ASYMPTOTIC ENUMERATION OF PERMUTATIONS AVOIDING GENERALIZED PATTERNS

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Abstract. Motivated by the recent proof of the Stanley-Wilf conjecture, we study the asymptotic behavior of the number of permutations avoiding a generalized pattern. Generalized patterns allow the requirement that some pairs of letters must be adjacent in an occurrence of the pattern in the permutation, and consecutive patterns are a particular case of them.

We determine the asymptotic behavior of the number of permutations avoiding a consecutive pattern, showing that they are an exponentially small proportion of the total number of permutations. For some other generalized patterns we give partial results, showing that the number of permutations avoiding them grows faster than for classical patterns but more slowly than for consecutive patterns.

1. Introduction

One of the most important breakthroughs in the subject of pattern-avoiding permutations has been the proof by Marcus and Tardos [18] of the so-called Stanley-Wilf conjecture, which had been open for over a decade. This is a basic result regarding the asymptotic behavior of the number of permutations that avoid a given pattern. It states that for any pattern σ there exists a constant λ such that, if $\alpha_n(\sigma)$ denotes the number of σ-avoiding permutations of size n, then $\alpha_n(\sigma) < \lambda^n$. The notion of pattern avoidance that this result is concerned with is the standard one, namely, where a permutation is said to avoid a pattern if it does not contain any subsequence which is order-isomorphic to it.

In [3], Babson and Steingrímsson introduced the notion of *generalized patterns*, which allows the requirement that certain pairs of letters of the pattern must be adjacent in any occurrence of it in the permutation. One particular case of these are *consecutive patterns*, which were independently studied by Elizalde and Noy [11]. For a subsequence of a permutation to be an occurrence of a consecutive pattern, its elements have to appear in adjacent positions of the permutation.

Analogously to the case of classical patterns, it is natural to study the asymptotic behavior of the number of permutations avoiding a generalized pattern. This problem is far from being understood. It follows from our work that for most generalized patterns the number of permutations avoiding them behaves very differently than in the case of classical patterns. In this paper we determine the asymptotic behavior for the case of consecutive patterns, showing that if σ is a consecutive pattern and $\alpha_n(\sigma)$ denotes the number of permutations of size n avoiding it, then $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)/n!}$ is a positive constant. For some particular generalized patterns we obtain the same asymptotic behavior, and for patterns of length 3 the problem is solved as well. However, the general case remains open, and it seems from our investigation that there is a big range of possible asymptotic behaviors. For some generalized patterns σ of length 4 we give asymptotic upper and lower bounds on $\alpha_n(\sigma)$.

The paper is structured as follows. In Section 2 we introduce the definitions and notation for generalized pattern avoidance. We also mention some generating function techniques that will be used in the paper, as well as previous results regarding consecutive patterns. In Section 3 we give the exponential generating functions for permutations avoiding a special kind of generalized patterns, extending the results from [11]. In Section 4 we study the asymptotic behavior as n goes to infinity of the number of permutations of size n avoiding a generalized pattern, solving the problem only in some cases. In Section 5 we give lower and upper bounds on the number of 12-34-avoiding permutations, and in Section 6 we obtain a similar result for the pattern 1-23-4. Finally, in Section 7 we discuss some open problems and further research.

2. Preliminaries

In this section we define most of the notation that will be used later on. We start introducing the notion of generalized pattern avoidance.

2.1. **Generalized patterns.** These patterns, which were introduced by Babson and Steingrímsson [3], extend the classical notion of pattern avoidance. We will denote by $\mathcal{S}_n$ the symmetric group on $\{1, 2, \ldots, n\}$. Let n, m be two positive integers with $m \leq n$, and let $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathcal{S}_n$ be a permutation. A generalized pattern σ is obtained from a permutation $\sigma_1\sigma_2 \cdots \sigma_m \in \mathcal{S}_m$ by choosing, for each $j = 1, \ldots, m-1$, either to insert a dash - between σ_j and σ_{j+1} or not. More formally, $\sigma = \sigma_1\varepsilon_1\sigma_2\varepsilon_2 \cdots \varepsilon_{m-1}\sigma_m$, where each ε_j is either the symbol - or the empty string.

With this notation, we say that π *contains* (the generalized pattern) σ if there exist indices $i_1 < i_2 < \ldots < i_m$ such that

(i) for each $j = 1, \ldots, m-1$, if ε_j is empty then $i_{j+1} = i_j + 1$, and
(ii) $\rho(\pi_{i_1}\pi_{i_2} \cdots \pi_{i_m}) = \sigma_1\sigma_2 \cdots \sigma_m$, where ρ is the reduction consisting in relabeling the elements with $\{1, \ldots, m\}$ so that they keep the same order relationships they had in π. (Equivalently, this means that for all indices a and b, $\pi_{i_a} < \pi_{i_b}$ if and only if $\sigma_a < \sigma_b$.)

In this case, $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_m}$ is called an *occurrence* of σ in π.

If π does not contain σ, we say that π *avoids* σ, or that it is σ*-avoiding.* For example, the permutation $\pi = 3542716$ contains the pattern 12-4-3, and it has exactly one occurrence of it, namely the subsequence 3576. On the other hand, π avoids the pattern 12-43.

Observe that in the case where σ has dashes in all $m-1$ positions, we recover the classical definition of pattern avoidance, because in this case condition (i) holds trivially. On the other end, the case in which σ has no dashes corresponds to consecutive patterns. In this situation, an occurrence of σ in π has to be a consecutive subsequence. Consecutive patterns were introduced independently in [11], where the authors give generating functions for the number of occurrences of certain consecutive patterns in permutations. Several papers deal with the enumeration of permutations avoiding generalized patterns. In [7], Claesson presented a complete solution for the number of permutations avoiding any single 3-letter generalized pattern with exactly one adjacent pair of letters. Claesson and Mansour [8] (see also [17]) did the same for any pair of such patterns. In [10], Elizalde and Mansour studied the distribution of several statistics on permutations avoiding 1-3-2 and 1-23 simultaneously. On the other hand, Kitaev [14] investigated simultaneous avoidance of two or more 3-letter generalized patterns without dashes.

All the patterns that appear in this paper will be represented by the notation just described. In particular, a pattern $\sigma = \sigma_1\sigma_2 \cdots \sigma_m$ without dashes will denote a consecutive pattern. We will represent classical patterns by writing dashes between any two adjacent elements, namely, as σ_1-σ_2-$\cdots$-σ_m.

If σ is a generalized pattern, let $\mathcal{S}_n(\sigma)$ denote the set of permutations in $\mathcal{S}_n$ that avoid σ. Let $\alpha_n(\sigma) = |\mathcal{S}_n(\sigma)|$ be the number of such permutations, and let

$$A_\sigma(z) = \sum_{n\geq 0} \alpha_n(\sigma)\frac{z^n}{n!}$$

be the exponential generating function counting σ-avoiding permutations.

2.2. **Labeled classes and exponential generating functions.** Here we recall some basic machinery for exponential generating functions that will be used later. We direct the reader to [13] for a detailed account on combinatorial classes and the symbolic method. Let $\mathcal{A}$ be a class of labeled combinatorial objects and let $|\zeta|$ be the size of an object $\zeta \in \mathcal{A}$. If $\mathcal{A}_n$ denotes the objects in $\mathcal{A}$ of size n and $a_n = |\mathcal{A}_n|$, then the *exponential generating function*, EGF for short, of the class $\mathcal{A}$ is

$$A(z) = \sum_{\zeta\in\mathcal{A}} \frac{z^{|\zeta|}}{|\zeta|!} = \sum_{n\geq 0} a_n \frac{z^n}{n!}.$$

In our context, the size of a permutation is simply its length.

There is a direct correspondence between set-theoretic operations (or "constructions") on combinatorial classes and algebraic operations on EGFs. Table 1 summarizes this correspondence for the operations that are used in the paper. There "union" means union of disjoint copies, "labeled product" is the usual cartesian product enriched with the relabeling operation, and "set" forms sets in the usual sense. Particularly important for us is the construction "boxed product" $\mathcal{A} = \mathcal{B}^\square * \mathcal{C}$, which corresponds to the subset of $\mathcal{B} \star \mathcal{C}$ (the usual labeled product) formed by those pairs in which the smallest label lies in the $\mathcal{B}$ component. Another similar construction is the "double boxed product" $\mathcal{A} = \mathcal{B}^\boxtimes * \mathcal{C}$, which denotes the subset of $\mathcal{B} \star \mathcal{C}$ formed by those pairs in which both the smallest and the largest label lie in the $\mathcal{B}$ component.

Construction		*Operation on GF*
Union	$\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$A(z) = B(z) + C(z)$
Labeled product	$\mathcal{A} = \mathcal{B} \star \mathcal{C}$	$A(z) = B(z)C(z)$
Set	$\mathcal{A} = \Pi(\mathcal{B})$	$A(z) = \exp(B(z))$
Boxed product	$\mathcal{A} = \mathcal{B}^{\Box} \star \mathcal{C}$	$A(z) = \int_0^z (\frac{d}{dt} B(t)) \cdot C(t)\, dt$
Double boxed product	$\mathcal{A} = \mathcal{B}^{\boxtimes} \star \mathcal{C}$	$A(z) = \int_0^z \int_0^y (\frac{d^2}{dt^2} B(t)) \cdot C(t)\, dt\, dy$

TABLE 1. The basic combinatorial constructions and their translation into exponential generating functions.

2.3. **Consecutive patterns of length** 3. For patterns of length 3 with no dashes, it follows from the trivial reversal and complementation operations that $\alpha_n(123) = \alpha_n(321)$ and $\alpha_n(132) = \alpha_n(231) = \alpha_n(312) = \alpha_n(213)$. The EGFs for these numbers are given in the following theorem of Elizalde and Noy [11], which we will use later in the paper. The symbol $\sim$ between two sequences indicates that they have the same asymptotic behavior.

Theorem 2.1 ([11])**.** *We have*

$$A_{123}(z) = \frac{\sqrt{3}}{2} \frac{e^{z/2}}{\cos(\frac{\sqrt{3}}{2}z + \frac{\pi}{6})}, \qquad A_{132}(z) = \frac{1}{1 - \int_0^z e^{-t^2/2} dt}.$$

Their coefficients satisfy

$$\alpha_n(123) \sim \gamma_1 \cdot (\rho_1)^n \cdot n!, \qquad \alpha_n(132) \sim \gamma_2 \cdot (\rho_2)^n \cdot n!,$$

where $\rho_1 = \frac{3\sqrt{3}}{2\pi}$, $\gamma_1 = e^{3\sqrt{3}\pi}$, $(\rho_2)^{-1}$ *is the unique positive root of* $\int_0^z e^{-t^2/2} dt = 1$, *and* $\gamma_2 = e^{(\rho_2)^{-2}/2}$, *the approximate values being*

$$\rho_1 = 0.8269933, \quad \gamma_1 = 1.8305194, \quad \rho_2 = 0.7839769, \quad \gamma_2 = 2.2558142.$$

Furthermore, for every $n \geq 4$, *we have*

$$\alpha_n(123) > \alpha_n(132).$$

3. PATTERNS OF THE FORM 1-σ

In this section we study a very particular class of generalized patterns, namely those that start with 1-, followed by a consecutive pattern (i.e., without dashes).

Proposition 3.1. *Let* $\sigma = \sigma_1\sigma_2 \cdots \sigma_k \in \mathcal{S}_k$ *be a consecutive pattern, and let* 1-σ *denote the generalized pattern* 1-$(\sigma_1 + 1)(\sigma_2 + 1) \cdots (\sigma_k + 1)$. *Then,*

$$A_{1\text{-}\sigma}(z) = \exp\left(\int_0^z A_\sigma(t)\, dt\right).$$

Proof. Given a permutation π, let $m_1 > m_2 > \cdots > m_r$ be the values of its left-to-right minima (recall that π_i is a left-to-right minimum of π if $\pi_j > \pi_i$ for all $j < i$). We can write $\pi = m_1 w_1 m_2 w_2 \cdots m_r w_r$, where each w_i is a (possibly empty) subword of π, each of whose elements is greater than m_i. We claim that π avoids 1-σ if and only if each of the blocks w_i (more precisely, its reduction $\rho(w_i)$) avoids the consecutive pattern σ. Indeed, it is clear that if one of the blocks w_i contains σ, then m_i together with the occurrence of σ forms an occurrence of 1-σ. Conversely, if π contains 1-σ, then the elements of π corresponding to σ have to be adjacent, and none of them can be a left-to-right minimum (since the element corresponding to '1' has to be to their left), therefore they must be all inside the same block w_i for some i.

If we denote by $\mathcal{A}$ the class of permutations avoiding σ, then, in the notation of Table 1, the class of permutations avoiding 1-σ can be expressed as

$$\Pi(\{z\}^{\Box} \star \mathcal{A}),$$

where $\{z\}^{\Box} \star \mathcal{A}$ corresponds to a block $m_i w_i$, with the box indicating that the left-to-right minimum has the smallest label. The set construction arises from the fact given a collection of blocks $m_i w_i$, there is a unique way to order them, namely with the left-to-right minima in decreasing order. The expression $A_{1\text{-}\sigma}(z) = \exp(\int_0^z A_\sigma(t)\,dt)$ follows now from this construction. □

Proposition 3.1 also appears independently in a preprint of Kitaev [15].

Example. The only permutation avoiding $\sigma = 12$ (resp. $\sigma = 21$) is the decreasing (resp. increasing) one. Therefore, by Proposition 3.1,

$$A_{1\text{-}23}(z) = A_{1\text{-}32}(z) = \exp\left(\int_0^z e^t dt\right) = e^{e^z-1},$$

the EGF for Bell numbers, which agrees with the result in [7].

Example. For the consecutive patterns 132, 231, 312 and 213, the generating function for the number of permutations avoiding either of them is given in Theorem 2.1 (which follows from [11, Theorem 4.1]). Now, by Proposition 3.1, we get the following expression:

$$A_{1\text{-}243}(z) = A_{1\text{-}342}(z) = A_{1\text{-}423}(z) = A_{1\text{-}324}(z) = \exp\left(\int_0^z \frac{dt}{1-\int_0^z e^{-u^2/2}du}\right).$$

Example. The EGF for permutations avoiding 123 or 321 is also given in Theorem 2.1. Proposition 3.1 implies now that

$$A_{1\text{-}234}(z) = A_{1\text{-}432}(z) = \exp\left(\frac{\sqrt{3}}{2}\int_0^z \frac{e^{t/2}\,dt}{\cos(\frac{\sqrt{3}}{2}t+\frac{\pi}{6})}\right).$$

Combined with the results of [11], Proposition 3.1 gives expressions for the EGFs $A_{1\text{-}\sigma}(z)$ where σ has one of the following forms:

$\sigma = 123\cdots k$,

$\sigma = k(k-1)\cdots 21$,

$\sigma = 12\cdots a\,\tau\,(a+1)$,

$\sigma = (a+1)\,\tau\,a(a-1)\cdots 21$,

$\sigma = k(k-1)\cdots(k+1-a)\,\tau'\,(k-a)$,

$\sigma = (k-a)\,\tau'\,(k+1-a)(k+2-a)\cdots k$,

where k, a are positive integers with $a \le k-2$, τ is any permutation of $\{a+2, a+3, \cdots, k\}$ and τ' is any permutation of $\{1, 2, \cdots, k-a-1\}$.

4. Asymptotic enumeration

Here we discuss the behavior of the numbers $\alpha_n(\sigma)$ as n goes to infinity, for a given generalized pattern σ. We use the symbol $\sim$ to indicate that two sequences of numbers have the same asymptotic behavior (i.e., we write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$), and we use the symbol $\ll$ to indicate that a sequence is asymptotically smaller than another one (i.e., we write $a_n \ll b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$).

Let us first consider the case of consecutive patterns.

Theorem 4.1. *Let $k \ge 3$ and let $\sigma \in \mathcal{S}_k$ be a consecutive pattern.*

(i) *There exist constants $0 < c, d < 1$ such that*

$$c^n n! < \alpha_n(\sigma) < d^n n!$$

for all $n \ge k$.

(ii) *There exists a constant* $0 < w \leq 1$ *such that*

$$\lim_{n\to\infty}\left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n} = w.$$

Note that c, d and w depend only on σ. Compare this result with the conjecture of Warlimont [21] that for any consecutive pattern σ there exist constants $\gamma > 0$ and $w < 1$ such that $\alpha_n(\sigma)/n! \sim \gamma\, w^n$.

Proof. The key observation is that, for any consecutive pattern σ,

$$\alpha_{m+n}(\sigma) \leq \alpha_m(\sigma)\alpha_n(\sigma)\binom{m+n}{n}. \tag{1}$$

To see this, just observe that a σ-avoiding permutation of length $m+n$ induces two juxtaposed σ-avoiding permutations of lengths m and n.

By induction on $n \geq k$ one gets

$$\alpha_{m+n}(\sigma) < d^m m!\, d^n n! \binom{m+n}{n} = d^{m+n}(m+n)!$$

for some positive $d < 1$.

For the lower bound, let $\tau = \rho(\sigma_1\sigma_2\sigma_3)$ be the reduction of the first three elements of σ. Clearly $\mathcal{S}_n(\tau) \subseteq \mathcal{S}_n(\sigma)$ for all n, since an occurrence of σ in a permutation produces also an occurrence of τ, hence $\alpha_n(\tau) \leq \alpha_n(\sigma)$. But the fact that $\sigma \in \mathcal{S}_3$ implies that $\alpha_n(\sigma)$ equals either $\alpha_n(123)$ or $\alpha_n(132)$. In any case, by Theorem 2.1 we have that

$$\alpha_n(\sigma) \geq \alpha_n(132) > c^n n!$$

for some $c > 0$.

To prove part (ii), we can express (1) as

$$\frac{\alpha_{m+n}(\sigma)}{(m+n)!} \leq \frac{\alpha_m(\sigma)}{m!}\frac{\alpha_n(\sigma)}{n!}$$

and apply *Fekete's lemma* (see [20, Lemma 11.6] or [12]) to the function $n!/\alpha_n(\sigma)$ to conclude that $\lim_{n\to\infty}\left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n}$ exists. Calling it w, then part (i) implies that $w \leq 1$ and $w \geq \lim_{n\to\infty}\left(\frac{\alpha_n(132)}{n!}\right)^{1/n} = 0.7839769$. □

In order to study the asymptotic behavior of $\alpha_n(\sigma)$ for a generalized pattern σ we separate the problem into the following three cases. Assume from now on that $k \geq 3$ and that σ is a generalized pattern of length k. We use the word *slot* to refer to the place between two adjacent elements of σ, where there can be a dash or not.

• **Case 1.** *The pattern* σ *has dashes between any two adjacent elements, i.e.,* $\sigma = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$.

These are just the classical patterns, which have been widely studied in the literature. The asymptotic behavior of the number of permutations avoiding them is given by the Stanley-Wilf conjecture, which has been recently proved by Marcus and Tardos [18], after several authors had given partial results over the last few years [1, 2, 5, 16].

Theorem 4.2 (*Stanley-Wilf conjecture*, proved in [18])**.** *For every classical pattern* $\sigma = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$, *there is a constant* λ *(which depends only on* σ*) such that*

$$\alpha_n(\sigma) < \lambda^n$$

for all $n \geq 1$.

On the other hand, it is clear that $\alpha_n(\sigma) \geq \alpha_n(\rho(\sigma_1\text{-}\sigma_2\text{-}\sigma_3)) = \mathbf{C}_n \sim \frac{1}{\sqrt{\pi n}}\,4^n$, where $\mathbf{C}_n$ denotes the n-th *Catalan number*. As shown by Arratia [2], Theorem 4.2 is equivalent to the statement that $\lim_{n\to\infty}\sqrt[n]{\alpha_n(\sigma)}$ exists. The value of this limit has been computed for several classical patterns: it is clearly 4 for patterns of length 3, it is known [19] to be $(k-1)^2$ for $\sigma = 1\text{-}2\text{-}\cdots\text{-}k$, it has been shown [4] to be 8 for $\sigma = 1\text{-}3\text{-}4\text{-}2$, and it has recently been proved by Bóna [6] to be nonrational for certain patterns.

• **Case 2.** *The pattern σ has two consecutive slots without a dash (equivalently, three consecutive elements without a dash between them), i.e.,* $\sigma = \cdots\sigma_i\sigma_{i+1}\sigma_{i+2}\cdots$.

Proposition 4.3. *Let σ be a generalized pattern having three consecutive elements without a dash. Then there exist constants $0 < c, d < 1$ such that*

$$c^n n! < \alpha_n(\sigma) < d^n n!$$

for all $n \geq k$.

Proof. For the upper bound, notice that if a permutation contains the consecutive pattern $\sigma_1\sigma_2\sigma_3\cdots\sigma_k$ obtained by removing all the dashes in σ, then it also contains σ. Therefore, $\alpha_n(\sigma) \leq \alpha_n(\sigma_1\sigma_2\sigma_3\cdots\sigma_k)$ for all n, and now the upper bound follows from part (i) of Theorem 4.1.

For the lower bound, we use that $\alpha_n(\sigma) \geq \alpha_n(\rho(\sigma_i\sigma_{i+1}\sigma_{i+2})) \geq \alpha_n(132) > c^n n!$, where $\sigma_i\sigma_{i+1}\sigma_{i+2}$ are three consecutive elements in σ without a dash. □

• **Case 3.** *The pattern σ has at least a slot without a dash, but not two consecutive slots without dashes.*

This case includes all the patterns not considered in Cases 1 and 2. The asymptotic behavior of $\alpha_n(\sigma)$ for these patterns is not known in general. The case of patterns of length 3 is covered by the following result, due to Claesson [7]. Let $\mathbf{B}_n$ denote the n-th *Bell number*, which counts the number of partitions of an n-element set.

Proposition 4.4 ([7])**.** *Let σ be a generalized pattern of length 3 with one dash.*

(i) *If* $\sigma \in \{1\text{-}23, 3\text{-}21, 32\text{-}1, 12\text{-}3, 1\text{-}32, 23\text{-}1, 3\text{-}12, 21\text{-}3\}$, *then* $\alpha_n(\sigma) = \mathbf{B}_n$.

(ii) *If* $\sigma \in \{2\text{-}13, 2\text{-}31, 31\text{-}2, 13\text{-}2\}$, *then* $\alpha_n(\sigma) = \mathbf{C}_n$.

It is known that the asymptotic behavior of the Catalan numbers is given by $\mathbf{C}_n \sim \frac{1}{\sqrt{\pi n}} 4^n$. For the Bell numbers, one has the formula

$$\mathbf{B}_n \sim \frac{1}{\sqrt{n}} \lambda(n)^{n+1/2} e^{\lambda(n)-n-1},$$

where $\lambda(n)$ is defined by $\lambda(n)\ln(\lambda(n)) = n$. Another useful description of the asymptotic behavior of $\mathbf{B}_n$ is the following:

$$\frac{\ln \mathbf{B}_n}{n} = \ln n - \ln\ln n + O\left(\frac{\ln\ln n}{\ln n}\right).$$

This shows in particular that $\delta^n \ll \mathbf{B}_n \ll c^n n!$ for any constants $\delta, c > 0$.

For patterns σ of length $k \geq 4$ that have slots without a dash, but not two consecutive slots without dashes, not much is known in general about the number of permutations avoiding them. It follows from Cases 1 and 2 that

$$\delta^n < \alpha(\sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k) \leq \alpha_n(\sigma) \leq \alpha(\sigma_1\sigma_2\cdots\sigma_k) < d^n n!$$

for some constants $\delta > 0$ and $d < 1$. Clearly, if σ contains one of the patterns in part (i) of Proposition 4.4, then the lower bound can be improved to $\mathbf{B}_n$. However, determining the precise asymptotic behavior of $\alpha_n(\sigma)$ seems to be a difficult problem. In the rest of the paper we discuss a few partial results in this direction.

The next statement is about permutations of the form 1-σ.

Corollary 4.5. *Let σ be a consecutive pattern, and let 1-σ be defined as in Proposition 3.1. Then,*

$$\lim_{n\to\infty}\left(\frac{\alpha_n(1\text{-}\sigma)}{n!}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n}.$$

Proof. By Proposition 3.1 we know that $A_{1\text{-}\sigma}(z) = \exp\left(\int_0^z A_\sigma(t)\,dt\right)$. Since the exponential is an analytic function over the whole complex plane, we obtain that $A_{1\text{-}\sigma}(z)$ has the same radius of convergence as $A_\sigma(z)$, from where the result follows. □

5. The pattern 12-34

The next proposition gives an upper and a lower bound for the numbers $\alpha_n(\text{12-34})$. Given two formal power series $F(z) = \sum_{n\geq 0} f_n z^n$ and $G(z) = \sum_{n\geq 0} g_n z^n$, we use the notation $F(z) < G(z)$ to indicate that $f_n < g_n$ for all n, and $F(z) \ll G(z)$ to indicate that $f_n \ll g_n$.

Proposition 5.1. *For $k \geq 1$, let*

$$\begin{aligned} h_k &= 1 + \frac{1}{2} + \cdots + \frac{1}{k}, \\ b_k(z) &= \sum_{i=0}^{k} \binom{k}{i}^2 \left[z + 2(h_{k-i} - h_i)\right] e^{iz}, \\ c_k(z) &= \frac{e^{(k+1)z}}{k+1} - \sum_{i=0}^{k} \binom{k}{i}\binom{k+1}{i} \left[z + 2(h_{k-i} - h_i) + \frac{1}{k+1-i}\right] e^{iz}, \\ S(z) &= \sum_{k\geq 1} b_k(z) + \sum_{k\geq 1} c_k(z). \end{aligned}$$

Then

$$e^{S(z)} < A_{\text{12-34}}(z) < e^{S(z)+e^z+z-1}.$$

If we write $e^{S(z)} = \sum l_n \frac{z^n}{n!}$ and $e^{S(z)+e^z+z-1} = \sum u_n \frac{z^n}{n!}$ to denote the coefficients of the series giving the lower and the upper bound respectively, then the graph in Figure 1 shows the values of $\sqrt[n]{\alpha_n(\text{12-34})/n!}$ for $n \leq 13$, bounded between the values $\sqrt[n]{l_n/n!}$ and $\sqrt[n]{u_n/n!}$ for $n \leq 120$. The two horizontal dotted lines are at height 0.7839769 and 0.8269933, which are $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)/n!}$ for $\sigma = 132$ and $\sigma = 123$ respectively, given by Theorem 2.1. From this plot it seems conceivable that $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\text{12-34})/n!} = 0$, although we have not succeeded in proving this.

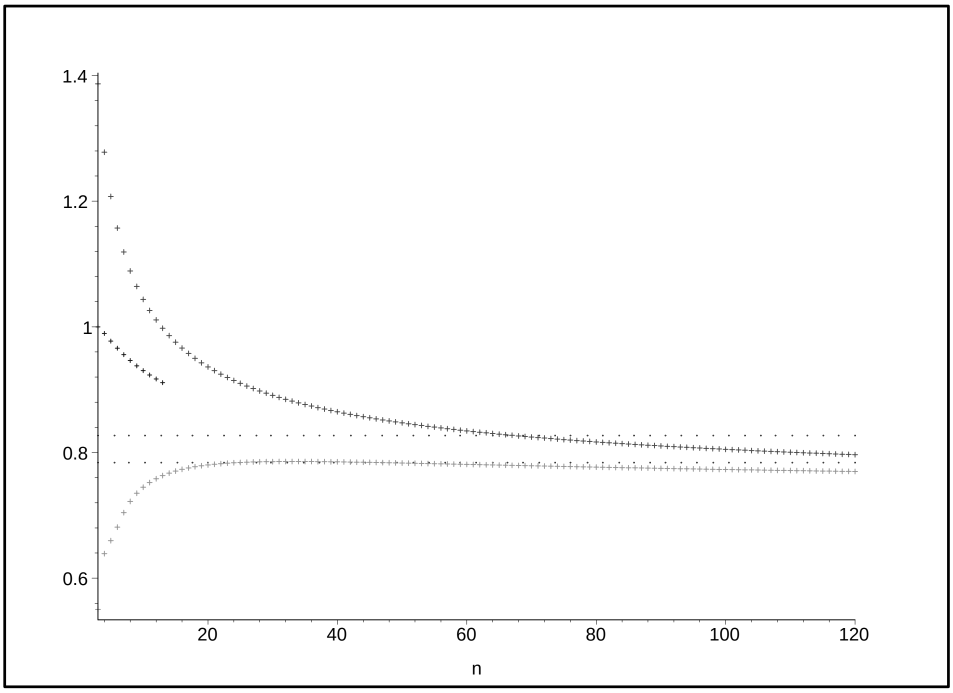

Figure 1. The first values of $\sqrt[n]{\alpha_n(\text{12-34})/n!}$ between the lower and the upper bound given by Proposition 5.1.

Note that the lower bound, together with the fact that $S(z) \gg e^z - 1$ (which follows from the definition), shows that $A_{12\text{-}34}(z) > e^{S(z)} \gg e^{e^z-1}$, which means that $\alpha_n(12\text{-}34) \gg \mathbf{B}_n$, that is, the number of 12-34-avoiding permutations is asymptotically larger than the Bell numbers.

Proof. Let π be a permutation that avoids 12-34. This means that it has no two ascents such that the second one starts at a higher value than where the first one ends. We can write $\pi = B_0 a_1 B_1 a_2 B_2 a_3 B_3 \cdots$, where a_1 and the element preceding it form the first ascent of π, a_2 and the element preceding it form the first ascent such that $a_2 < a_1$, a_3 and the element preceding it form the first ascent such that $a_3 < a_2$, and so on. By definition, B_0 is a non-empty decreasing word whose last element is less than a_1, and each B_i with $i \geq 1$ can be written uniquely as a sequence $B_i = w_{i,0} w_{i,1} w_{i,2} \cdots w_{i,r_i}$ for some $r_i \geq 1$ (r_i can be 0 if $w_{i,0}$ is nonempty) with the following properties:

(i) each $w_{i,j}$ is a decreasing word,
(ii) for $j \geq 1$, $w_{i,j}$ is nonempty and its first element is bigger than a_i,
(iii) the last element of each $w_{i,j}$ is less than a_i,
(iv) the last element of B_i is less than a_{i+1}.

These properties ensure that π avoids 12-34 (since no B_i has an ascent above a_i), and that the decomposition is unique.

Ideally we would like to use this decomposition to find a generating function for the numbers $\alpha_n(12\text{-}34)$. Unfortunately, the structure of the decomposition is a bit too complicated to find an exact formula. Instead, we will add and remove restrictions to simplify this description, which allows us to give lower and upper bounds respectively.

To find an upper bound, we will count permutations of the form $\pi = B_0 a_1 B_1 a_2 B_2 a_3 B_3 \cdots$, where the B_i and a_i satisfy the properties above, except for the requirement (iv) that the last element of each B_i has to be less than a_{i+1}. Omitting this requirement we are overcounting permutations, and thus we get an upper bound. The first step now is to find the EGF for a block K_i of the form $a_i B_i$, where B_i satisfies properties (i), (ii) and (iii) from above.

Let us first assume that $w_{i,0}$ is empty, that is, $B_i = w_{i,1} w_{i,2} \cdots w_{i,r_i}$. We compute the EGF for $K_i = a_i B_i$ where r_i is fixed, by induction on r_i. If $r_i = 0$, then we have that $K_i = a_i$, so the EGF is $b_0(z) := z$. If $r_i = 1$, then $K_i = a_i w_{i,1}$, where $w_{i,1}$ is a decreasing word starting above a_i and ending below it. The EFG for $w_{i,1}$ is e^z. Now, to incorporate the condition that the largest and the smallest labels of K_i lie in $w_{i,1}$, we use the double boxed product construction described in Section 2.2. A double derivative is needed to mark the two special elements. We get that the EGF for such a block is

$$\int_0^z \int_0^y t \left(\frac{d^2}{dt^2} e^t \right) dt\, dy = \int_0^z \int_0^y t e^t dt\, dy = (z-2)e^z + z + 2 = b_1(z).$$

Let now $r_i = 2$. The case in which both the largest and the smallest label of $K_i = a_i w_{i,1} w_{i,2}$ are contained in $w_{i,2}$ corresponds to the EGF

$$\int_0^z \int_0^y b_1(t) \left(\frac{d^2}{dt^2} e^t \right) dt\, dy. \tag{2}$$

If we write each $w_{i,j}$ as $w_{i,j}^+ w_{i,j}^-$, separating the elements above and below a_i ($w_{i,j}^+$ and $w_{i,j}^-$ respectively), then the largest element of K_i can be either in $w_{i,1}^+$ or in $w_{i,2}^+$, and the smallest element of K_i can be either in $w_{i,1}^-$ or in $w_{i,2}^-$. Thus, all the possibilities are obtained from the case counted by the EGF (2) by permuting the upper and lower parts of the $w_{i,1}$ and $w_{i,2}$ in the four possible different ways. It follows that the EGF for K_i when $r_i = 2$ is

$$4 \int_0^z \int_0^y b_1(t)\, e^t dt\, dy = (z-3)e^{2z} + 4ze^z + z + 3 = b_2(z).$$

In general, if $b_{k-1}(z)$ is the EGF for the case $r_i = k-1$, then the EGF for the case $r_i = k$ is given by

$$b_k(z) = k^2 \int_0^z \int_0^y b_{k-1}(t)\, e^t dt\, dy.$$

It is straightforward to check that the functions $b_k(z)$ defined in the statement of the proposition satisfy this recurrence.

The case where $w_{i,0}$ is nonempty can be treated similarly. Now we have $B_i = w_{0,i}w_{i,1}w_{i,2}\cdots w_{i,r_i}$. If $r_i = 0$, the EGF for $a_i w_{0,i}$ is $c_0(z) := e^z - 1 - z$ (since the block has at least 2 elements). If $r_i = 1$, then a block of the form $a_i w_{0,i} w_{i,1}$ can be obtained from the case where the largest and the smallest element are in $w_{i,1}$ by permuting $w_{0,i}$ and $w^-_{i,1}$ if necessary. This yields the EGF

$$2\int_0^z \int_0^y c_0(t)\left(\frac{d^2}{dt^2}e^t\right) dt\,dy = \frac{e^{2z}}{2} + 2(1-z)e^z - z - \frac{5}{2} = c_1(z).$$

In general, for nonempty $w_{i,0}$, if $c_{k-1}(z)$ is the EGF for the case $r_i = k-1$, then the EGF for the case $r_i = k$ is given by

$$c_k(z) = k(k+1)\int_0^z \int_0^y c_{k-1}(t)\,e^t dt\,dy.$$

This is the recurrence satisfied by the functions $c_k(z)$ defined in the statement of the proposition.

The generating function for a set of blocks $K_i = a_i B_i$ of the form just described is

$$\exp\left(\sum_{k\geq 0} b_k(z) + \sum_{k\geq 0} c_k(z)\right) = \exp(S(z) + z + e^z - 1 - z).$$

From such a set there is a unique way to form a sequence $a_1B_1a_2B_2a_3B_3\cdots$ where $a_1 > a_2 > a_3 > \cdots$. Finally, we multiply by e^z to take into account the initial decreasing segment B_0 of the permutation $\pi = B_0a_1B_1a_2B_2a_3B_3\cdots$, again relaxing the condition that its last element should be smaller than a_1. This gives the upper bound $e^z\exp(S(z) + e^z - 1) = \exp(S(z) + e^z + z - 1)$.

Now we use a similar reasoning to obtain a lower bound. We have seen that $b_k(z)$ counts blocks of the form $a_i w_{i,1}w_{i,2}\cdots w_{i,k}$, where each $w_{i,j}$ is a decreasing word starting above a_i and ending below it. If $k \geq 1$, using the notation $w_{i,k} = w^+_{i,k}w^-_{i,k}$ to separate the elements that are bigger than a_i from those that are smaller, we can move the last part of the block to the beginning and write $L_i := w^-_{i,k}a_iw_{i,1}w_{i,2}\cdots w^+_{i,k}$. Similarly, a block of the form $a_iw_{i,0}w_{i,1}w_{i,2}\cdots w_{i,k}$ like the ones counted by $c_k(z)$ with $k \geq 1$ can be reordered as $L'_i := w^-_{i,k}a_iw_{i,0}w_{i,1}w_{i,2}\cdots w^+_{i,k}$. The EGF that counts sets of pieces of the forms given by L_i and L'_i is

$$\exp\left(\sum_{k\geq 1} b_k(z) + \sum_{k\geq 1} c_k(z)\right) = \exp(S(z)).$$

Ordering the pieces of such a set by decreasing order of the a_i, the sequence that they form by juxtaposition is a 12-34-avoiding permutation. Besides, no such permutation is obtained in more than one way by this construction. However, notice that not every 12-34-avoiding permutation is produced by this process, hence this construction gives only a lower bound. □

The decomposition of 12-34-avoiding permutations given in the proof of Proposition 5.1 can be generalized to permutations avoiding a pattern of the form 12-σ. If $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in \mathcal{S}_k$ is a consecutive pattern, 12-σ denotes the generalized pattern 12-$(\sigma_1+2)(\sigma_2+2)\cdots(\sigma_k+2)$.

Any permutation π that avoids 12-σ can be uniquely decomposed as $\pi = B_0a_1B_1a_2B_2a_3B_3\cdots$, where a_1 and the element preceding it form the first ascent of π, a_2 and the element preceding it form the first ascent such that $a_2 < a_1$, a_3 and the element preceding it form the first ascent such that $a_3 < a_2$, and so on. Then, by definition, B_0 is a non-empty decreasing word whose last element is less than a_1, and each B_i with $i \geq 1$ can be written uniquely as a sequence $B_i = w_{i,0}U_{i,1}w_{i,1}U_{i,2}w_{i,2}\cdots U_{i,r_i}w_{i,r_i}$ for some $r_i \geq 1$ (r_i can be 0 if $w_{i,0}$ is nonempty) with the following properties:

(i) each $w_{i,j}$ is a decreasing word all of whose elements are less than a_i,
(ii) each $U_{i,j}$ is a nonempty permutation avoiding σ, all of whose elements are greater than a_i,
(iii) $w_{i,j}$ is nonempty for $j \geq 1$,
(iv) the last element of B_i is less than a_{i+1}.

From this decomposition the following result follows immediately.

Proposition 5.2. *If σ, τ are two consecutive patterns satisfying $A_\sigma(z) = A_\tau(z)$, then $A_{12\text{-}\sigma}(z) = A_{12\text{-}\tau}(z)$.*

The structure of 21-σ-avoiding permutations (defined analogously) can be described using the same ideas, and it is not hard to see that the following result holds as well.

Proposition 5.3. *If σ is a consecutive pattern, then $A_{12\text{-}\sigma}(z) = A_{21\text{-}\sigma}(z)$.*

6. The pattern 1-23-4

Similarly to what we did for the pattern 12-34, analyzing the structure of permutations avoiding 1-23-4 we can give lower and upper bounds for the numbers $\alpha_n(\text{1-23-4})$. Let $\mathbf{C}^{\exp}(z) := \sum_{n\geq 0} \mathbf{C}_n \frac{z^n}{n!}$ be the EGF for the Catalan numbers.

Proposition 6.1. *We have that*

$$\frac{1}{2}\int_0^z e^{2e^y-2}\,dy - \frac{z}{2} < A_{\text{1-23-4}}(z) < \mathbf{C}^{\exp}(e^z-1).$$

Writing $\frac{1}{2}\int_0^z e^{2e^y-2}\,dy - \frac{z}{2} = \sum l_n \frac{z^n}{n!}$ and $\mathbf{C}^{\exp}(e^z-1) = \sum u_n \frac{z^n}{n!}$ to denote the coefficients of the series giving the lower and the upper bound respectively, then the values of $\sqrt[n]{l_n/n!}$ and $\sqrt[n]{u_n/n!}$ for $n \leq 90$ are plotted in Figure 2, bounding the values of $\sqrt[n]{\alpha_n(\text{1-23-4})/n!}$ for $n \leq 11$.

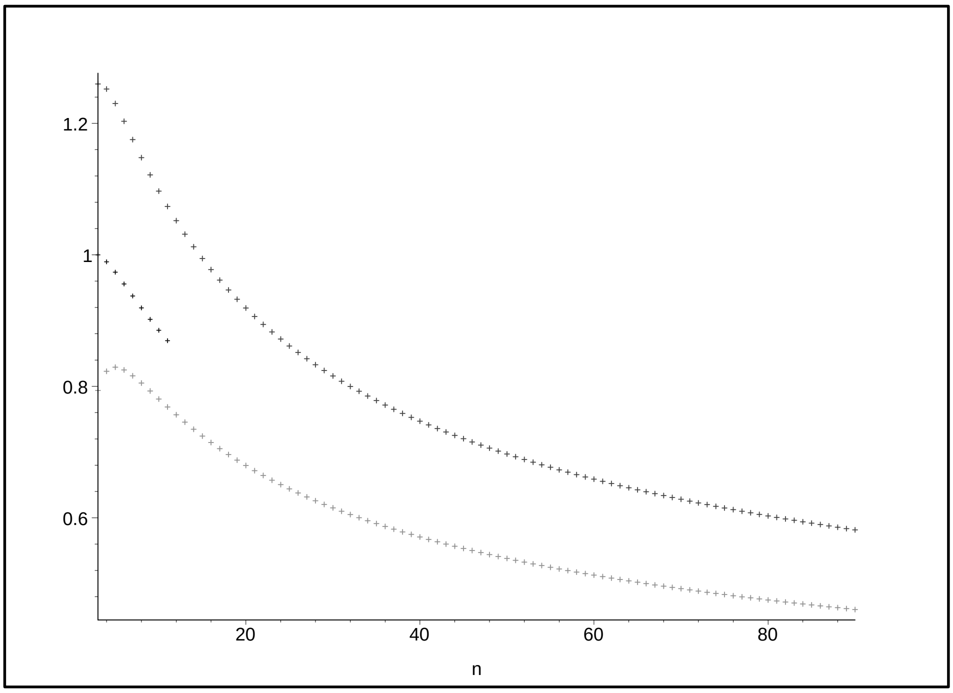

Figure 2. The first values of $\sqrt[n]{\alpha_n(\text{1-23-4})/n!}$ between the lower and the upper bound given by Proposition 6.1.

Note that the lower bound implies that $\alpha_n(\text{1-23-4}) \gg \mathbf{B}_n$, since $e^{2e^z-2} \gg e^{e^z-1}$.

Proof. Let π be a permutation that avoids 1-23-4. Let $a_1 > a_2 > a_3 > \cdots > a_r$ be the left-to-right minima of π, and let $b_1 > b_2 > b_3 > \cdots > b_s$ be its right-to-left maxima (recall that π_i is a right-to-left maximum of π if $\pi_j < \pi_i$ for all $j > i$). Then, marking the positions of the left-to-right minima and right-to-left maxima, we can write $\pi = c_1 w_1 c_2 w_2 \cdots c_{r+s-1} w_{r+s-1} c_{r+s}$, where $c_i \in \{a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s\}$ for all i (in fact the number of c_i's could be less than $r+s$ if some element is simultaneously a left-to-right minimum and a right-to-left maximum). Note that $c_1 = a_1$ and $c_{r+s} = b_s$. Now, the condition that π

avoids 1-23-4 is equivalent to the fact that each w_i is a (possibly empty) decreasing word. Indeed, if there was an ascent inside one of the w_i, then together with the closest left-to-right minimum to the left of w_i and the closest right-to-left maximum to the right of w_i, it would form an occurrence of 1-23-4. On the other hand, it is clear that if all w_i are decreasing, then no such occurrence can exist.

We use this decomposition to obtain upper and lower bounds for $\alpha_n(\text{1-23-4})$. Let us first show the lower bound. For that we count only a special type of 1-23-4-avoiding permutations, namely the ones where all the left-to-right minima come before all the right-to-left maxima. Such a π can be written as $\pi = a_1w_1a_2w_2\cdots a_rw_rb_1w_{r+1}b_2w_{r+2}\cdots w_{r+s-1}b_s$, where for $1 \le i \le r$ the elements of the decreasing words w_i have values between a_i and b_1, and for $r \le i \le r+s-1$ the elements of w_i have values between a_r and b_{i+1}. The EGF for the part $a_1w_1a_2w_2\cdots a_{r-1}w_{r-1}$ is e^{e^z-1}, since it is an arbitrary 1-23-avoiding permutation (see the example following Proposition 3.1). Similarly, the EGF for the part $w_{r+1}b_2w_{r+2}\cdots w_{r+s-1}b_s$ is also e^{e^z-1} (it can be viewed as a set of blocks of the form $w_{r+i}b_{i+1}$, each one contributing e^z-1, arranged by decreasing order of the b_i's). The decreasing word w_r contributes e^z. Now, to get the EGF for the whole permutation $a_1w_1a_2w_2\cdots a_rw_rb_1w_{r+1}b_2w_{r+2}\cdots w_{r+s-1}b_s$ we use the boxed product construction to require that the biggest element of the block is b_1 and the smallest one is a_r. The EGF that we obtain is

$$\int_0^z \int_0^y e^{e^t-1}\left(\frac{d}{dt}t\right) e^t \left(\frac{d}{dt}t\right) e^{e^t-1}\,dt\,dy = \frac{1}{2}\int_0^z (e^{2e^y-2}-1)\,dy,$$

which gives a lower bound for the coefficients of $A_{\text{1-23-4}}(z)$.

To find the upper bound, consider first permutations of the form $\pi = c_1w_1c_2w_2\cdots c_{r+s-1}w_{r+s-1}c_{r+s}$ where all the w_i are empty. Such permutations, where every element is either a left-to-right minimum or a right-to-left maximum, are precisely those avoiding 1-2-3, which are counted by the Catalan numbers. Thus, the EGF for such permutations is $\mathbf{C}^{\exp}(z)$.

The next step is to insert a decreasing word w_i after each c_i. If c_i is a left-to-right minimum, we require that the elements of w_i are bigger than c_i, so the EGF for the block c_iw_i is e^z-1. We omit the requirement that the elements of w_i have to be smaller than the nearest right-to-left maximum to the right of w_i; this is why we only get an upper bound. Similarly, if c_j is a right-to-left maximum, we require that the elements of w_j are smaller than c_j, so the EGF for the block c_jw_j is also e^z-1. We also omit the requirement that after the last right-to-left maximum there is no decreasing word. Replacing each c_i for a block c_iw_i as just described translates in terms of generating functions into substituting e^z-1 for the variable z in $\mathbf{C}^{\exp}(z)$. This gives the stated upper bound. □

The upper bound given in the above proposition yields the following corollary.

Corollary 6.2. *We have that*

$$\lim_{n\to\infty}\left(\frac{\alpha_n(\text{1-23-4})}{n!}\right)^{1/n} = 0.$$

Proof. The power series $\mathbf{C}^{\exp}(z)$ can be bounded by

$$\mathbf{C}^{\exp}(z) < \sum_{n\ge 0} 4^n \frac{z^n}{n!} = e^{4z},$$

which converges for all z. Therefore, so does $\mathbf{C}^{\exp}(e^z-1)$, which is an upper bound for $A_{\text{1-23-4}}(z)$. The result follows now from the fact that if $\sum_n f_nz^n$ is an analytic function in the whole complex plane, then $\lim_{n\to\infty}\sqrt[n]{f_n}=0$ (see [13, Chapter 4] for a discussion). □

If $\sigma = \sigma_1\sigma_2\cdots\sigma_{k-2}\in\mathcal{S}_{k-2}$ is a consecutive pattern, let 1-σ-k denote the generalized pattern 1-$(\sigma_1+1)(\sigma_2+1)\cdots(\sigma_{k-2}+1)$-$k$. The decomposition of 1-23-4-avoiding permutations given in the proof of the above proposition can be generalized to permutations avoiding any pattern of the form 1-σ-k.

Any permutation π that avoids 1-σ-k can be uniquely decomposed as $\pi = c_1w_1c_2w_2\cdots c_{m-1}w_{m-1}c_m$, where the c_i are all the left-to-right minima and right-to-left maxima of π, and each w_i is a permutation that avoids σ, all of whose elements are bigger than the closest left-to-right minimum to its left and smaller than the closest right-to-left maximum to its right.

Using exactly the same reasoning as in the proof of Proposition 6.1, we obtain the following lower and upper bounds for the numbers $\alpha_n(1\text{-}\sigma\text{-}k)$.

Proposition 6.3. *Let $\sigma \in \mathcal{S}_{k-2}$ be a consecutive pattern, and let $1\text{-}\sigma\text{-}k$ be defined as above. Then,*

$$\int_0^z \int_0^u e^{2\int_0^y A_\sigma(t)dt+y}\, dy\, du < A_{1\text{-}\sigma\text{-}k}(z) < \mathbf{C}^{\exp}\left(\int_0^z A_\sigma(t)\, dt\right).$$

Corollary 6.4. *With the same definitions as in the above proposition,*

$$\lim_{n\to\infty}\left(\frac{\alpha_n(1\text{-}\sigma\text{-}k)}{n!}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n}.$$

Proof. The upper and lower bounds for $A_{1\text{-}\sigma\text{-}k}(z)$ given in Proposition 6.3 are analytic functions of $A_\sigma(z)$, since essentially they only involve exponentials and integrals. Therefore, $A_{1\text{-}\sigma\text{-}k}(z)$ and $A_\sigma(z)$ have the same radius of convergence, hence the limits above coincide. $\square$

Finally, the following proposition is an immediate consequence of the structure of $1\text{-}\sigma\text{-}k$-avoiding permutations discussed above. In particular, it implies that $A_{1\text{-}23\text{-}4}(z) = A_{1\text{-}32\text{-}4}(z)$.

Proposition 6.5. *If σ, τ are two consecutive patterns in $\mathcal{S}_{k-2}$ satisfying $A_\sigma(z) = A_\tau(z)$, then $A_{1\text{-}\sigma\text{-}k}(z) = A_{1\text{-}\tau\text{-}k}(z)$.*

7. Other patterns

In Section 6 we have proved that $\alpha_n(1\text{-}23\text{-}4) \gg \mathbf{B}_n$ and that $\alpha_n(1\text{-}23\text{-}4) \ll c^n n!$ for any constant $c > 0$. For the pattern 12-34, we showed in Section 5 that the analogue to the first statement holds as well, and the second one seems to be true from numerical computations. It remains as an open problem to describe precisely the asymptotic behavior of $\alpha_n(\sigma)$ for these two patterns, and for several remaining generalized patterns of length 4.

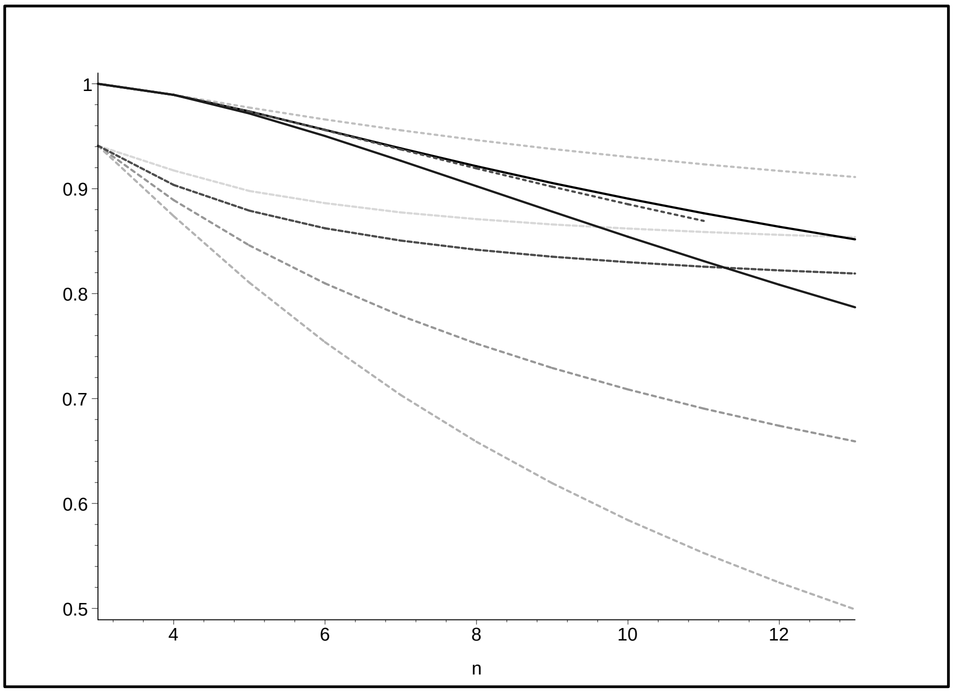

FIGURE 3. The first values of $\sqrt[n]{\alpha_n(\sigma)/n!}$ for several generalized patterns σ.

In Figure 3 we have plotted the initial values (connected by lines) of the sequences $\sqrt[n]{\alpha_n(\sigma)/n!}$ for other cases that appear to have some interest. The two dotted lines at the bottom of the graph correspond to the sequences $\sqrt[n]{\mathbf{C}_n/n!}$ and $\sqrt[n]{\mathbf{B}_n/n!}$, which are known to tend to 0 as n goes to infinity. The two dashed lines that start at the same point (around 0.941) and tend to a constant correspond to the sequences $\sqrt[n]{\alpha_n(132)/n!}$ and $\sqrt[n]{\alpha_n(123)/n!}$, for which their limits are known by Theorem 2.1 to be 0.7839769 and 0.8269933 respectively. Among the lines starting at 1, the two dotted ones correspond to the patterns 1-23-4 (the lower line) and 12-34 (the upper line) discussed in the previous sections.

Of the two solid lines, the one below corresponds to the pattern 3-14-2. This pattern has a special interest because all of its subpatterns of length 3 are among those in part (ii) of Proposition 4.4. Since it does not contain any of the patterns in part (i), we cannot say that $\alpha_n(\text{3-14-2}) \geq \mathbf{B}_n$ for all n. In fact, comparing the slopes in Figure 3 it seems quite plausible that $\alpha_n(\text{3-14-2})$ grows more slowly than $\mathbf{B}_n$, and proving this is an interesting open question. The other solid line in the plot corresponds to the pattern 13-24, for which we do not know the asymptotic behavior either.

This paper is the first attempt to study the asymptotic behavior of the numbers $\alpha_n(\sigma)$ where σ is an arbitrary generalized pattern. Despite the fact that we have been unable to provide a precise description of this behavior in most cases, we hope that our work shows the intricateness of the problem and the amount of questions that it opens. The main goal of further research in this direction would be to give a complete classification of all generalized patterns according to the asymptotic behavior of $\alpha_n(\sigma)$ as n goes to infinity.

Another interesting open problem is to find the value of $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)/n!}$ for patterns σ in Case 2, for which this limit is known to be a constant. The analogous problem for patterns in Case 1, namely finding $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)}$ for classical patterns σ, is a current direction of research as it remains open for most patterns as well (see [4, 6]).

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