Computing L(1, χ)

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Outline

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2. Preliminary Estimates for L $(1, \chi)$
3. Euler-MacLaurin Summation
4. Example: Determining L$(1, \chi)$ for χ (mod 3)
5. Computing L$(1, \chi)$ for other moduli ≤ 10
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7. Estimates for L$(1, \chi)$ when $x = 1000$

Introduction

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A **Dirichlet Character** χ modulo m is a function $\chi : \mathbb{Z} \to \mathbb{C}$ that is periodic, completely multiplicative and satisfies the following property: If $\gcd(a, m) > 1$ then $\chi(a) = 0$ and if $\gcd(a, m) = 1$ then $\chi(a) \neq 0$.

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- We say that a Dirichlet character is **principal** if it assumes the value 1 on arguments coprime to the modulus and 0 otherwise.

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- We can analytically continue $\mathrm{L}(s,\chi)$ to a meromorphic function on the whole complex plane.

- In class, we used the fact that $\mathrm{L}(1,\chi)$ is nonvanishing to prove Dirichlet's Theorem on Primes in Arithmetic Progressions.

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- When χ is a principal Dirichlet character (mod m), we know that $\mathrm{L}(1,\chi) = \sum_{\substack{n=1\\(n,m)=1}}^{\infty} \frac{1}{n}$, which diverges.
- We are interested in accurately computing $\mathrm{L}(1,\chi)$ for nonprincipal Dirichlet characters with modulus up to 10.

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- So $\mathrm{L}(1,\ \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \sum_{n=1}^{\infty} (\frac{1}{4n+1} - \frac{1}{4n+3})$

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- So $\mathrm{L}(1,\ \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \sum_{n=1}^{\infty} (\frac{1}{4n+1} - \frac{1}{4n+3})$
- We can re-write the last expression as $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \text{arctan } 1 = \frac{\pi}{4}$. Hence, $\mathrm{L}(1, \chi) = \frac{\pi}{4}$.

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- Unfortunately, $\mathrm{L}(1, \chi)$ does not appear to have such a simple expression when we consider other Dirichlet characters.

- Instead, we will try to estimate $\mathrm{L}(1, \chi)$ by rewriting the infinite series in terms of a finite sum plus a small error term:

$$\mathrm{L}(1, \chi) = \sum_{n \leq x} \left(\frac{\chi(1)}{mn+1} + \cdots + \frac{\chi(m-1)}{mn+(m-1)} \right), \ \chi \text{ nonprincipal.}$$

Example: $\chi \pmod 3$

For the next few slides, let χ be the nonprincipal character (mod 3):

$$\chi(n) = \begin{cases} 0, & n \equiv 0(3) \\ 1, & n \equiv 1(3) \\ -1, & n \equiv -1(3) \end{cases}$$

A Naive Estimate

- We can easily write $\mathrm{L}(1, \chi) = \sum_{n \leq x} \frac{\chi(n)}{n} + O(\frac{1}{x})$,

  since $\mathrm{L}(1, \chi) = \frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + - \cdots$.

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- Using this expression, we could determine $\mathrm{L}(1,\ \chi)$ to 6 decimal places by summing to $x = 1,000,000$.

A Better Estimate

- Instead, let's rewrite $L(1, \chi) = \sum_{n=0}^{\infty}(\frac{1}{3n+1} - \frac{1}{3n+2})$ as $\sum_{n=0}^{\infty}(\frac{1}{(3n+1)(3n+2)}) = \sum_{n\leq x}\frac{1}{(3n+1)(3n+2)} + E(x)$.

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- We will estimate $E(x)$ with the integral $\int_x^{\infty} \frac{dt}{(3t+1)(3t+2)}$. The integral evaluates to $\frac{1}{3} \log(1 + \frac{1}{3x+1})$.

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- The Taylor series for $\frac{1}{3}\log(1 + \frac{1}{3x+1}) = \frac{1}{3}(\frac{1}{3x+1} - \frac{1}{2}(\frac{1}{3x+1})^2 + \cdots)$

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- Thus, $\mathrm{L}(1, \chi) = \sum_{n \leq x} \frac{1}{(3n+1)(3n+2)} + \frac{1}{9x} + O(\frac{1}{x^2})$

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- Q: Can we do better?

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- Q: Can we do better?

- A: Yes, but in order to do so, we will use a more sophisticated technique called Euler-MacLaurin Summation.

Euler-MacLaurin Summation

- In order to obtain a better estimate for L$(1, \chi)$, we need to be able to more accurately measure the difference between $\displaystyle\sum_{n=x}^{\infty} \frac{1}{(3n+1)(3n+2)}$ and $\int_{x}^{\infty} \frac{1}{(3t+1)(3t+2)}$.

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- Euler-MacLaurin summation will give us an expression for the difference between the sum and the integral in terms of higher derivatives of $f(t) = \frac{1}{(3t+1)(3t+2)}$ at the end points of the interval $[x, \infty]$.

Euler-MacLaurin Summation

- We will derive the Euler-MacLaurin Summation formula for a function defined on the interval $[1, n]$. The same method will work on $[x, \infty]$ if we shift the argument of the function by a constant and take the limit as the right endpoint goes to infinity.

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  Let $d_n := \sum_{k=1}^{n-1} f(k) - \int_1^n f(t)dt.$

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- We can re-write $d_n = \sum_{k=1}^{n-1} I(k)$ where $I(k) = \int_k^{k+1} (f(k) - f(t))dt$.

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- We can use integration by parts, taking
  $u = f(k) - f(t), v = t - (k+1)$, to re-write
  $I(k) = \int_k^{k+1}(t - k - 1)f'(t)dt$.

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$$d_n = \sum_{k=1}^{n-1} I(k) = \sum_{k=1}^{n-1} \int_k^{k+1} (t - \lfloor t \rfloor - 1) f'(t) dt.$$

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- But this is just $\int_1^n (t - \lfloor t \rfloor) f'(t)dt - \int_1^n f'(t)dt$, which is the same as $\int_1^n (t - \lfloor t \rfloor) f'(t)dt + f(1) - f(n)$, by the Fundamental Theorem of Calculus.

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- Rearranging terms and using the definition of d_n, we have:
$$\sum_{k=1}^{n} f(k) = \int_1^n f(t)dt + \int_1^n (t - \lfloor t \rfloor)f'(t)dt + f(1).$$

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- The last two terms on the right of $\sum_{k=1}^{n} f(k) = \int_1^n f(t)dt + \int_1^n (t - \lfloor t \rfloor) f'(t)dt + f(1)$ represent the difference between the sum and the integral.

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- The integral term in the error can now be written: $\int_1^n (t - \lfloor t \rfloor) f'(t)dt = \int_1^n (t - \lfloor t \rfloor - \frac{1}{2}) f'(t)dt + \frac{1}{2}\int_1^n f'(t)dt$. The last term of the right-hand side is $\frac{1}{2}(f(n) - f(1))$.

Euler-MacLaurin Summation

- The factor $x - \lfloor x \rfloor - \frac{1}{2}$ has the value $-\frac{1}{2}$ when $x \in \mathbb{Z}$. We modify this factor slightly to make it vanish at the integers (this will be helpful when we use integration by parts later on):

$$\mathrm{P}_1(t) = \begin{cases} t - \lfloor t \rfloor - \dfrac{1}{2}, & t \notin \mathbb{Z} \\ 0, & t \in \mathbb{Z} \end{cases}$$

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- Putting everything together, we arrive at...

First Derivative Euler-MacLaurin Summation

Theorem

For any function f with a continuous derivative on the interval [1,n], we have $\sum_{k=1}^{n} f(k) = \int_1^n f(t)dt + \frac{1}{2}(f(n) + f(1)) + \int_1^n P_1(t)f'(t)dt.$

Second Derivative Euler-MacLaurin Summation

- We define $P_2(t)$ to be a function whose derivative is $2P_1(t)$ at all noninteger values of t, ie. $P_2(t) = 2\int_0^t P_1(u)du + c$, where c is a constant that we will determine later.

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- Since P_1 has period 1 and $\int_0^1 P_1(u)du = 0$, then $P_2(t+1) - P_2(t) = 2\int_t^{t+1} P_1(u)du = 0$, i.e. P_2 has period 1.

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- Because of periodicity, P_2 has constant value $c = P_2(0)$ at the integers.

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- On [0,1], $P_2(t) = t^2 - t + c$. The integral of $P_2(t)$ on this interval is $c - \frac{1}{6}$, so we choose $c = \frac{1}{6}$. Thus, $P_2(0) = \frac{1}{6}$.

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- Applying integration by parts to the First Derivative Form of the Euler-MacLaurin Summation Formula, we arrive at...

Second Derivative Euler-MacLaurin Summation

Theorem

For any function f with a continuous second derivative on the interval [1, n], we have $\sum_{k=1}^{n} f(k) =$

$$\int_1^n f(t)dt + \frac{1}{2}[f(n) + f(1)] + \frac{1}{2}P_2(0)[f'(n) - f'(1)] - \frac{1}{2}\int_1^n P_2(t)f''(t)dt.$$

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- As with the previous case, P_3 has period 1 and vanishes at the integers. Moreover, $P_3(0) = 0$ since $P_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$ on [0,1].

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- Integration by parts over $[1, n]$ gives us $\int_1^n P_2(t)f''(t)dt = -\frac{1}{3}\int_1^n P_3(t)f'''(t)dt$, provided that f''' is continuous.

Third Derivative Euler-MacLaurin Summation

Combining $\int_1^n P_2(t)f''(t)dt = -\frac{1}{3}\int_1^n P_3(t)f'''(t)dt$ with the 2nd Derivative Form of the Euler-MacLaurin Formula yields:

Theorem

For any function f with a continuous third derivative on the interval [1,n], we have $\sum_{k=1}^{n} f(k) =$

$$\int_1^n f(t)dt + \frac{1}{2}[f(n) - f(1)] + \frac{1}{2}P_2(0)[f'(n) - f'(1)] - \frac{1}{3!}\int_1^n P_3(t)f'''(t)dt.$$

Example: Determining L(1, χ) for χ (mod 3)

- When χ is the nonprincipal Dirichlet character (mod 3), we have $$\mathrm{L}(1,\ \chi) = \sum_{n \le x-1} \frac{1}{(3n+1)(3n+2)} + E(x).$$

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- To reduce error, we will assume that x is an integer. Thus, $L(1,\ \chi) = \displaystyle\sum_{n \le x-1} \frac{1}{(3n+1)(3n+2)} + \sum_{n \ge x} \frac{1}{(3n+1)(3n+2)}$.

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- When χ is the nonprincipal Dirichlet character (mod 3), we have $L(1,\chi) = \sum_{n \le x-1} \frac{1}{(3n+1)(3n+2)} + E(x)$.

- To reduce error, we will assume that x is an integer. Thus, $L(1,\chi) = \sum_{n \le x-1} \frac{1}{(3n+1)(3n+2)} + \sum_{n \ge x} \frac{1}{(3n+1)(3n+2)}$.

- Using the Third Derivative Form of the Euler-MacLaurin Summation formula, we can write $\sum_{n \ge x} \frac{1}{(3n+1)(3n+2)} = \lim_{m \to \infty} (\int_x^m f(t)dt + \frac{1}{2}[f(m) + f(x)] + \frac{1}{2}P_2(0)[f'(m) - f'(x)] - \frac{1}{3!}\int_x^m P_3(t)f'''(t)dt)$.

Example: Determining L(1, χ) for χ (mod 3)

- We have already shown that $\lim_{m\to\infty}\int_x^m f(t)dt = \frac{1}{3}\log\left(1+\frac{1}{3x+1}\right)$.

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- Since $P_2(0)=\frac{1}{6}$, we have:

  $\frac{1}{2}P_2(0)\lim_{m\to\infty}[f'(m)-f'(x)] = \frac{1}{12}\left(\frac{3}{(1+3x)(2+3x)^2}+\frac{3}{(1+3x)^2(2+3x)}\right)$.

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- We can also compute $\frac{1}{2} \lim_{m\to\infty}[f(m) + f(x)] = \frac{1}{18x^2+18x+4}$.

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- Thus, our error term will have the same order as f''.

Example: Determining L(1, χ) for χ (mod 3)

- Now, $f''(x) = \frac{18}{81x^4+189x^3+162x^2+60x+8} + \frac{18}{81x^4+162x^3+117x^2+36x+4} + \frac{18}{81x^4+135x^3+81x^2+21x+2}$.

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- The explicit constant for O is $\frac{1}{18}$, since $\mid E(x) \mid = \frac{1}{3!} \mid \int_x^\infty P_3(t) f'''(t) dt \mid \leq \frac{1}{12}(\frac{3 \cdot 18}{81x^4}) = \frac{1}{18}(\frac{1}{x^4})$.

Nonprincipal χ (mod 3)

Putting together the results from the previous two slides, we have shown:

$$L(1,\,\chi) = \Sigma_{n \leq x-1}\left(\frac{1}{3n+1} - \frac{1}{3n+2}\right) + \frac{1}{3}\log\left(1 + \frac{1}{3x+1}\right)$$

$$+\frac{1}{18x^2+18x+4} + \frac{1}{4}\left(\frac{1}{(1+3x)(2+3x)^2} + \frac{1}{(1+3x)^2(2+3x)}\right) + O(1/x^4)$$

Extension to Other Nonprincipal Characters

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- We can use the same process to estimate other nonprincipal characters with modulus ≤ 10.
- For each nonprincipal Dirichlet character $\chi \pmod{m}$, let $f(n) = \frac{\chi(1)}{mn+1} + \frac{\chi(2)}{mn+2} + \cdots + \frac{\chi(m-1)}{mn+(m-1)}$.

Primitive Characters

Let χ be a character (mod m).
Let χ_0 be the principal character (mod km).
Then $\chi\chi_0$ is a character (mod km).

A character χ' is **primitive** (mod km) if it cannot be represented as $\chi\chi_0$, where χ, χ_0 satisfy the conditions above.

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- So $\frac{L(1,\chi)}{L(1,\chi\chi_0)} = \prod_{\substack{p \mid k \\ p \nmid m}} (1 - \frac{\chi(p)}{p})^{-1}.$

Primitive Characters

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- $L(1, \chi) = \prod_{p}(1 - \frac{\chi(p)}{p})^{-1} = \prod_{p \nmid m}(1 - \frac{\chi(p)}{p})^{-1}.$
- $L(1, \chi\chi_0) = \prod_{p}(1 - \frac{\chi\chi_0(p)}{p})^{-1} = \prod_{p \nmid km}(1 - \frac{\chi(p)}{p})^{-1}.$
- So $\frac{L(1,\chi)}{L(1,\chi\chi_0)} = \prod_{\substack{p \mid k \\ p \nmid m}}(1 - \frac{\chi(p)}{p})^{-1}.$
- Since this product is finite, we can easily estimate $L(1, \chi\chi_0)$ if we have an estimate for $L(1, \chi)$.

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- Then L(1, $\chi\chi_0$) $= \frac{L(1,\chi)}{(1-\frac{\chi(2)}{2})^{-1}} = \frac{3}{2}$L(1, χ).

L(1, χ) Estimates when $x = 1000$

Let χ be primitive, non-principal (mod m).
We have the following estimates for L(1, χ) when $m \leq 10$:

m	L(1, χ) at $x = 1000$	Accuracy (decimals)
3	0.6045997880780727	15$\ast$
4	0.78539816339744831377	16*
5	$0.8645663859100509 + 0.2040731061181289i$	≥ 13
5	0.4304089329666 7226	19$\ast$
7	$0.8040221564886499 + 0.3985253776222 1485i$	≥ 13
7	$0.5377473513795157 + 0.1052975355414652i$	≥ 13
7	1.187410411723726	16$\ast$
8	1.11072073453951 5	≥ 13
8	0.6232252713538019	≥ 16
9	$1.1360538075944662 + 0.413442376401646i$	≥ 13
9	$0.5894898173762613 + 0.1730977885063 4772i$	≥ 13

*Computed by comparing with the Class Number Formula.

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Theorem (Granville, Soundararajan)

For any primitive Dirichlet character χ (mod q), if χ is cube-free then $|L(1,\chi)| \leq \{\frac{17}{70} + o(1)\} \log q$.

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Theorem (Siegel)

$\forall \epsilon > 0$, *there exists a constant* $C_\epsilon > 0$ *such that* $|L(1, \chi)| \geq C_\epsilon q^{-\epsilon}$.

More Precision?

- As we saw in the previous slides, we can use the Third Derivative Euler Summation formula to find estimates that are accurate to at least 13 decimal places if $x = 1000$.

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- As we saw in the previous slides, we can use the Third Derivative Euler Summation formula to find estimates that are accurate to at least 13 decimal places if $x = 1000$.
- One natural question is to ask whether there is a general Euler Summation formula that would enable us to find an estimate that is accurate to any number of decimal places.

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For any function f with a continuous $(2l+1)^{st}$ *derivative on the interval* $[1, n]$*, we have* $\sum_{k=x}^{m} f(k) = \int_x^m f(t)dt + \frac{1}{2}[f(m) - f(x)] + \sum_{r=1}^{l} \frac{P_{2r}(0)}{2r!}[f^{(2r-1)}(m) - f^{(2r-1)}(x)] + \frac{1}{(2l+1)!}\int_x^m P_{2l+1}(t) f^{(2l+1)}(t)dt.$

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- This formula is expressed in terms of odd derivatives since $P_{2r+1}(0) = 0$ when $2r+1 \geq 3$.

The Bad News

The amount of time that it takes to compute $\mathrm{L}(1, \chi)$ increases each time we compute the next higher order derivative at $x = 1000$.

For example, if we use the $(2l + 1)^{st}$ Euler-MacLaurin Summation for nonprincipal χ (mod 3) with $l = 1, \cdots, 10$:

Order	Time (in seconds)
3	0.02512
5	0.142578
7	0.71495
9	1.73616
11	3.81776
13	8.48304
15	17.5361
17	43.6196
19	80.9763
21	154.386

As you can see, there is a tradeoff between accuracy and efficiency!