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Old and young leaves on plane trees

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Abstract

A leaf of a plane tree is called an old leaf if it is the leftmost child of its parent, and it is called a young leaf otherwise. In this paper we enumerate plane trees with given numbers of old leaves and young leaves. The formula is obtained combinatorially via two bijections between plane trees and 2-Motzkin paths which map young leaves to red horizontal steps, and old leaves to up steps. We derive some implications for the enumeration of restricted permutations with respect to certain statistics such as pairs of consecutive deficiencies, double descents, and ascending runs. Finally, our main bijection is applied to obtain refinements of two identities of Coker, involving refined Narayana numbers and the Catalan numbers.

1. Introduction

Plane trees, also referred to as ordered trees, are basic objects frequently used in combinatorics. Many enumerative results about them appear throughout the literature. For example, a well-known interpretation of the Narayana numbers is that they count the number of plane trees with a fixed number of leaves. In this paper we classify the leaves of a plane tree into two different kinds, distinguishing between old leaves and young leaves.

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This definition, which is introduced in Section 2, naturally gives rise to a refinement of the Narayana numbers.

These refined Narayana numbers also appear in the enumeration of 2-Motzkin paths with respect to the numbers of up steps and red horizontal steps. Such paths were introduced in [1], and their structure has proved to be useful in the study of lattice paths, noncrossing partitions, plane trees [6], and other combinatorial objects and identities. Our paper gives yet another example of the applicability of 2-Motzkin paths. The key to several of our results is a new bijection between plane trees and 2-Motzkin paths, with very convenient properties. It provides a combinatorial derivation of the expression for the number of plane trees with given numbers of old and young leaves.

Partly motivated by our work, Chen, Yan and Yang [4] give combinatorial interpretations of two identities involving the Narayana numbers and Catalan numbers, due to Coker [5]. While the proof in [4] uses a different bijection, the authors note that our bijection provides a combinatorial proof as well. Here we will show that a more detailed analysis of the bijection and its properties gives refinements of the two identities of Coker, as well as bijective proofs of these refinements.

The paper is structured as follows. In Section 2 we review some definitions and notation about plane trees, Dyck paths, Motzkin paths, and 2-Motzkin paths. We also introduce the concepts of old leaves and young leaves of a plane tree. In Section 3 we give the generating function for plane trees with variables marking the number of old leaves and the number of young leaves, as well as exact formulas for the number of plane trees of a given size when the numbers of old and young leaves are fixed. In Section 4 we present two bijections from the set of plane trees with n edges to the set of 2-Motzkin paths of length $n-1$. Some interesting properties of these bijections are studied in Section 5. We show that they map old and young leaves of trees into statistics on 2-Motzkin paths that are easier to deal with. In Section 6 we describe some bijections between plane trees and permutations avoiding patterns of length 3, and investigate what old and young leaves are mapped to by these bijections. This implies that the distributions of certain parameters on restricted permutations are given by the same formulas enumerating plane trees with respect to old and young leaves. Finally, in Section 7 we apply our bijection to obtain refinements of two combinatorial identities due to Coker [5] and proved combinatorially by Chen, Yan and Yang [4].

2. Preliminaries

2.1. Plane trees

A *plane tree* T can be defined recursively (see for example [11, Appendix]) as a finite set of vertices such that one distinguished vertex r is called the *root* of T, and the remaining vertices are put into an ordered partition $(T_1, T_2, \ldots, T_m)$ of m disjoint non-empty sets, each of which is a plane tree. We will draw plane trees with the root on the top level, with edges connecting it to the roots of $T_1, T_2, \ldots, T_m$, which will be drawn from left to right on the second level. For each vertex v, the nodes in the next lower level connected to v by an edge are called the *children* or *successors* of v, and v is called the *parent* of its children. Clearly each vertex other than r has exactly one parent. A vertex of T is called a *leaf* if it has no children (by convention, we assume that the *empty* tree, formed by a single node, has no leaves).

We denote by $\mathcal{T}_n$ the set of (unlabeled) plane trees with n edges. It is well known that $|\mathcal{T}_n| = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number, and that the number of trees with n edges and k leaves is the Narayana number $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

We classify the leaves of a plane tree into old and young leaves. We say that a leaf is an *old leaf* if it is the leftmost child of its parent, and that it is a *young leaf* otherwise. For example, the tree in Fig. 1 has four young leaves, drawn with black filled circles, and three old leaves, drawn with empty circles. The enumeration of plane trees with respect to the numbers of old and young leaves is done in Section 3.

2.2. Lattice paths

We review the definitions of Dyck, Motzkin, and 2-Motzkin paths. They are all lattice paths in $\mathbb{Z}^2$ starting at $(0, 0)$, ending on the x-axis, and never going below this axis. A *Dyck path* consists of steps $U = (1, 1)$ and $D = (1, -1)$. In a *Motzkin path* we also allow horizontal steps $H = (1, 0)$, so that the path is a sequence of steps U, D and H. A *2-Motzkin path* consists of up and down steps, and horizontal steps that can be colored either red or blue. We use R to denote a red step, and B a blue step. In the pictures in this paper, red steps will be drawn with a dashed line to make them distinguishable from bluef steps, which will be drawn with a solid line. The length of any of these paths is the total number of steps.

We shall denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$, by $\mathcal{M}_n$ the set of Motzkin paths of length n, and by $\mathcal{N}_n$ the set of 2-Motzkin paths of length n. The number of paths of each kind is given by $|\mathcal{D}_n| = C_n$, $|\mathcal{M}_n| = M_n$, and $|\mathcal{N}_n| = C_{n+1}$, where $M_n = \sum_{k=0}^{n}\binom{n}{2k} C_k$ is the n-th Motzkin number.

The generating function for Catalan numbers is $C(z) = \sum_{n\geq 0} C_n z^n = \frac{1-\sqrt{1-4z}}{2z}$, and that for Motzkin numbers is $M(z) = \sum_{n\geq 0} M_n z^n = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$.

3. Enumeration of trees with respect to old and young leaves

Here we give an expression for the generating function:

$$G(t, s, z) = \sum_{T} t^{\#\text{old leaves of } T} s^{\#\text{young leaves of } T} z^{\#\text{edges of } T},$$

where the sum is over all plane trees T.

Theorem 1. *Let $G(t, s, z)$ be defined as above. We have*

$$G(t, s, z) = \frac{1 + z - sz - \sqrt{1 - 2(1 + s)z + (1 - 4t + 2s + s^2)z^2}}{2z}.$$

Proof. We will find an equation for $G = G(t, s, z)$ using a decomposition of plane trees. Let T be any plane tree, and let m be the number of children of the root. If $m = 0$, then the tree has no edges, and its contribution to the generating function G is 1. If $m \geq 1$, let

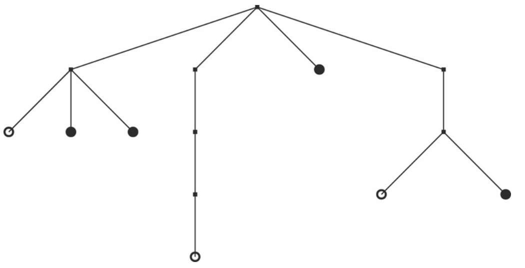

Fig. 1. A tree with three old leaves and four young leaves.

$T_1, T_2, \ldots, T_m$ be the sequence of plane trees hanging from left to right from the children of the root. If T_1 has no edges, then it creates an old leaf of T; otherwise all the old (resp. young) leaves of T_1 become old (resp. young) leaves of T. Therefore, the contribution to the generating function of T_1 and the edge connecting it to the root is $z(G-1+t)$. For $i \geq 2$, old and young leaves of T_i become leaves of T of the same kind as well. However, if T_i has no edges, then an additional young leaf of T is created. Thus, the contribution to the generating function of each T_i with $i \geq 2$ and the edge connecting it to the root is $z(G-1+s)$. It follows that for $m \geq 1$, the contribution of the plane trees whose root has degree m is $z^m(G-1+t)(G-1+s)^{m-1}$. Summing over all $m \geq 0$ we obtain

$$G = 1 + \frac{z(G-1+t)}{1-z(G-1+s)}. \tag{1}$$

Isolating G, the formula follows. □

Proposition 2. (1) *The number of plane trees with n edges, i old leaves, and j young leaves is*

$$\frac{1}{n}\binom{n}{i}\binom{n-i}{j}\binom{n-i-j}{i-1}.$$

(2) *The number of plane trees with n edges and k old leaves is*

$$\frac{2^{n-2k+1}}{k}\binom{n-1}{2k-2}\binom{2k-2}{k-1}.$$

(3) *The number of plane trees with n edges and k young leaves is*

$$\binom{n-1}{k}M_{n-k-1}.$$

First proof. If we let $G_0 = G(t,s,z) - 1$, Eq. (1) can be written as $G_0 = z[G_0^2 + (s+1)G_0 + t]$. Applying the Lagrange inversion formula, we obtain that, for $n > 0$, the coefficient of z^n in $G(t,s,z)$ is $\frac{1}{n}[x^{n-1}](x^2 + (s+1)x + t)^n$, where $[x^{n-1}]$ denotes the coefficient of x^{n-1}. It follows that the coefficient of $t^i s^j z^n$ in $G(t,s,z)$ is $\frac{1}{n}\binom{n}{i}\binom{n-i}{j}\binom{n-i-j}{i-1}$, which is the first expression. For the other two expressions, apply

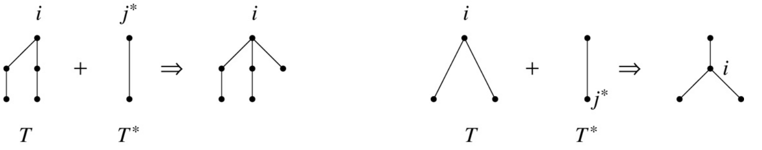

Fig. 2. Horizontal merge and vertical merge.

the Lagrange inversion to the same equation, after making the substitutions $s = 1$ and $t = 1$ respectively. $\square$

Second proof. We can give a bijective proof of the first part of Proposition 2 as follows. In [3], the author gives a bijective algorithm to decompose any labeled plane tree with n edges (where the set of vertex labels is $\{1, 2, \ldots, n+1\}$) into a set F of n matches with labels $\{1, \ldots, n, n+1, (n+2)^*, \ldots, (2n)^*\}$, where a match is a rooted tree with two vertices. The reverse procedure of the decomposition algorithm is the following merging algorithm. We start with a set F of matches on $\{1, \ldots, n+1, (n+2)^*, \ldots, (2n)^*\}$. A vertex labeled with a mark $*$ is called a marked vertex.

(1) Find the tree T with the smallest root in which no vertex is marked. Let i be the root of T.
(2) Find the tree T^* in F that contains the smallest marked vertex. Let j^* be this marked vertex.
(3) If j^* is the root of T^*, then merge T and T^* by identifying i and j^*, keep i as the new vertex, and place the subtrees of T^* to the right of T. The operation is called a *horizontal merge*. If j^* is a leaf of T^*, then replace j^* with T in T^*. This operation is called a *vertical merge*. See Fig. 2.
(4) Repeat the above procedure until F becomes a labeled tree.

For any labeled plane tree with n edges, i old leaves, and j young leaves, the corresponding set F of n matches consists of i matches without marked vertices, j matches with marked roots and unmarked leaves, and all leaves in the remaining matches are marked vertices. Thus, we can count the number of labeled plane trees with n edges, i old leaves, and j young leaves as follows:

$$\begin{aligned}&\binom{n+1}{2i}\frac{(2i)!}{i!}\binom{n+1-2i}{j}\binom{n-1}{j}j!\binom{n-1-j}{n-i-j}(n-i-j)!\\&\quad=\frac{(n+1)!}{n}\binom{n}{i}\binom{n-i}{j}\binom{n-i-j}{i-1}.\end{aligned}$$

Now, to count unlabeled plane trees we just divide by $(n+1)!$, obtaining the desired formula.

Summing for all j we obtain the formula in part (2) of the proposition, and summing for all i we derive the third formula. $\square$

Particular cases of this proposition give rise to the following two statements. The second one appeared already in [7] as a manifestation of the Motzkin numbers.

Corollary 3. (1) *The number of plane trees in $\mathcal{T}_n$ with exactly one old leaf is* 2^{n-1}.
(2) *The number of plane trees in $\mathcal{T}_n$ with no young leaves is* M_{n-1}.

4. Two bijections between plane trees and 2-Motzkin paths

In this section we present two bijections Ψ and Ψ' between the set of plane trees with n edges and the set of 2-Motzkin paths of length $n-1$. These bijections have the convenient property that they map old and young leaves of the tree to certain statistics of the 2-Motzkin path that are very easy to deal with, as shown in the next section. This will allow us to give bijective proofs of Corollary 3 and some parts of Proposition 2. The two bijections have very similar properties, and in fact one of them would be enough to prove the results in the next section. However, they are defined in quite different ways, and we feel that presenting both bijections gives a better insight into how old and young leaves correspond to statistics on paths.

Let us start by describing the bijection Ψ. It consists of three steps. Given a plane tree $T \in \mathcal{T}_n$ (assume $n \geq 1$), we first transform it into a Dyck path using the following well-known bijection, which we denote as θ. Starting from the root, traverse the edges of the tree in preorder from right to left. To each edge passed on the way down there corresponds a step U, and to each edge passed on the way up there corresponds a step D. This gives us a Dyck path $\theta(T)$ of length $2n$.

The next step is to replace each peak UD of the path followed by a U step with a red horizontal step R. That is, we traverse the path $\theta(T)$ from left to right replacing each UDU with RU. This gives us a Motzkin path with steps U, D and R, whose length is variable.

Finally, we need to transform this Motzkin path into a 2-Motzkin path $\Psi(T)$ of length $n-1$. The bijection that we will use for this purpose is essentially the same one as was described by Callan [2] between UDU-free Dyck paths and Motzkin paths, where we "ignore" the steps R of our path and let the new level steps all be B steps. Notice that after the transformation in the previous paragraph, every peak UD in our Motzkin path is followed by a D step, unless it is at the end of the path. This last transformation is done as follows. Place a mark on each U that is followed by a D, on each D that is followed by another D, and on the D at the end of the path. Next, change each unmarked U whose matching D is marked into an B. (The matching D is the step that is encountered directly east of U.) Lastly, delete all the marked steps.

After this procedure we obtain a 2-Motzkin path $\Psi(T)$ with $n-1$ steps. For example, for the tree T in Fig. 1, applying the first part of the bijection we get the Dyck path in Fig. 3. Replacing each UDU with RU, we get the Motzkin path in Fig. 4. In the third part of the bijection, we mark the steps that are thicker in Fig. 5. Changing each unmarked U with a marked matching D to a B, we get $UBR\dot{U}\dot{D}\dot{D}DRUBB\dot{U}\dot{D}\dot{D}\dot{D}DBRR\dot{U}\dot{D}\dot{D}$, where the dots indicate the marked steps. Finally, deleting the marked steps, we obtain the 2-Motzkin path in Fig. 6.

It is clear that the first two steps of this map are reversible, that is, from the Motzkin path with steps U, D and R it is easy to recover the tree. The fact that the last step is a bijection as well follows from the description of the inverse given in [2]. The only difference here is that we need to disregard the steps R that we have now in the path, since they are not affected by this part of the bijection.

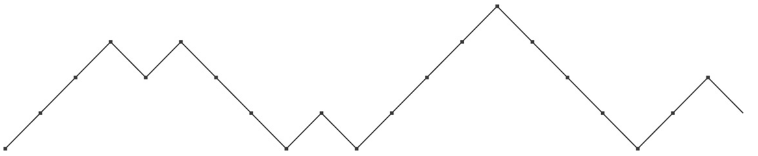

Fig. 3. The Dyck path $\theta(T)$ for T in Fig. 1.

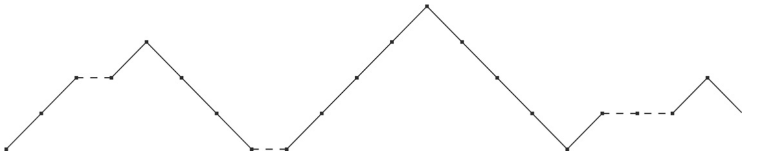

Fig. 4. The Motzkin path $UURUDDDRUUUUDDDDURRUDD$.

Fig. 5. The Motzkin path with some steps marked.

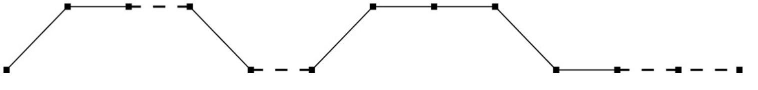

Fig. 6. The 2-Motzkin path $\Psi(T) = UBRDRUBBDBRR$.

Now we describe another bijection Ψ' between $\mathcal{T}_n$ and the set of 2-Motzkin paths of length $n-1$. We can construct Ψ' recursively. Given a plane tree T, consider the decomposition given in Fig. 7, where $(e_1, e_2, \ldots, e_k)$ is the path obtained starting at the root and successively descending to the rightmost child until we reach a leaf. $T_1, T_2, \ldots, T_k$ are possibly empty subtrees hanging from the vertices of this path. When $i \neq k$, if the subtree T_i consists of a single vertex, then we encode the edge e_i with B; otherwise, e_i is encoded with a U and a D. When $i = k$, if the subtree T_k consists of a single vertex, then the edge e_k does not produce any step in the encoding; otherwise, e_k is encoded with R. We traverse the path from e_1 to e_k and construct $\Psi'(T)$ as follows. If the encoding of e_i is B or R, then we record $Q_i = B$ or $Q_i = R\Psi'(T_i)$ respectively. If the encoding of e_i is a U and a D, then we denote $Q_i = U\Psi'(T_i)D$. Joining these segments Q_i from 1 to k, we obtain a 2-Motzkin path $M = Q_1Q_2\cdots Q_k$.

Here is an alternative way to describe Ψ'. Given a plane tree T with n edges, label its vertices with U, D, B or R while traversing it in preorder. For an internal vertex, if it is

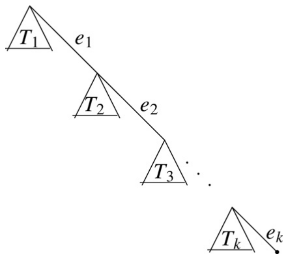

Fig. 7. Decomposition of a plane tree.

not the leftmost child of its parent we label it with U; otherwise we label the vertex with B. A young leaf is labeled with R and an old leaf with D, except the last old leaf that we encounter, which is left unlabeled. Thus all vertices get a label except the root and the last old leaf.

To construct the 2-Motzkin path we traverse the vertices of the tree in a different order and read the labels. Suppose that the root of T has k children $v_1, v_2, \ldots, v_k$ and that T_i is the subtree with root v_i. Then we traverse first the vertices $v_k, v_{k-1}, \ldots, v_1$ in this order, and then traverse $T_1, T_2, \ldots, T_k$ recursively. It can be shown that the path obtained in this way is $\Psi'(T)$.

5. Consequences of the bijections

The main properties of Ψ and Ψ' are given in the following proposition. We state it only for Ψ, but exactly the same result holds if we replace Ψ with Ψ'. The proof for Ψ' follows easily from its recursive description.

Proposition 4. *Let T be a plane tree with $n \geq 1$ edges, and let $P = \Psi(T)$ be the corresponding 2-Motzkin path. We have*

(1) *# of old leaves of $T = 1 +$ # of U steps of P,*
(2) *# of young leaves of $T =$ # of R steps of P.*

Proof. Let us first take a look at how old and young leaves are transformed by the first part θ of the bijection, which consists in reading T in preorder from right to left and building a Dyck path out of it. It is clear that each leaf of T produces a peak in $\theta(T)$. Now, a young leaf of T corresponds to a peak UD followed by a U step, whereas an old leaf of T gives rise to a peak UD not followed by a U.

The second part of the bijection transforms each peak UD followed by a U into a red step R, and these steps remain unchanged by the third part of the bijection. This proves (2). The remaining peaks of the Dyck path are followed either by a D or by nothing, and they are not affected by the second part of the bijection, so these are the only peaks in the Motzkin path. In the final part, we place a mark on each D that is followed by another D or by nothing, and the only D's that are not erased are the unmarked ones. Therefore, the number of U steps (equivalently, the number of D steps) in $\Psi(T)$ equals the number of D's in the Motzkin path that are left unmarked. The D steps in the Motzkin path can be grouped in sequences of consecutive D's, each such sequence immediately following a peak (note that the path has no occurrences of RD, so each D is in one of these sequences). In the sequence of D's following the rightmost peak all the steps are marked. For each remaining peak, among the D steps in the consecutive sequence following it, all but the last one are marked. Thus, only one D step survives for each peak other than the rightmost one. In other words, the number of D steps in $\Psi(T)$ is the number of peaks of the Motzkin path minus one. This implies (1). $\square$

By means of the bijection Ψ and the properties described above, we can now give a combinatorial proof of Corollary 3. To prove the first part, observe that by property (1) of Proposition 4, Ψ induces a bijection between plane trees with exactly one old leaf and 2-Motzkin paths with no U steps. But these paths are just sequences of horizontal steps, each of which can be colored red or blue. Thus, the number of plane trees on n edges with exactly one old leaf is 2^{n-1}.

A direct proof of this nice fact, without using bijections to lattice paths, can be given as follows. Let T be a tree with n edges and exactly one old leaf, call it ℓ. We can find ℓ by following the path that starts at the root and always continues to the leftmost child. Let P be this path. Then ℓ must be at the end of P. Now we claim that the remaining nodes of T are leaves hanging from the nodes of P other than ℓ. Indeed, if a node of P had a child not in P with successors, then following the path that starts at this child and continues always to the leftmost child, we would end at another old leaf, which is a contradiction. Reciprocally, if only leaves are hanging from P, then no more old leaves appear. Now, the number of trees consisting of a path P with leaves hanging from its nodes is clearly 2^{n-1}. Indeed, one can think of it as a composition of n, say $n = a_1 + a_2 + \cdots$, where a_i is the number of children of the i-th node of P.

More generally, we can use our bijection to give a combinatorial proof of the second part of Proposition 2, namely the number of plane trees with n edges and k old leaves is $\frac{2^{n-2k+1}}{k}\binom{n-1}{2k-2}\binom{2k-2}{k-1}$. By the first property of Ψ given above, we have to count the number of 2-Motzkin paths of length $n-1$ with $k-1$ U steps. To produce such a path, we can choose in $\binom{n-1}{2k-2}$ ways the positions of the $k-1$ U's and $k-1$ D's in the path. Deciding which of these positions will be filled with a U or with a D is equivalent to choosing a Dyck path with $2k-2$ steps, and this can be done in $\frac{1}{k}\binom{2k-2}{k-1}$ ways. The remaining $n-2k+1$ positions are horizontal steps, which can be colored red or blue in 2^{n-2k+1} ways.

To show the second part of Corollary 3 combinatorially, notice that property (2) of Proposition 4 implies that Ψ maps plane trees with no young leaves into 2-Motzkin paths with no R steps. These are just Motzkin paths with steps U, D and B. Therefore, the number of plane trees on n edges with no young leaves equals the number of Motzkin paths with $n-1$ steps, which is M_{n-1}.

More generally, the same property of Ψ can be used to prove the last part of Proposition 2, namely the number of plane trees with n edges and k young leaves is $\binom{n-1}{k}M_{n-k-1}$. Indeed, now the problem is equivalent to counting 2-Motzkin paths of length $n-1$ with k R steps. We can choose in $\binom{n-1}{k}$ ways where these R steps go, and then the remaining $n-k-1$ steps can be filled with a Motzkin path with steps U, D and B.

Remark. Another combinatorial proof of part (3) of Proposition 2 can be obtained using the result mentioned in [6] (and proved also in [12]), that $\binom{n-1}{k} M_{n-k-1}$ counts the number of Dyck paths of length $2n$ with k DUD's.

The description of Ψ implicitly contains a bijection between Dyck paths and 2-Motzkin paths. There is a simpler bijection, perhaps the most standard one, that transforms a 2-Motzkin path of length $n-1$ into a Dyck path of length $2n$, by first applying the following rules:

$$U \to UU, \qquad D \to DD, \qquad R \to UD, \qquad B \to DU,$$

and then inserting a U at the beginning and a D at the end of the path. Applying Ψ followed by this bijection, young leaves of the tree are mapped to peaks at even height in the Dyck path. This shows that the statistic ‘number of young leaves’ in $\mathcal{T}_n$ is equidistributed with the statistic ‘number of peaks at even height’ in $\mathcal{D}_n$.

6. Some statistics on restricted permutations

Using some known bijections between Dyck paths and permutations avoiding a pattern of length 3, the parameters counting the number of old and young leaves in plane trees correspond to certain statistics on restricted permutations. Given a pattern σ, we denote by $\mathcal{S}_n(\sigma)$ the set of permutations in the symmetric group $\mathcal{S}_n$ avoiding σ. It is well known that if σ is any pattern of length 3, then $|\mathcal{S}_n(\sigma)| = C_n$, the n-th Catalan number [9].

We begin with a few definitions. Let π be a permutation. We say that π_i is an *excedance* if $\pi_i > i$, that it is a *weak excedance* if $\pi_i \geq i$, and that it is a *deficiency* if $\pi_i < i$. The distribution of excedances and deficiencies in permutations avoiding patterns of length 3 was studied in [8]. A *left-to-right minimum* of π is an element π_i such that $\pi_i < \pi_j$ for all $j < i$. We define a *double descent* of π as a sequence of three consecutive decreasing elements $\pi_i > \pi_{i+1} > \pi_{i+2}$ (equivalently, two consecutive descents). A *double ascent* is defined analogously. An *ascending run* is a maximal increasing sequence of (at least two) consecutive elements of π, i.e., $\pi_i < \pi_{i+1} < \cdots < \pi_{i+k}$, with $k \geq 1$.

Proposition 5. *There is a bijection* $\varphi_1 : \mathcal{T}_n \longrightarrow \mathcal{S}_n(321)$ *such that, if* $T \in \mathcal{T}_n$ *and* $\pi := \varphi_1(T) \in \mathcal{S}_n(321)$*, then*

(1) *# of young leaves of* T = *# of pairs of consecutive weak excedances of* π*,*
(2) *# of old leaves of* T = *# of weak excedances of* π *not followed by another weak excedance.*

Proof. We use a bijection ψ between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$ which is similar to the one given by Krattenthaler [10] from $\mathcal{S}_n(123)$ to $\mathcal{D}_n$. Here is a way to describe it. Let $\pi \in \mathcal{S}_n(321)$, and let $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_k}$ be its weak excedances, from left to right. Define $\psi(\pi)$ to be the path that starts with π_{i_1} up steps, then has, for each j from 2 to k, $i_j - i_{j-1}$ down steps followed by $\pi_{i_j} - \pi_{i_{j-1}}$ up steps, and finally ends with $n+1-i_k$ down steps. It can be checked that this is indeed a bijection between 321-avoiding permutations and Dyck paths.

Our bijection φ_1 is defined as $\varphi_1 = \psi^{-1} \circ \theta$. Recall that θ reads a plane tree in preorder from right to left and creates a Dyck path.

We see that young leaves of T correspond to occurrences of UDU in the path $\theta(T)$, and that old leaves of T are mapped by θ to either a UDD or a terminal (i.e., at the end of the path) UD. Now, if $\pi \in \mathcal{S}_n(321)$, a UDU is obtained in $\psi(\pi)$ precisely when we have a weak excedance followed by another weak excedance, which causes one of the descending slopes to have length $i_j - i_{j-1} = 1$. Similarly, a UDD corresponds to a weak excedance followed by a deficiency (i.e., an element that is not a weak excedance), and a terminal UD corresponds to the weak excedance $\pi_n = n$. □

For example, if T is the tree in Fig. 1, with $\theta(T)$ given in Fig. 3, then the corresponding permutation is $\varphi_1(T) = (3, 4, 1, 2, 5, 9, 6, 7, 8, 11, 12, 13, 10) \in \mathcal{S}_{12}(321)$. It has four pairs of consecutive weak excedances, namely $(3, 4)$, $(5, 9)$, $(11, 12)$ and $(12, 13)$, and three weak excedances not followed by another weak excedance, namely 4, 9 and 13.

A similar result for 132-avoiding permutations is given next. For $\pi \in \mathcal{S}_n$, let $(n+1)\pi$ (resp. $\pi(n+1)$) be the permutation in $\mathcal{S}_{n+1}$ obtained by inserting $n+1$ at the beginning (resp. at the end) of π.

Proposition 6. *There is a bijection* $\varphi_2 : \mathcal{T}_n \longrightarrow \mathcal{S}_n(132)$ *such that, if* $T \in \mathcal{T}_n$ *and* $\pi := \varphi_2(T) \in \mathcal{S}_n(132)$*, then*

(1) *# of young leaves of* $T =$ *# of double descents of* $(n+1)\pi$,
(2) *# of old leaves of* $T =$ *# of ascending runs of* $\pi(n+1)$.

Proof. We use the bijection from $\mathcal{S}_n(132)$ to $\mathcal{D}_n$ denoted by Φ that appears in Krattenthaler [10]. Given $\pi \in \mathcal{S}_n(132)$, let $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_k}$ be its left-to-right minima, from left to right. Then $\Phi(\pi)$ is the Dyck path that starts with $n + 1 - \pi_{i_1}$ up steps, then has, for each j from 2 to k, $i_j - i_{j-1}$ down steps followed by $\pi_{i_{j-1}} - \pi_{i_j}$ up steps, and finally ends with $n + 1 - i_k$ down steps. It can be checked that this is indeed a bijection between 132-avoiding permutations and Dyck paths. The bijection we are looking for is $\varphi_2 := \Phi^{-1} \circ \theta$.

Each young leaf of T produces an occurrence of UDU in $\theta(T)$. Such an occurrence appears in $\Phi(\pi)$ for each pair of consecutive left-to-right minima. These two elements, together with the entry of $(n+1)\pi$ immediately to their left, form a decreasing sequence of three consecutive elements (a double descent). To see that these are the only double descents of $(n+1)\pi$, notice that from the structure of 132-avoiding permutations it follows that if $\pi_j > \pi_{j+1}$ is a descent of π, then π_{j+1} must be a left-to-right minimum.

The reasoning for old leaves is similar. They correspond to occurrences of UDD and possibly a UD at the end or, equivalently, to occurrences of UDD in $\theta(T)D$ (i.e., the Dyck path $\theta(T)$ with a D step appended at the end). Each of these occurrences marks the start of a maximal sequence of at least two consecutive D steps in $\theta(T)D$, and each such sequence corresponds to an ascending run of $\pi(n+1)$. □

For example, if T is again the tree in Fig. 1, then the corresponding 132-avoiding permutation is $\pi = \varphi_2(T) = (11, 10, 12, 13, 9, 5, 6, 7, 8, 3, 2, 1, 4)$. Note that $(n+1)\pi = (14, \pi)$ has four double descents, namely $(14, 11, 10)$, $(13, 9, 5)$, $(8, 3, 2)$ and $(3, 2, 1)$, and $(\pi, 14)$ has three ascending runs, namely $(10, 12, 13)$, $(5, 6, 7, 8)$ and $(1, 4, 14)$.

There is another well-known bijection between plane trees and Dyck paths, which we denote as δ. Given a tree T, traverse it in preorder (from left to right) and build

$\delta(T)$ as follows. For each node with r children, draw r up steps followed by one down step; draw a D for each leaf except for the last leaf, for which we do not draw anything. For example, the path corresponding to the tree in Fig. 1 is $\delta(T) = UUUUDUUUDDDDUDUDUDDDUDUUDD$.

Define a *drop* of a Dyck path to be a maximal succession of at least two consecutive D steps, and a *triple fall* to be an occurrence of DDD. Then the bijection δ maps each old leaf of T to a drop of $\delta(T)D$, and each young leaf to a triple fall of $\delta(T)D$. For the above example, $\delta(T)D$ has three drops and four triple falls.

Following arguments similar to the ones in Propositions 5 and 6, but using the bijection δ instead of θ, we obtain the next two results.

Proposition 7. *There is a bijection $\varphi_3 : \mathcal{T}_n \longrightarrow \mathcal{S}_n(321)$ such that, if $T \in \mathcal{T}_n$ and $\pi := \varphi_3(T) \in \mathcal{S}_n(321)$, then*

(1) *# of young leaves of T = # of pairs of consecutive deficiencies of π $(+1$ if $\pi_n < n)$,*
(2) *# of old leaves of T = # of weak excedances of π not followed by another weak excedance.*

Proposition 8. *There is a bijection $\varphi_4 : \mathcal{T}_n \longrightarrow \mathcal{S}_n(132)$ such that, if $T \in \mathcal{T}_n$ and $\pi := \varphi_4(T) \in \mathcal{S}_n(132)$, then*

(1) *# of young leaves of T = # of double ascents of $\pi(n+1)$,*
(2) *# of old leaves of T = # of ascending runs of $\pi(n+1)$.*

7. Refinements of two combinatorial identities

In [5] Coker established the following two identities, involving the Narayana and the Catalan numbers:

$$\sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} 4^{n-k} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} 4^k 5^{n-2k-1}, \tag{2}$$

$$\sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} x^{2k}(1+x)^{2n-2k} = x^2 \sum_{k=0}^{n-1} C_{k+1} \binom{n-1}{k} x^k (1+x)^k. \tag{3}$$

He stated the open problem of finding a combinatorial interpretation of these identities. In [4], Chen, Yan and Yang proved these identities combinatorially. In this section we use the properties of Ψ given in Proposition 4 to obtain refinements of the identities (2) and (3).

Theorem 9. *For $n \geq 1$, we have*

$$\sum_{i=1}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{n-2i+1} \frac{1}{n}\binom{n}{i}\binom{n-i}{j}\binom{n-i-j}{i-1} x^{i-1} y^j$$
$$= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} x^k (1+y)^{n-2k-1}. \tag{4}$$

Proof. It will be convenient to use the term *critical leaf* to denote the last old leaf that we encounter when we traverse a plane tree in preorder. Given a plane tree T with n edges, assign weights to the vertices of T as follows: young leaves are given weight y, old leaves other than the critical one are given weight x, and the rest of the vertices (including the critical leaf) are given weight 1. The weight of T is the product of the weights of its vertices. Then, the left hand side of (4) is the sum of the weights of all plane trees with n edges.

By Proposition 4, Ψ is a weight preserving bijection between the set of weighted plane trees on n edges, with weights given as above, and the set of weighted 2-Motzkin paths of length $n-1$ where weights are assigned as follows: U steps are given weight x, R steps are given weight y, and all the remaining steps are given weight 1, defining the weight of a 2-Motzkin path to be the product of weights of its steps. We claim that the right hand side of (4) is the sum of the weights of all 2-Motzkin paths of length $n-1$. Indeed, let $k \le \lfloor (n-1)/2 \rfloor$ and consider the weighted 2-Motzkin paths with k up steps and k down steps. These up and down steps from a Dyck path of length $2k$, and the positions of these $2k$ steps can be chosen in $\binom{n-1}{2k}$ ways. They contribute x^k to the weight of the path. The remaining $n-2k-1$ steps are either R or B steps. Since R steps have weight y and B steps have weight 1, the total contribution of the horizontal steps in paths with k up steps is $(1+y)^{n-2k-1}$. This justifies the right hand side. $\square$

With the substitution $y = x$ in Eq. (4) we recover the result proved in [4], and the particular case $y = x = 4$, together with the symmetry of the Narayana numbers, yields Eq. (2). A refinement of the second identity (3) is given next.

Theorem 10. *For $n \ge 1$, we have*

$$\sum_{i=1}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{n-2i+1} \frac{1}{n} \binom{n}{i} \binom{n-i}{j} \binom{n-i-j}{i-1} x^{2(i-1)} y^j z^{n-2i-j+1} = \sum_{k=0}^{n-1} C_{k+1} \binom{n-1}{k} x^k (y+z-2x)^{n-1-k}. \tag{5}$$

Proof. Recall the definition of the critical leaf from the proof of Theorem 9. Given a plane tree T with n edges, assign weights to the vertices of T in the following way. Old leaves other than the critical one are given weight x, the parents of such leaves are given weight x as well, young leaves are given weight y, the critical leaf and its parent are given weight 1, and the rest of the vertices are given weight z. As before, the weight of T is the product of the weights of its vertices. Notice that two different old leaves cannot have the same parent, so the weight of a tree with i old leaves and j young leaves is $x^{2(i-1)} y^j z^{n-2i-j+1}$. The left hand side of (5) is the sum of the weights of all plane trees with n edges.

By Proposition 4, a tree with i old leaves and j young leaves is mapped by Ψ to a 2-Motzkin path with $i-1$ up steps, $i-1$ down steps, j horizontal R steps, and $n-2i-j+1$ horizontal B steps. To make Ψ a weight preserving bijection between plane trees on n edges with the above weights and 2-Motzkin paths of length $n-1$, we assign weights to the steps of a 2-Motzkin path by giving weight x to U and D steps, weight y to R steps, and weight z to B steps.

Consider now the set of 3-Motzkin paths of length $n-1$, where horizontal steps can be either red, blue or green (call them R, B and G steps respectively). Assign weights to the steps by giving weight $y+z-2x$ to G steps and weight x to all the other steps. Again, the weight of a path is the product of weights of its steps. This weight assignment for 3-Motzkin paths has the property that the sum of the weights of an R step, a B step and a G step equals the sum of the weights of an R step and a B step in the assignment for 2-Motzkin paths above (namely $x+y$), and also that U and D steps have the same weight x in both assignments. This implies that the sum of weights over all 2-Motzkin paths with the above weight assignment equals the sum of weights over all 3-Motzkin paths with this new assignment. Therefore, it remains to show that the right hand side of (5) is the total sum of the weights of 3-Motzkin paths of length $n-1$. But this is clear because if we fix the number of G steps of a 3-Motzkin path to be $n-1-k$, then the positions of these G steps can be chosen in $\binom{n-1}{k}$ ways. The remaining steps, U, D, R and B, form a 2-Motzkin path of length k, and the number of such paths is C_{k+1}. □

To recover identity (3) we only need to substitute $x(1+x)$ for x, x^2 for y, and $(1+x)^2$ for z in Eq. (5), and notice that a tree with i old leaves and j young leaves has $k=i+j$ leaves in total.

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