} + \frac{Z}{2Z} x \log x \ll_A \frac{x}{(\log x)^A}.$$

Thus, with

$$Z' := \frac{1}{2 \log A} \log \log x,$$

Lemma 3.1 allows us to assume that

$$(3.4) \quad \omega(n) \geq Z'.$$

We now show that most of the primes dividing  $n$  make a large contribution to  $\Omega(\sigma(n)) = \Omega(m)$ . We claim we can assume that both of the following hold:

- (i) There are fewer than  $Z'/4$  primes  $p$  for which  $p^2 \mid n$ .
- (ii) There are fewer than  $Z'/4$  primes  $p$  dividing  $n$  for which

$$(3.5) \quad \Omega(p + 1) \leq 16A \log A.$$

With  $K := \lceil Z'/4 \rceil$ , the number of  $n \leq x$  which are exceptions to (i) is, by the multinomial theorem,

$$\leq x \sum_{\substack{d \leq x, \text{ squarefree} \\ \omega(d)=K}} \frac{1}{d^2} \leq \frac{x}{K!} \left( \sum_{p \leq x} \frac{1}{p^2} \right)^K \leq x(e/K)^K < x/(\log x)^A,$$

once  $x$  is large. (We use here that  $\sum p^{-2} < 1$  and the elementary inequality  $K! \geq (K/e)^k$ .) To handle (ii), we observe that from Lemma 3.3 and partial summation, the sum of the reciprocals of all  $p$  satisfying (3.5) converges. Let  $S$  denote this sum. Then the number of exceptions to (ii) is, for large  $x$ ,

$$\leq \frac{x}{K!} \left( \sum_{\substack{p \leq x \\ p \text{ satisfies (3.5)}}} \frac{1}{p} \right)^K \leq x(eS/K)^K < x/(\log x)^A.$$

Hence, we can indeed assume (i) and (ii).

From (3.4), it now follows that there are at least  $Z' - 2\frac{Z'}{4} = \frac{Z'}{4}$  primes  $p$  for which  $p \parallel n$  and for which  $\Omega(p+1) > 16A \log A$ . Hence,

$$\Omega(m) = \Omega(\sigma(n)) \geq \sum_{p \parallel n} \Omega(p+1) > 16A \log A \cdot \frac{Z'}{4} = 2A \log \log x.$$

But by Lemma 3.2, the number of  $m \leq x$  with  $\Omega(m)$  this large is  $\ll_A x/(\log x)^A$ . This completes the proof of Theorem 1.3 for large  $x$ . If  $x$  is bounded in terms of  $A$ , then the theorem is trivial.  $\square$

**Remark.** The method given here can be pushed to yield the more explicit result that the count of  $m \leq x$  that occur as additive endpoints is smaller than

$$x / \exp \left( c_8 \log \log x \frac{\log \log \log x}{\log \log \log \log x} \right).$$

**3.2. The lower bound in Theorem 1.3.** The lower bound in Theorem 1.3 will be deduced from the following proposition, which may be of interest outside of this context.

**Proposition 3.4.** *Let  $A > 0$ . There is a constant  $c = c(A)$  so that the following holds. If  $x$  is sufficiently large, say  $x > x_0(A, c)$ , then any subset  $\mathcal{S} \subset [1, x]$  with*

$$\#\mathcal{S} \leq x / \exp(c(\log \log x)^3)$$

*satisfies*

$$\#\sigma^{-1}(\mathcal{S}) \leq x/(\log x)^A.$$

*Here  $\sigma^{-1}(\mathcal{S})$  denotes the set of  $n$  with  $\sigma(n) \in \mathcal{S}$ .*

**Remark.** It is perhaps surprising that one cannot improve the upper bound on  $\#\sigma^{-1}(\mathcal{S})$  very much, even if one assumes that  $\mathcal{S}$  consists of only a single element! Indeed, plausible conjectures about the distribution of smooth shifted primes  $p+1$  (such as what would follow from the Elliott–Halberstam conjecture) imply that for all large  $x$ , there is a singleton set  $\mathcal{S} \subset [1, x]$  with  $\#\sigma^{-1}(\mathcal{S}) > x^{1-\epsilon}$ . (Here  $\epsilon > 0$  is arbitrary but fixed.) For the Euler  $\varphi$ -function, this result is due to Erdős [1] (see also the exposition of Pomerance [10]); the  $\sigma$ -version can be proved similarly, replacing  $p-1$  with  $p+1$  when necessary.

To apply Proposition 3.4 to the case of the practical numbers, it is convenient to recall Gronwall's determination of the maximal order of the sum-of-divisors function  $\sigma$  [5, Theorem 323, p. 350].

**Lemma 3.5.** *We have  $\limsup_{n \rightarrow \infty} \frac{\sigma(n)}{n \log \log n} = e^\gamma$ .*

*Proof of the lower bound in Theorem 1.3.* Let  $x$  be large. By Lemma 3.5, if  $n \leq \frac{x}{2 \log \log x}$ , then  $\sigma(n) \leq x$ . (We use here that  $e^\gamma < 2$ .) Thus, with  $\mathcal{S}$  the set of additive endpoints not exceeding  $x$ ,

$$\#\sigma^{-1}(\mathcal{S}) \geq PR \left( \frac{x}{2 \log \log x} \right) \gg \frac{x}{(\log x)(\log \log x)},$$

using the lower estimate in (1.3) for the last step. The desired lower bound on  $\#\mathcal{S}$  now follows from (the contrapositive of) Proposition 3.4, with  $A = 1.1$ .  $\square$

The rest of this section is devoted to the proof of Proposition 3.4. The proof rests on a  $\sigma$ -analogue of a result for the Euler function appearing in a paper of Luca and the first author [7, Lemma 2.1].

**Lemma 3.6.** *Let  $x \geq 3$ . Let  $d$  be a squarefree natural number with  $d \leq x$ . The number of  $n$  for which  $d \mid \sigma(n)$  and  $\sigma(n) \leq x$  is*

$$\leq \frac{x}{d} (c_9 \log x)^{3\omega(d)}.$$

*Proof.* If  $d = 1$ , the result is clear. Suppose that  $d > 1$ . Let  $n$  be an integer for which  $\sigma(n) \in [1, x]$  is a multiple of  $d$ . Write the prime factorization of  $n$  in the form  $n = \prod_i p_i^{e_i}$ . Since  $d \mid \sigma(n)$ , there is a factorization  $d = d_1 d_2 \cdots$  for which each  $d_i \mid \sigma(p_i^{e_i})$ . Discarding those terms with  $d_i = 1$  and relabeling, we can assume that  $d = d_1 \cdots d_l$ , where each  $d_i > 1$ . Clearly,  $l \leq \omega(d)$ .

We now fix the factorization  $d = d_1 \cdots d_l$  and count the number of corresponding  $n$ . This count does not exceed

$$(3.6) \quad x \prod_{i=1}^l \left( \sum_{\substack{p^e: \sigma(p^e) \leq x \\ d_i \mid \sigma(p^e)}} \frac{1}{p^e} \right).$$

We proceed to estimate the inner sum in (3.6). If  $d_i \mid \sigma(p^e)$ , then  $\sigma(p^e) = d_i m$ , with  $m \leq x/d_i$ . Since  $\sigma(p^e) = 1 + p + \cdots + p^e \leq 2p^e$ ,

$$\sum_{\substack{p^e: \sigma(p^e) \leq x \\ d_i \mid \sigma(p^e)}} \frac{1}{p^e} \leq \frac{2}{d_i} \sum_{m \leq x/d_i} \frac{1}{m} \sum_{p^e: \sigma(p^e) = md_i} 1.$$

For each fixed  $e \geq 1$ , there is at most one prime  $p$  with  $\sigma(p^e) = md_i$ ; moreover, since  $md_i \leq x$ , there are no such  $p$  once  $e > \log x / \log 2$ . Thus,

$$\frac{2}{d_i} \sum_{m \leq x/d_i} \frac{1}{m} \sum_{p^e: \sigma(p^e) = md_i} 1 \ll \frac{\log x}{d_i} \sum_{m \leq x/d_i} \frac{1}{m} \ll \frac{(\log x)^2}{d_i}.$$

Inserted back into (3.6), we find that for a certain absolute constant  $C > 1$ , the number of  $n$  corresponding to the given factorization is at most

$$x \prod_{i=1}^l \frac{C(\log x)^2}{d_i} = \frac{x}{d} C^l (\log x)^{2l} \leq \frac{x}{d} C^{\omega(d)} (\log x)^{2\omega(d)}.$$

Finally, we sum over unordered factorizations of  $d$  into parts  $> 1$ . Since  $d$  is squarefree, the number of such factorizations is precisely  $B_{\omega(d)}$ , where  $B_k$  denotes the  $k$ th Bell number

(the number of set partitions of a  $k$ -element set). Thinking combinatorially, we have the crude bound  $B_k \leq k^k$ , and so the total number of  $n$  which arise is

$$\leq \omega(d)^{\omega(d)} \left( \frac{x}{d} C^{\omega(d)} (\log x)^{2\omega(d)} \right) = \frac{x}{d} (C\omega(d)(\log x)^2)^{\omega(d)}.$$

By definition, we have  $\omega(d) \leq \Omega(d) \leq \log x / \log 2$ , where the final inequality follows from the simple observation that  $2^{\Omega(d)} \leq d \leq x$ . This proves our lemma with  $c_9 = (C/\log 2)^{1/3}$ .  $\square$

**Lemma 3.7.** *Fix  $A \geq 3$ . The number of  $n \leq x$  for which*

$$(3.7) \quad \Omega(\sigma(n)) \geq 8A^2(\log \log x)^2$$

*is  $o(x/(\log x)^A)$ , as  $x \rightarrow \infty$ .*

*Proof.* We may suppose that  $\omega(n) \leq 2A \log \log x$ . Indeed, Lemma 3.2 shows that for  $x \geq 3$ , the number of  $n \leq x$  not satisfying the stronger inequality  $\Omega(n) \leq 2A \log \log x$  is

$$\ll \frac{A \log \log x}{2^{A \log \log x}} x \log x \ll_A \frac{x \log \log x}{(\log x)^{2A \log 2 - 1}}.$$

Since  $A \geq 3$ , the exponent  $2A \log 2 - 1 > A$ , and so this upper bound is  $o(x/(\log x)^A)$ .

Writing  $\Omega(\sigma(n)) = \sum_{p^e \parallel n} \Omega(\sigma(p^e))$ , we thus deduce that if (3.7) holds, then

$$(3.8) \quad \Omega(\sigma(p^e)) \geq \frac{8A^2(\log \log x)^2}{2A \log \log x} = 4A \log \log x$$

for some prime power  $p^e \parallel n$ .

Suppose first that  $e > 1$ . Then (for large  $x$ ) the squarefull part of  $n$  is of size at least

$$(3.9) \quad p^e \geq \frac{1}{2} \sigma(p^e) \geq \frac{1}{2} 2^{\Omega(\sigma(p^e))} \geq \frac{1}{2} 2^{4A \log \log x} > (\log x)^{5A/2}.$$

But then the number of possibilities for  $n \leq x$  is  $\ll x/(\log x)^{5A/4}$ , and so in particular is  $o(x/(\log x)^A)$ . On the other hand, if  $e = 1$ , then (3.8) implies that  $n$  is divisible by some prime  $p$  with  $\Omega(p+1) \geq 4A \log \log x$ . For each such  $p$ , the number of corresponding  $n$  is  $\leq x/p < 2x/(p+1)$ . Summing over  $p$ , we find that the total number of such  $n \leq x$  is at most

$$2x \sum_{\substack{d \leq x \\ \Omega(d) \geq 4A \log \log x}} \frac{1}{d}.$$

Put  $Z := 4A \log \log x$ ; by partial summation, along with Lemma 3.2 and the final inequality in (3.9), this upper bound is

$$\ll x \frac{Z}{2^Z} \int_2^x \frac{\log t}{t} dt \ll x (\log x)^2 \frac{Z}{2^Z} \ll_A x \frac{(\log x)^2 \log \log x}{(\log x)^{5A/2}},$$

and so is  $o(x/(\log x)^A)$ , as  $x \rightarrow \infty$ . This completes the proof.  $\square$

**Lemma 3.8.** *Let  $x \geq 3$ , and let  $z \geq 1$ . The number of  $n \leq x$  with  $\sigma(n)$  divisible by  $p^2$  for some prime  $p > z$  is  $\ll x(\log x)^2 z^{-1/2}$ .*

*Proof.* If  $p^2 \mid \sigma(n)$ , then either  $p \mid \sigma(q^e)$  for a proper prime power  $q^e$  exactly dividing  $n$ , or there are two distinct primes  $q_1$  and  $q_2$  exactly dividing  $n$  with  $q_1, q_2 \equiv -1 \pmod{p}$ . In the former case,  $n$  has a squarefull divisor of size  $\geq q^e \geq \frac{1}{2} \sigma(q^e) > p/2 > z/2$ . The

number of such  $n$  is  $\ll xz^{-1/2}$ , which is acceptable for us. For a given  $p$ , the number of  $n$  arising in the second case is

$$\leq x \left( \sum_{\substack{q \leq x \\ q \equiv -1 \pmod{p}}} \frac{1}{q} \right)^2 \leq x \left( \sum_{j \leq x/p} \frac{1}{pj-1} \right)^2 \ll x(\log x)^2 p^{-2}.$$

Summing over  $p > z$ , we find that the total number of  $n$  that can arise from this case is  $\ll x(\log x)^2 z^{-1}$ , which is also acceptable.  $\square$

*Proof or Proposition 3.4.* We may suppose that our fixed constant  $A$  satisfies  $A \geq 5$ . We will show that for such  $A$ , the proposition holds with  $c(A) = 50A^3$ .

Let  $\mathcal{S}_1$  consist of those  $m \in \mathcal{S}$  for which either

- (i)  $m \leq x/(\log x)^{2A}$ , or
- (ii)  $\Omega(m) \geq 8A^2(\log \log x)^2$ , or
- (iii)  $p^2 \mid m$  for some  $p > (\log x)^{3A}$ .

We let  $\mathcal{S}_2$  consist of the remaining elements of  $\mathcal{S}$ . By Lemmas 3.7 and 3.8, the size of  $\sigma^{-1}(\mathcal{S}_1)$  is  $o(x/(\log x)^A)$  as  $x \rightarrow \infty$  (uniformly in the choice of  $\mathcal{S}$ ).

We turn now to  $\mathcal{S}_2$ . To each  $m \in \mathcal{S}_2$ , we associate the divisor  $m'$  of  $m$  defined by

$$m' := \prod_{\substack{p^e \parallel m \\ p > (\log x)^{3A}}} p.$$

Then  $m'$  is squarefree, and

$$(3.10) \quad \omega(m') \leq \Omega(m) < 8A^2(\log \log x)^2.$$

Moreover, assuming that  $x$  is large, since  $m > (x/\log x)^{2A}$ ,

$$(3.11) \quad m' \geq m/((\log x)^{3A})^{\Omega(m)} > \frac{x/(\log x)^{2A}}{\exp(24A^3(\log \log x)^3)} > x/\exp(25A^3(\log \log x)^3).$$

We bound the number of  $\sigma$ -preimages of  $m$  from above by the number of  $n$  for which  $\sigma(n) \in [1, x]$  is a multiple of  $m'$ . By Lemma 3.6, along with (3.10) and (3.11), the number of such  $n$  is

$$\begin{aligned} &\leq \frac{x}{m'} (c_9 \log x)^{3\omega(m')} \leq \exp(25A^3(\log \log x)^3) (c_9 \log x)^{24A^2(\log \log x)^2} \\ &\leq \exp(49A^3(\log \log x)^3), \end{aligned}$$

say. Summing over the elements of  $\mathcal{S}_2$ , we find that

$$\#\sigma^{-1}(\mathcal{S}_2) \leq \exp(49A^3(\log \log x)^3) \cdot \#\mathcal{S}_2.$$

So if we assume that  $\#\mathcal{S} \leq x/\exp(50A^3(\log \log x)^3)$ , then  $\#\sigma^{-1}(\mathcal{S}_2) = o(x/(\log x)^A)$ , as  $x \rightarrow \infty$ . Combined with our earlier estimate on the size of  $\sigma^{-1}(\mathcal{S}_1)$ , this shows that  $\#\sigma^{-1}(\mathcal{S}) \leq x/(\log x)^A$  once  $x$  is sufficiently large.  $\square$

#### 4. PROOF OF THEOREM 1.5

The key to the proof of Theorem 1.5 is the following inequality of Robin [12, Théorème 2].

**Lemma 4.1.** *For each natural number  $n \geq 3$ ,*

$$\sigma(n) \leq e^\gamma n \log \log n + 0.6483 \frac{n}{\log \log n}.$$

*Proof of Theorem 1.5.* Suppose for the sake of contradiction that  $f(n) \geq HS(n)$  but that  $n$  is not practical. We assume to begin with that  $n > 14$ , treating small  $n$  at the end of the proof. Let  $d$  be the practical component of  $n$ , and write  $n = dq$ . Then  $q > 1$ , and

$$P^-(q) > \sigma(d) + 1 = f(n) + 1 > HS(n) > n^{1/2},$$

where in fact the last inequality holds for all  $n > 6$ . It follows that  $q$  is prime and  $P^-(q) = q$ . Hence,  $HS(n) < q = n/d$ , and so

$$d < \frac{n}{HS(n)}.$$

Also, since  $\sigma(d) = f(n) \geq HS(n)$ , we have

$$q = n/d \leq \frac{n}{d} \frac{\sigma(d)}{HS(n)}.$$

Multiplying the last two displayed inequalities shows that

$$n = dq \leq \frac{\sigma(d)}{d} \left( \frac{n}{HS(n)} \right)^2 = \frac{\sigma(d)}{d} n (e^\gamma \log \log n)^{-1},$$

and so

$$(4.1) \quad \frac{\sigma(d)}{d} \geq e^\gamma \log \log n.$$

Since  $q > n^{1/2}$  and  $n = dq$ , we have that  $q > d$ , and so

$$\log \log n = \log \log (qd) > \log \log (d^2) = \log \log d + \log 2;$$

thus, (4.1) gives

$$(4.2) \quad \begin{aligned} \frac{\sigma(d)}{d} &\geq e^\gamma \log \log d + e^\gamma \log 2 \\ &> e^\gamma \log \log d + 1.2345. \end{aligned}$$

We now derive a contradiction to Robin's inequality. We can assume that  $d \geq 6$ ; otherwise,  $\sigma(d)/d \leq 7/4$ , and (4.1) then implies that  $n \leq 14$ , contrary to hypothesis. By Lemma 4.1,

$$\frac{\sigma(d)}{d} \leq e^\gamma \log \log d + \frac{0.6483}{\log \log d}.$$

Combining this inequality with (4.2), we obtain  $0.6483/\log \log d > 1.2345$ . But this fails for all  $d \geq 6$ . This contradiction completes the proof for  $n > 14$ .

It remains to treat the cases when  $3 < n \leq 14$ . For odd  $n > 3$ , the hypotheses of the theorem are never satisfied, since  $f(n) = 1 < HS(5) \leq HS(n)$ . So the only possible exceptions to the theorem have  $n$  even. The non-practical even values of  $n \leq 14$  are  $n = 10$  and  $n = 14$ , and in both cases,  $f(n) = 3 < HS(n)$ , so the theorem holds.  $\square$

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## REFERENCES

- [1] P. Erdős, *On the normal number of prime factors of  $p - 1$  and some related problems concerning Euler's  $\varphi$ -function*, Quart. J. Math. **6** (1935), 205–213.
- [2] ———, *On a Diophantine equation*, Mat. Lapok **1** (1950), 192–210.
- [3] H. Halberstam and K. F. Roth, *Sequences*, second ed., Springer-Verlag, New York, 1983.
- [4] R. R. Hall and G. Tenenbaum, *Divisors*, Cambridge Tracts in Mathematics, vol. 90, Cambridge University Press, Cambridge, 1988.
- [5] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, sixth ed., Oxford University Press, Oxford, 2008.
- [6] M. Hausman and H. N. Shapiro, *On practical numbers*, Comm. Pure Appl. Math. **37** (1984), no. 5, 705–713.
- [7] F. Luca and P. Pollack, *An arithmetic function arising from Carmichael's conjecture*, J. Théor. Nombres Bordeaux **23** (2011), 697–714.
- [8] F. Luca and C. Pomerance, *Irreducible radical extensions and Euler-function chains*, Combinatorial number theory, de Gruyter, Berlin, 2007, pp. 351–361.
- [9] M. Margenstern, *Les nombres pratiques: théorie, observations et conjectures*, J. Number Theory **37** (1991), no. 1, 1–36.
- [10] C. Pomerance, *Two methods in elementary analytic number theory*, Number theory and applications (Banff, AB, 1988), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 265, Kluwer Acad. Publ., Dordrecht, 1989, pp. 135–161.
- [11] K. Prachar, *Primzahlverteilung*, Springer-Verlag, Berlin, 1957.
- [12] G. Robin, *Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann*, J. Math. Pures Appl. (9) **63** (1984), no. 2, 187–213.
- [13] E. Saias, *Entiers à diviseurs denses. I*, J. Number Theory **62** (1997), no. 1, 163–191.
- [14] W. Sierpiński, *Sur une propriété des nombres naturels*, Ann. Mat. Pura Appl. (4) **39** (1955), 69–74.
- [15] A. K. Srinivasan, *Practical numbers*, Current Science **17** (1948), 179–180.
- [16] B. M. Stewart, *Sums of distinct divisors*, Amer. J. Math. **76** (1954), 779–785.
- [17] G. Tenenbaum, *Sur un problème de crible et ses applications*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 1, 1–30.
- [18] ———, *Introduction to analytic and probabilistic number theory*, Cambridge Studies in Advanced Mathematics, vol. 46, Cambridge University Press, Cambridge, 1995.
- [19] ———, *Sur un problème de crible et ses applications. II. Corrigendum et étude du graphe divisoriel*, Ann. Sci. École Norm. Sup. (4) **28** (1995), no. 2, 115–127.

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