Measuring symmetry in lattice paths and partitions

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Measuring symmetry

For some combinatorial objects, one can study the subset of those that are *symmetric*, such as

- symmetric Dyck paths,
- self-conjugate partitions,
- palindromic compositions,
- symmetric binary trees,
- etc.

To refine this idea, we introduce the notion of *degree of symmetry*, a combinatorial statistic that measures how close an object is to being symmetric.

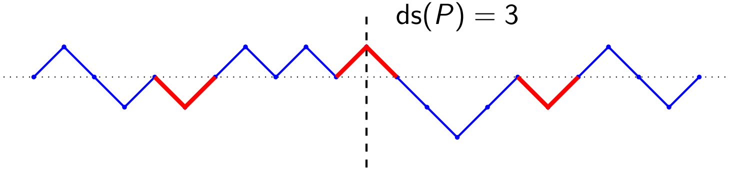
The degree of symmetry of lattice paths

Grand Dyck paths: $\mathcal{GD}_n = \{$paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)\}$

Dyck paths: $\mathcal{D}_n =$ paths in $\mathcal{GD}_n$ that do not go below the x-axis

Definition

The *degree of symmetry* of a path $P \in \mathcal{GD}_n$, denoted by $\mathrm{ds}(P)$, is the number of steps in the first half of p that are mirror images of steps in the second half.



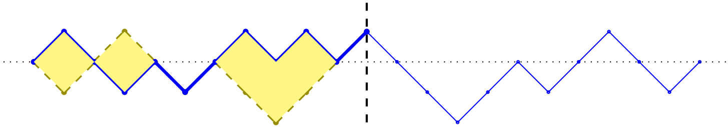
The generating function for grand Dyck paths

Theorem

The GF for grand Dyck paths by their degree of symmetry is

$$\sum_{n\geq 0}\sum_{P\in\mathcal{GD}_n} s^{\mathrm{ds}(P)}z^n = \frac{1}{2(1-s)z+\sqrt{1-4z}}.$$

The reason for this simple generating function is that when we fold a grand Dyck path along the middle, the blocks of steps that do not coincide form *parallelogram polyominoes*, which are counted by the Catalan numbers.



Another measure of symmetry

One can also measure symmetry of a grand Dyck path by the number of *symmetric vertices*: vertices in the first half that are mirror images of vertices in the second half.

Theorem

The GF for grand Dyck paths by their number of symmetric vertices is

$$\sum_{n\geq 0}\sum_{P\in\mathcal{GD}_n} v^{\mathrm{sv}(P)} z^n = \frac{1}{1-v+v\sqrt{1-4z}}.$$

Denote by $\mathrm{ret}(P)$ the number of returns of P to the x-axis. The following result also has a bijective proof:

Corollary

The statistics sv *and* ret *are equidistributed on* $\mathcal{GD}_n$.

The generating function for Dyck paths

In contrast to the simplicity of the GF for grand Dyck paths by their degree of symmetry, the GF for Dyck paths

$$D(s,z)=\sum_{n\geq 0}\sum_{P\in\mathcal{D}_n} s^{\mathrm{ds}(P)}z^n$$

is unwieldy. We rephrase the problem in terms of walks in the plane, and then apply some transformations on these walks.

$\mathcal{W}_n^1=\{$walks in the first quadrant starting at $(0,0)$, ending on diagonal, and having n steps $\nearrow\nwarrow\swarrow\searrow\}$

$\mathcal{W}_n^2=\{$walks in the first octant starting at $(0,0)$, ending on diagonal, and having n steps $\nearrow\nwarrow\swarrow\searrow$, with 2 colors for $\searrow$ leaving diagonal$\}$

$\mathcal{W}_n^3=\{$walks in the first quadrant starting at $(0,0)$, ending on x-axis, and having n steps $\nwarrow\searrow\leftarrow\rightarrow$, with 2 colors for $\nwarrow$ leaving x-axis$\}$

From Dyck paths to walks in the plane

We build a sequence of bijections:

$$\mathcal{D}_n \xrightarrow{\text{combine halves}} \mathcal{W}^1_n \xrightarrow{\text{fold along y=x}} \mathcal{W}^2_n \xrightarrow{(x,y)\mapsto(y,\frac{x-y}{2})} \mathcal{W}^3_n$$

walks in	first octant	first quadrant	first octant	first quadrant
allowed steps	↗ ↘	↖ ↗ ↙ ↘	↖ ↗ ↙ ↘	↖ ← → ↘
length	$2n$	n	n	n
ending on	x-axis	diagonal	diagonal	x-axis
2 colors for			↘ leaving diagonal	↖ leaving x-axis
ds counts	symmetric steps	steps on diagonal	steps on diagonal	steps on x-axis

Computing $D(s,z)$ is equivalent to counting walks in $\mathcal{W}^3_n$ with respect to the number of steps on the x-axis.

Let $W(x,y,s,z)$ be the GF for walks like those in $\mathcal{W}^3_n$ but with an arbitrary endpoint (whose coordinates are marked by x,y), where s marks the number of steps on the x-axis.

The generating function for Dyck paths

Theorem

The GF for Dyck paths by their degree of symmetry is $D(s,z) = W(1,0,s,z)$, where $W(x,y) := W(x,y,s,z)$ satisfies the functional equation

$$\begin{aligned}\left(xy - z(y+x^2)(1+y)\right) W(x,y) &= xy - zy(1+y)W(0,y) \\ &+z\left(y^2 - x^2 + (s-1)y(x^2+1)\right) W(x,0) \\ &-zy(y+s-1)W(0,0).\end{aligned}$$

Computations by Alin Bostan using this equation suggest:

Conjecture

$D(s,z)$ is D-finite in z but not algebraic.

Partitions by self-conjugate parts

Let $\mathcal{P}$ be the set of all integer partitions, i.e., $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$.

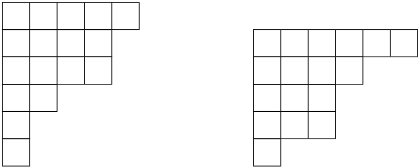
λ' = conjugate of λ, obtained by transposing its Young diagram.

Define the degree of symmetry of $\lambda \in \mathcal{P}$ as

$$\text{ds}(\lambda) = |\{i : \lambda_i = \lambda'_i\}|.$$

Example

If $\lambda = (5, 4, 4, 2, 1, 1)$, then $\lambda' = (6, 4, 3, 3, 1)$, and so $\text{ds}(\lambda) = 2$.



Partitions by self-conjugate parts

For $\lambda \in \mathcal{P}$, let $\mathrm{sp}(\lambda) = \lambda_1 + \lambda'_1$ denote the semiperimeter of its Young diagram.

Theorem

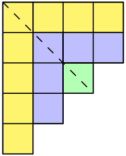
Two GF for partitions by their degree of symmetry:

$$\sum_{\lambda \in \mathcal{P}} s^{\mathrm{ds}(\lambda)} z^{\max\{\lambda_1, \lambda'_1\}} = \frac{1 - sz}{2(1-s)z + \sqrt{1-4z}}.$$

$$\sum_{\lambda \in \mathcal{P}} s^{\mathrm{ds}(\lambda)} z^{\mathrm{sp}(\lambda)} = 1 + \frac{z^2\left((1-s)(1-2z) - \sqrt{1-4z^2}\right)}{(2z-1)\left(2(1-s)z^2 + \sqrt{1-4z^2}\right)}.$$

Partitions by self-conjugate hooks

Another measure of symmetry of a partition λ is the number of self-conjugate *diagonal hooks*, denoted by $\mathrm{ds}^{\ulcorner}(\lambda)$.

$\lambda =$  has 3 diagonal hooks, 2 of which are self-conjugate, so $\mathrm{ds}^{\ulcorner}(\lambda) = 2$

Theorem

$$\sum_{\lambda \in \mathcal{P}} s^{\mathrm{ds}^{\ulcorner}(\lambda)} z^{\max\{\lambda_1, \lambda'_1\}} = \frac{1-z}{(1-s)z + \sqrt{1-4z}}.$$

Corollary

$$|\{\lambda \in \mathcal{P} : \lambda_1 \le n,\ \lambda'_1 \le n,\ \mathrm{ds}^{\ulcorner}(P) = k\}| = |\{P \in \mathcal{GD}_n : P \text{ has } k \text{ peaks at height } 1\}|.$$

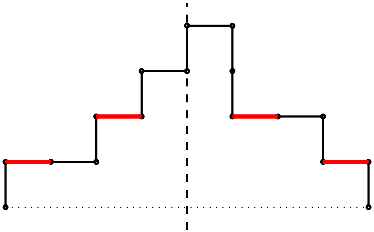
Unimodal compositions

Unimodal compositions with a centered maximum are sequences of positive integers $(a_1, a_2, \ldots, a_k)$ s.t.

$$1 \leq a_1 \leq \cdots \leq a_{\lfloor (k+1)/2 \rfloor}, \quad a_{\lceil (k+1)/2 \rceil} \geq \cdots \geq a_{k-1} \geq 1.$$

Similarly to how partitions are represented as Young diagrams, compositions can be represented as *bargraphs*:

$(1,1,2,3,4,2,2,1) \mapsto$



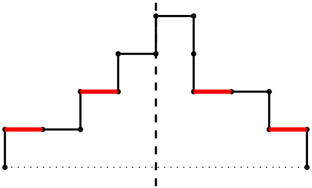
The *degree of symmetry* is the number of $i \leq k/2$ s.t. $a_i = a_{k+1-i}$.

Unimodal compositions

$\mathcal{U}$ = unimodal bargraphs with a centered maximum

For $B \in \mathcal{U}$, let

$e(B)$ = number of east steps
$n(B)$ = number of north steps
$\mathrm{ds}(B)$ = degree of symmetry



$\mathrm{ds}(B) = 2, e(B) = 8, n(B) = 4$

Theorem

$$\sum_{B\in\mathcal{U}} s^{\mathrm{ds}(B)} x^{e(B)} y^{n(B)} = \frac{y(1+x-y)}{(1-s)x^2 + \sqrt{((x+1)^2-y)((x-1)^2-y)}} - y.$$

Some open questions

- Prove that the GF for Dyck paths by the degree of symmetry is *D*-finite but not algebraic.
- Enumerate partitions by the degree of symmetry and the *area* (instead of the semiperimeter).
- Study the degree of symmetry of other combinatorial objects; for sequences and words, there is work in progress with Emeric Deutsch.
- Study refined enumerations of walks with small steps in the quarter plane with an additional variable marking some parameter (e.g. the number of certain type of steps).