FIXED POINTS AND EXCEDANCES IN RESTRICTED PERMUTATIONS

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Abstract

In this paper we prove that among the permutations of length n with i fixed points and j excedances, the number of 321-avoiding ones equals the number of 132-avoiding ones, for all given $i, j \leq n$. We use a new technique involving diagonals of non-rational generating functions.

This theorem generalizes a recent result of Robertson, Saracino and Zeilberger, for which we also give another, more direct proof.

Résumé

Dans cet article nous prouvons que parmi les permutations de longueur n ayant i points fixes et j excédances, le nombre de celles qui évitent 321 est le même que le nombre de celles qui évitent 132, pour tous $i, j \leq n$. Nous employons une technique utilisant la série diagonale de fonctions génératrices non rationelles.

Ce théorème généralise un résultat récent de Robertson, Saracino et Zeilberger, dont nous donnons une démonstration différente et plus directe.

1. Introduction

Let n, m be two positive integers with $m \leq n$, and let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_m \in \mathcal{S}_m$. We say that π *contains* σ if there exist indices $i_1 < i_2 < \ldots < i_m$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ is in the same relative order as $\sigma_1\sigma_2\cdots\sigma_m$. If π does not contain σ, we say that π is *σ-avoiding*. For example, if $\sigma = 132$, then $\pi = 24531$ contains σ, because $\pi_1\pi_3\pi_4 = 253$. However, $\pi = 42351$ is σ-avoiding.

We say that i is a *fixed point* of a permutation π if $\pi_i = i$, and that i is an *excedance* of π if $\pi_i > i$. Denote by $\mathrm{fp}(\pi)$ and $\mathrm{exc}(\pi)$ the number of fixed points and the number of excedances of π respectively. Denote by $\mathcal{S}_n(\sigma)$ the set of σ-avoiding permutations in $\mathcal{S}_n$. We are interested in the distribution of the number of fixed points and excedances among the permutations in $\mathcal{S}_n(\sigma)$.

For the case of patterns of length 3, it is known ([7]) that regardless of the pattern $\sigma \in \mathcal{S}_3$, $|\mathcal{S}_n(\sigma)| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number. Bijective proofs of this fact are given in [8, 10, 13, 16].

In the recent paper [11], pattern-avoiding permutations are studied with respect to the number of fixed points, and an interesting refinement is presented. It is shown that given $i \leq n$, the number of 321-avoiding permutations of length n with i fixed points equals the number of 132-avoiding permutations of length n with i fixed points.

In this paper we prove a further refinement of this result, namely that it still holds when we fix not only the number of fixed points but also the number of excedances. In other words, the bivariate distribution of fixed points and excedances is the same in both 321-avoiding permutations and in 132-avoiding permutations.

One of the key points in the proof is to use bijections between pattern avoiding permutations and Dyck paths. Recall that a *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$

between $(0,0)$ and $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ which never goes below the x-axis. Sometimes it will be convenient to encode each up-step by a letter u and each down-step by d, obtaining an encoding of the Dyck path as a *Dyck word*. We shall denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$, and by $\mathcal{D}=\bigcup_{n\geq 0}\mathcal{D}_n$ the class of all Dyck paths. It is well-known that $|\mathcal{D}_n|=C_n$. If $D\in\mathcal{D}_n$, we will write $|D|=n$ to indicate the semilength of D. The generating function (*GF* for short) that enumerates Dyck paths according to their semilength is $\sum_{D\in\mathcal{D}} t^{|D|}=\sum_{n\geq 0} C_n t^n=\frac{1-\sqrt{1-4t}}{2t}$, which we denote by $C(t)$.

2. Statement of the main theorem

Here is the main result of this paper.

Theorem 2.1. *For any* $0\leq i,j\leq n$,

$$|\{\pi\in\mathcal{S}_n(321):\ \mathrm{fp}(\pi)=i,\ \mathrm{exc}(\pi)=j\}|=|\{\pi\in\mathcal{S}_n(132):\ \mathrm{fp}(\pi)=i,\ \mathrm{exc}(\pi)=j\}|.$$

Equivalently,

$$\sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}=\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}.$$

The proof of this theorem is done in two parts. First, in Section 3 we find the GF for the number of fixed points and excedances in 321-avoiding permutations. Then, in Section 4, we show that this GF also counts the number of fixed points and excedances in 132-avoiding permutations. To do the latter, we introduce an extra variable marking a new parameter in the GF. Then, using combinatorial properties, we deduce an identity that determines this GF. Finally, we conjecture an expression for it and check that our expression satisfies the identity, hence it is the correct GF.

3. Counting 321-avoiding permutations according to fixed points and excedances

The goal of this section is to find an expression for the GF

$$F_{321}(x,q,t):=\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}t^n.$$

Instead of counting fixed points and excedances directly in 321-avoiding permutations, we define the following bijection $\Phi_{\llcorner}$ between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$, suggested by Richard Stanley.

Given $\pi=\pi_1\pi_2\cdots\pi_n\in\mathcal{S}_n(321)$, let $a_i=\max\{j:\{1,2,\dots,j\}\subseteq\{\pi_1,\pi_2,\dots,\pi_i\}\}$ (j can be 0, in which case $\{1,2,\dots,j\}=\emptyset$), for each $1\leq i\leq n$. Now build the Dyck path $\Phi_{\llcorner}(\pi)$ by adjoining, for each i from 1 to n, one up-step followed by $\max\{a_i-\pi_i+1,0\}$ down-steps. For example, for $\pi=23147586$ we get $a_1=a_2=0$, $a_3=3$, $a_4=a_5=4$, $a_6=a_7=5$, $a_8=8$, and the corresponding Dyck path is given in Figure 1.

There is an alternative way to define this bijection. A *right-to-left minimum* of π is an element π_i such that $\pi_i<\pi_j$ for all $j>i$. Let $\pi_{i_1},\pi_{i_2},\dots,\pi_{i_k}$ be the right-to-left minima of π, from left to right. For example, the right-to-left minima of 23147586 are $1,4,5,6$. Then, $\Phi_{\llcorner}(\pi)$ is precisely the path that starts with i_1 up-steps, then has, for each j from 2 to k, $\pi_{i_j}-\pi_{i_{j-1}}$ down-steps followed by i_j-i_{j-1} up-steps, and finally ends with $n+1-\pi_{i_k}$ down-steps.

An easy way to picture this construction is to represent π as an $n\times n$ array with a cross on the squares (i,π_i). It is known that a permutation is 321-avoiding if and only if both

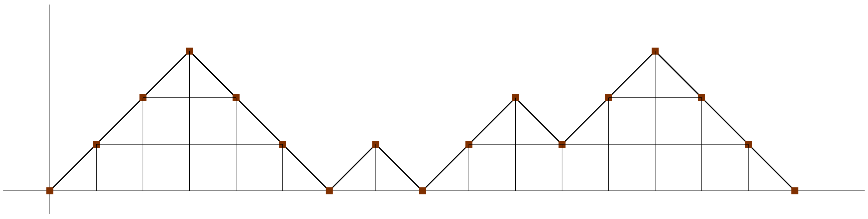

FIGURE 1. The Dyck path $\Phi_{\text{L}}(23147586)$.

the subsequence determined by its excedances and the one determined by the remaining elements are increasing. In this array representation, excedances correspond to crosses strictly to the right of the main diagonal. Note that the rest of the crosses are precisely the right-to-left minima. Consider the path with *down* and *right* steps along the edges of the squares that goes from the upper-left corner to the lower-right corner of the array leaving all the crosses to the right and remaining always as close to the main diagonal as possible. Then $\Phi_{\text{L}}(\pi)$ can be obtained from this path just by reading an up-step every time the path moves down, and a down-step every time the path moves to the right. Figure 2 shows a picture of this bijection, again for $\pi = 23147586$.

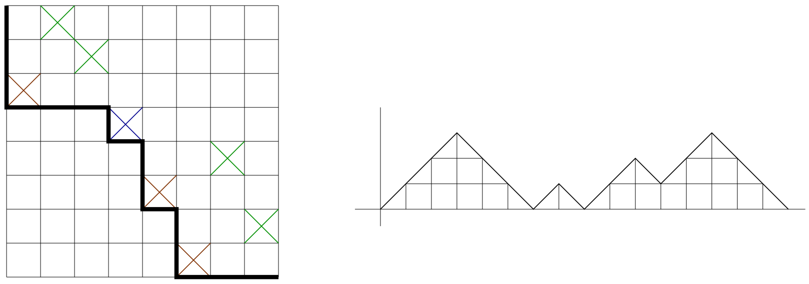

FIGURE 2. The bijection Φ_{L}.

Recall that a *peak* of a Dyck path $D \in \mathcal{D}$ is an up-step followed by a down-step (i.e., an occurrence of ud in the associated Dyck word). A *hill* is a peak at height 1, where the height is the y-coordinate of the top of the peak. Denote by $h(D)$ the number of hills of D. A *double rise* of a Dyck path is an up-step followed by another up-step (uu when seen as a word). Denote by $\text{dr}(D)$ the number of double rises of D.

It can easily be checked that Φ_{L} has the property that $\text{fp}(\pi) = h(\Phi_{\text{L}}(\pi))$ and $\text{exc}(\pi) = \text{dr}(\Phi_{\text{L}}(\pi))$. Therefore, counting 321-avoiding permutations according to the number fixed points and excedances is equivalent to counting Dyck paths according to the number of hills and double rises. More precisely,

$$F_{321}(x, q, t) = \sum_{D \in \mathcal{D}} x^{h(D)} q^{\text{dr}(D)} t^{|D|}.$$

We can give an equation for F_{321} using the symbolic method described in [5] and [12]. A recursive definition for the class $\mathcal{D}$ is given by the fact that every non-empty Dyck path D can be decomposed in a unique way as $D = uAdB$, where $A, B \in \mathcal{D}$. Clearly if A is empty, $h(D) = h(B) + 1$ and $\text{dr}(D) = \text{dr}(B)$, and otherwise $h(D) = h(B)$ and $\text{dr}(D) = \text{dr}(A) + \text{dr}(B) + 1$. Hence, we obtain the following equation for F_{321}:

$$F_{321}(x, q, t) = 1 + t(x + q(F_{321}(1, q, t) - 1))F_{321}(x, q, t). \tag{1}$$

Substituting first $x = 1$, we obtain that $F_{321}(1,q,t) = \frac{1+t(q-1)-\sqrt{1-2t(1+q)+t^2(1-q)^2}}{2qt}$. Now, solving (1) for $F_{321}(x,q,t)$ gives

$$F_{321}(x,q,t) = \frac{2}{1+t(1+q-2x)+\sqrt{1-2t(1+q)+t^2(1-q)^2}}. \tag{2}$$

To conclude this section, we want to remark that applying this method one can also obtain the GF that enumerates fixed points, excedances and descents in 321-avoiding permutations. It can be seen that the number of descents of a 321-avoiding permutation π (i.e., indices i for which $\pi_i > \pi_{i+1}$), denoted $\mathrm{des}(\pi)$, equals the number of occurrences of uud in the Dyck word of $\Phi_{\llcorner}(\pi)$. Using the same decomposition as before, we conclude that

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}p^{\mathrm{des}(\pi)}t^n$$
$$= \frac{2}{1+t(1+q-2x)+\sqrt{1-2t(1+q)+t^2((1+q)^2-4qp)}}.$$

4. Counting 132-avoiding permutations according to fixed points and excedances

Analogously to the previous section, we define

$$F_{132}(x,q,t) := \sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}t^n.$$

To prove Theorem 2.1 we have to show that $F_{321}(x,q,t) = F_{132}(x,q,t)$.

Instead of enumerating fixed points and excedances directly in 132-avoiding permutations, we use a bijection between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$, and then look at what are the statistics in Dyck paths that correspond to fp and *exc* after the bijection.

For any $D \in \mathcal{D}$, we define a *tunnel* of D to be a horizontal segment between two lattice points of D that intersects D only in these two points, and stays always below D. Tunnels are in obvious one-to-one correspondence with decompositions of the Dyck word $D = AuBdC$, where $B \in \mathcal{D}$ (no restrictions on A and C). In the decomposition, the tunnel is the segment that goes from the beginning of u to the end of d. If $D \in \mathcal{D}_n$, then D has exactly n tunnels, since such a decomposition can be given for each up-step of D.

A tunnel of $D \in \mathcal{D}_n$ is called a *centered tunnel* if the x-coordinate of its midpoint (as a segment) is n, that is, the tunnel is centered with respect to the vertical line through the middle of D. In terms of the decomposition $D = AuBdC$, this is equivalent to saying that A and C have the same length. Denote by $\mathrm{CT}(D)$ the set of centered tunnels of D, and let $\mathrm{ct}(D) = |\mathrm{CT}(D)|$.

A tunnel of $D \in \mathcal{D}_n$ is called a *left tunnel* if the x-coordinate of its midpoint is strictly less than n, that is, the midpoint of the tunnel is to the left of the vertical line through the middle of D. In terms of the decomposition $D = AuBdC$, this is equivalent to saying that the length of A is strictly smaller than the length of C. Denote by $\mathrm{lt}(D)$ the number of left tunnels of D. In Figure 3, there is one centered tunnel drawn with a solid line, and four left tunnels drawn with dotted lines.

We will use the bijection between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$ given by Krattenthaler in [8]. We denote it by φ_K. For $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n(132)$, $\varphi_K(\pi)$ is obtained by reading π from left to right and adjoining for each π_j as many up-steps as necessary followed by a down-step from height $h_j + 1$ to height h_j, where h_j is the number of elements in $\pi_{j+1}\cdots\pi_n$ which

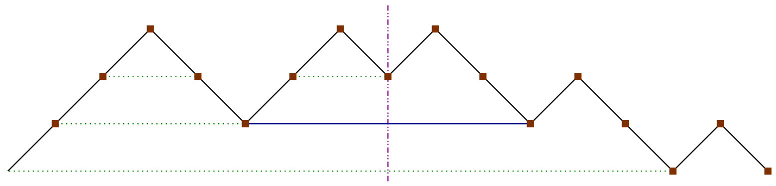

FIGURE 3. Centered and left tunnels.

are larger than π_j. As pointed out by Reifegerste in [9], this path is closely related to the diagram of π obtained from the $n \times n$ array representation of π by shading, for each cross, the cell containing it and the squares that are due south and due east of it. The diagram, defined as the region that remains unshaded, is determined by the path with *left* and *down* steps that goes from the upper-right corner to the lower-left corner, leaving all the crosses to the right, and staying always as close to the diagonal connecting these two corners as possible. If we go along this path reading an up-step every time it goes left and a down-step every time it goes down, we get $\varphi_K(\pi)$. Figure 4 shows an example when $\pi = 67435281$.

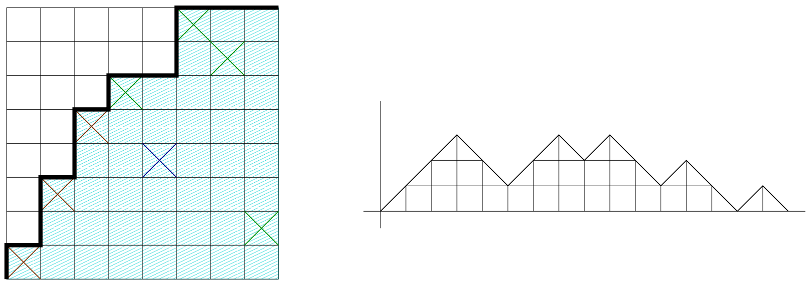

FIGURE 4. The bijection φ_K.

What is interesting of this bijection for our purposes is that it has the property that it maps fixed points to centered tunnels, and excedances to left tunnels. This can be seen using the diagram representation. There is an easy way to recover a permutation $\pi \in \mathcal{S}_n(321)$ from its diagram: row by row, put a cross in the leftmost shaded square such that there is exactly one dot in each column. Now, instead of looking directly at $\varphi_K(\pi)$, consider the path from the upper-right corner to the lower-left corner of the array of π. To each cross we can associate a tunnel in a natural way. Indeed, each cross produces a decomposition $\varphi_K(\pi) = AuBdC$ where B corresponds to the part of the path above and to the left of the cross. Here u corresponds to the horizontal step directly above the cross, and d to the vertical step directly to the left of the cross. Thus, fixed points, which correspond to crosses on the main diagonal, give centered tunnels, and excedances, which are crosses to the right of the main diagonal, give left tunnels. This means that $\mathrm{fp}(\pi) = \mathrm{ct}(\varphi_K(\pi))$ and $\mathrm{exc}(\pi) = \mathrm{lt}(\varphi_K(\pi))$. So, our problem is equivalent to counting Dyck paths according to centered and left tunnels, and the function we want to find becomes

$$F_{132}(x, q, t) = \sum_{D \in \mathcal{D}} x^{\mathrm{ct}(D)} q^{\mathrm{lt}(D)} t^{|D|}.$$

The decomposition of $\mathcal{D}$ that we used to enumerate hills and double rises no longer works here. Indeed, if we write $D = uAdB$ with $A, B \in \mathcal{D}$, then $\mathrm{ct}(A)$ and $\mathrm{ct}(B)$ do not give information about $\mathrm{ct}(D)$. However, to count only centered tunnels, we can use another decomposition.

Now we show how to obtain an expression for $F_{132}(x,1,t)$. We consider Dyck paths with *marked* centered tunnels. That is, we count pairs (D,S) where $D\in\mathcal{D}$ and $S\subseteq \mathrm{CT}(D)$. Each such pair is given weight $(x-1)^{|S|}t^{|D|}$, so that for a fixed D, the sum of weights of all pairs (D,S) will be $\sum_{S\subseteq \mathrm{CT}(D)}(x-1)^{|S|}t^{|D|}=((x-1)+1)^{|\mathrm{CT}(D)|}t^{|D|}=x^{\mathrm{ct}(D)}t^{|D|}$, which is precisely the weight that D has in $F_{132}(x,1,t)$.

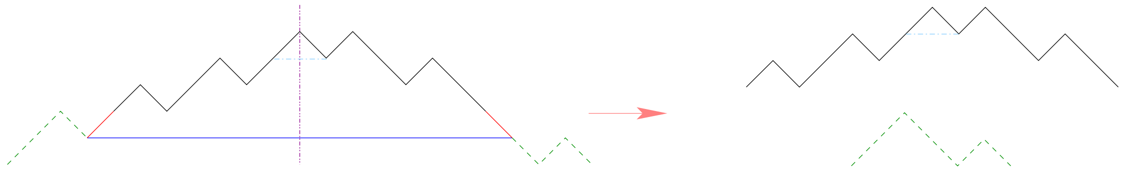

FIGURE 5. Decomposing Dyck paths with marked centered tunnels.

Dyck paths with no marked tunnels (i.e., pairs $(D,\emptyset)$) are enumerated by $C(t)=\frac{1-\sqrt{1-4t}}{2t}$, the GF for Catalan numbers. On the other hand, for an arbitrary Dyck path D with some centered tunnel marked (i.e., a pair (D,S) with $S\neq\emptyset$), we can consider the decomposition given by the longest marked tunnel, say $D=AuBdC$. Then, AC (seen as the concatenation of Dyck words) gives an arbitrary Dyck path with no marked centered tunnels, and B is an arbitrary Dyck path where some centered tunnels may be marked (Figure 5). This decomposition translates into the following equation for GFs:

$$F_{132}(x,1,t)=C(t)+(x-1)tC(t)F_{132}(x,1,t).$$

Solving it, we obtain

$$F_{132}(x,1,t)=\frac{2}{1+2t(1-x)+\sqrt{1-4t}},$$

which is precisely the expression that we had for $F_{321}(x,1,t)$ in (2). This gives a new and perhaps simpler proof of the main result in [11], namely that $|\{\pi\in\mathcal{S}_n(321):\mathrm{fp}(\pi)=i\}|=|\{\pi\in\mathcal{S}_n(132):\mathrm{fp}(\pi)=i\}|$ for all $i\le n$.

To enumerate left tunnels we will need a different approach. The first step is to generalize the concepts of centered and left tunnels, allowing the vertical line that we use as a reference to be shifted from the center of the Dyck path. For $D\in\mathcal{D}$ and $r\in\mathbb{Z}$, let $\mathrm{ct}_r(D)$ be the number of tunnels of D whose midpoint lies on the vertical line $x=n-r$ (we call this the *reference line*). Similarly, let $\mathrm{lt}_r(D)$ be the number of tunnels of D whose midpoint lies on the half-plane $x<n-r$. Notice that by definition, ct_0 and lt_0 are respectively the statistics *ct* and lt defined previously.

We also add a new variable v to F_{132} which marks the distance from the reference line to the actual middle of the path. Define

$$G(x,q,t,v):=\sum_{n,r\ge 0}\sum_{D\in\mathcal{D}_n}x^{\mathrm{ct}_r(D)}q^{\mathrm{lt}_r(D)}v^r t^n.$$

Our next goal is to find an equation that determines $G(x,q,t,v)$. The idea is to use again the decomposition of a Dyck path as $D=uAdB$, where $A,B\in\mathcal{D}$. The difference is that now the GFs involve sums not only over Dyck paths but also over the possible positions of the reference line.

Let

$$H_1(x,q,t,v):=\sum_{\substack{n\ge1\\ k\ge -n}}\sum_{A\in\mathcal{D}_{n-1}}x^{\mathrm{ct}_{-k}(uAd)}q^{\mathrm{ct}_{-k}(uAd)}v^k t^n \tag{3}$$

be the GF for the first part uAd of the decomposition, where the reference line can be anywhere to the right of the left end of the path (Figure 6). Similarly, let

$$H_2(x,q,t,v) := \sum_{\substack{n\geq 0\\ r\geq -n}} \sum_{B\in\mathcal{D}_n} x^{\mathrm{ct}_r(B)} q^{\mathrm{lt}_r(B)} v^r t^n \tag{4}$$

be the GF for the second part B of the decomposition, where now the reference line can be anywhere to the left of the right end of the path.

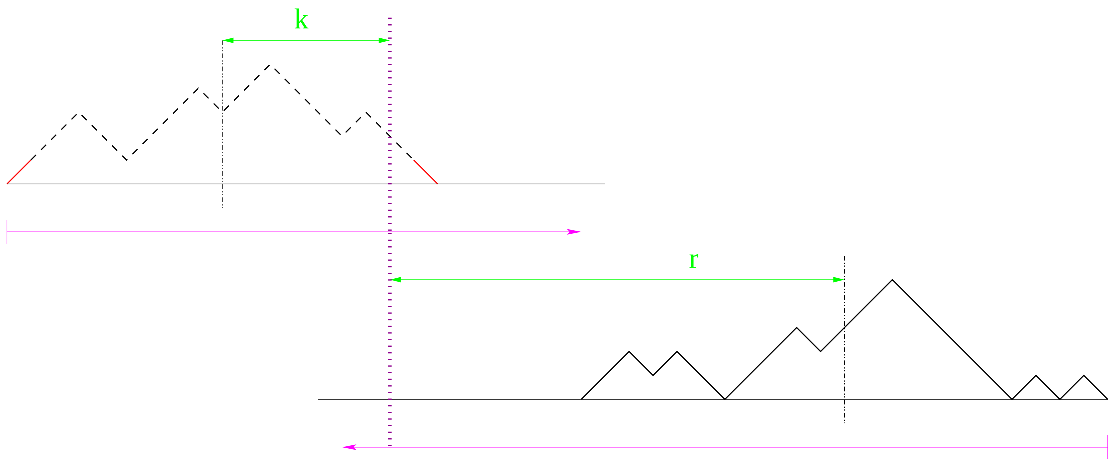

FIGURE 6. H_1 and H_2.

We would like to express the generating function for paths $uAdB$ in terms of H_1 and H_2. The product of these two GFs counts pairs (uAd, B), but if we want the reference line to coincide in uAd and in B, then the two parts are not necessarily placed next to each other (Figure 6). The exponent of v in H_1 indicates how far to the right the reference line is from the middle of the path uAd. The exponent of v in H_2 indicates how far to the left the reference line is from the middle of the path B. In the product H_1H_2, the exponent of v is the distance from the middle of the path uAd to the middle of the path B if we draw them so that the reference lines coincide. Now comes one of the key points of the argument. The terms that correspond to an actual path $D = uAdB$ are those in which the two parts are placed next to each other in the picture (B begins where uAd ends), and this happens precisely when the exponent of v is half the sum of lengths of uAd and B. But the semilength of each path is the exponent of t in the corresponding GF, so the sum of semilengths is the exponent of t in the product H_1H_2. Hence, the terms that correspond to actual paths $D = uAdB$ are exactly those in which the exponent of v equals the exponent of t (Figure 7). In generating function terminology, the GF consisting of only such terms is called a *diagonal*.

We also need another variable y to mark the distance between the reference line and the middle of the new path $D = uAdB$. Considering that D starts at $(0,0)$, the x-coordinate of the middle of the new path is given by the exponent of t in the product, which is the sum of the exponents of t in H_1 and H_2. The x-coordinate of the reference line is given by the exponent of t in H_1 plus the exponent of v in H_1. Hence, the difference between these two x-coordinates is given by the exponent of t in H_2 minus the exponent of v in H_1.

Let

$$P(x,q,t,v,y) := H_1(x,q,t,\frac{v}{y})H_2(x,q,ty,v),$$

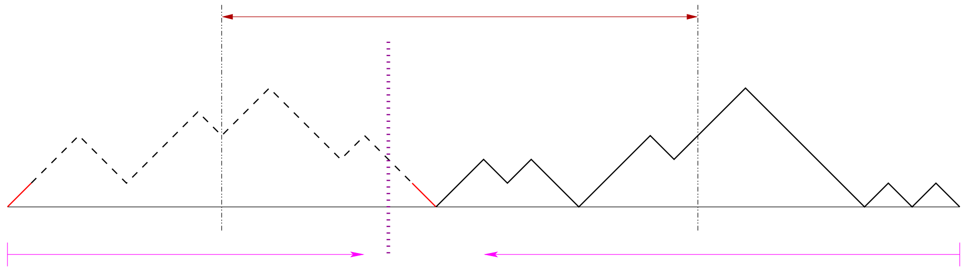

FIGURE 7. Terms with equal exponent in t and v.

and let its series expansion in v and t be

$$P(x,q,t,v,y)=\sum_{\substack{n\geq 0\\ j\geq -n}} P_{j,n}(x,q,y)v^j t^n.$$

The diagonal (in v and t) of P is defined by

$$\operatorname{diag}^z_{v,t}\ P := \sum_{n\geq 0} P_{n,n}(x,q,y)z^n.$$

Now, the above argument implies that this diagonal equals precisely

$$H_3(x,q,z,y) := \sum_{\substack{n\geq 1\\ -n\leq r\leq n}} \sum_{D\in\mathcal{D}_n} x^{\mathrm{ct}_r(D)} q^{\mathrm{lt}_r(D)} y^r z^n, \tag{5}$$

that is, the sum over arbitrary non-empty (since uAd was non-empty) Dyck paths D, where the reference line can be anywhere between the left end and the right end of the path.

We have found an equation that relates H_1, H_2 and H_3, thus proving the following lemma.

Lemma 4.1. *Let H_1, H_2 and H_3 be defined respectively by (3), (4), and (5). Then,*

$$\operatorname{diag}^z_{v,t}\ H_1(x,q,t,\frac{v}{y})H_2(x,q,ty,v) = H_3(x,q,z,y). \tag{6}$$

The next step is to express these three GFs in terms of G, so that (6) will in fact give an equation for G. First, note that given $D\in\mathcal{D}_n$, if D^R is the Dyck path obtained by reflecting D over the vertical line $x=n$, then we have that $\mathrm{ct}_{-r}(D)=\mathrm{ct}_r(D^R)$ and $\mathrm{lt}_{-r}(D)=n-\mathrm{lt}_r(D^R)-\mathrm{ct}_r(D^R)$, since the total number of tunnels of D^R is n. Thus,

$$\begin{aligned}\sum_{n,r\geq 0}\sum_{D\in\mathcal{D}_n} x^{\mathrm{ct}_{-r}(D)} q^{\mathrm{lt}_{-r}(D)} v^r t^n &= \sum_{n,r\geq 0}\sum_{D\in\mathcal{D}_n} \left(\frac{x}{q}\right)^{\mathrm{ct}_r(D^R)} \left(\frac{1}{q}\right)^{\mathrm{lt}_r(D^R)} v^r (qt)^n \\ &= G(\frac{x}{q},\frac{1}{q},qt,v). \end{aligned}\tag{7}$$

Also, note that if $|D|=n$ and $r\geq n$, then $\mathrm{ct}_r(D)=\mathrm{lt}_r(D)=\mathrm{ct}_{-r}(D)=0$ and $\mathrm{lt}_{-r}(D)=n$. In particular,

$$\sum_{\substack{n\geq 0\\ r>n}}\sum_{D\in\mathcal{D}_n} x^{\mathrm{ct}_r(D)} q^{\mathrm{lt}_r(D)} v^r t^n = \sum_{\substack{n\geq 0\\ r>n}} C_n v^r t^n = \sum_{n\geq 0} C_n \frac{v^{n+1}}{1-v} t^n = \frac{v}{1-v}C(tv). \tag{8}$$

For H_1 we can write

$$H_1(x,q,t,v) = \sum_{\substack{n\geq 0\\ k\geq -n-1}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_{-k}(uAd)} q^{\mathrm{lt}_{-k}(uAd)} v^k t^{n+1}$$

$$= t\left[\sum_{\substack{n\geq 0\\ k>0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_{-k}(uAd)} q^{\mathrm{lt}_{-k}(uAd)} v^k t^n + \sum_{n\geq 0} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_0(uAd)} q^{\mathrm{lt}_0(uAd)} t^n \right. \tag{9}$$

$$\left. + \sum_{\substack{n\geq 0\\ 0<r\leq n+1}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_r(uAd)} q^{\mathrm{lt}_r(uAd)} v^{-r} t^n \right].$$

For $k>0$, $\mathrm{ct}_{-k}(uAd) = \mathrm{ct}_{-k}(A)$ and $\mathrm{lt}_{-k}(uAd) = \mathrm{lt}_{-k}(A)+1$, so the first sum on the right hand side of (9) equals

$$q\sum_{\substack{n\geq 0\\ k>0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_{-k}(A)} q^{\mathrm{lt}_{-k}(A)} v^k t^n = q\left[\sum_{\substack{n\geq 0\\ k\geq 0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_{-k}(A)} q^{\mathrm{lt}_{-k}(A)} v^k t^n \right.$$

$$\left. - \sum_{n\geq 0} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_0(A)} q^{\mathrm{lt}_0(A)} t^n \right] = q\left[G(\frac{x}{q},\frac{1}{q},qt,v) - G(x,q,t,0)\right],$$

by (7). For the second sum in (9), note that $\mathrm{ct}_0(uAd) = \mathrm{ct}_0(A)+1$ and $\mathrm{lt}_0(uAd) = \mathrm{lt}_0(A)$, so the sum equals

$$x\sum_{n\geq 0} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_0(A)} q^{\mathrm{lt}_0(A)} t^n = xG(x,q,t,0).$$

Using that for $r>0$ $\mathrm{ct}_r(uAd) = \mathrm{ct}_r(A)$ and $\mathrm{lt}_r(uAd) = \mathrm{lt}_r(A)$, the third sum in (9) can be written as

$$\sum_{\substack{n\geq 0\\ r>0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_r(A)} q^{\mathrm{lt}_r(A)} v^{-r} t^n - \sum_{\substack{n\geq 0\\ r>n+1}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_r(A)} q^{\mathrm{lt}_r(A)} v^{-r} t^n$$

$$= G(x,q,t,v^{-1}) - G(x,q,t,0) - \frac{1}{v(v-1)}C(tv^{-1}),$$

by (8). Thus,

$$H_1(x,q,t,v) = \tag{10}$$

$$= t\left[qG(\frac{x}{q},\frac{1}{q},qt,v) + (x-q-1)G(x,q,t,0) + G(x,q,t,\frac{1}{v}) + \frac{1}{v(1-v)}C(\frac{t}{v})\right].$$

For H_2, a very similar reasoning implies that

$$H_2(x,q,t,v) = G(x,q,t,v) - G(x,q,t,0) + G(\frac{x}{q},\frac{1}{q},qt,\frac{1}{v}) + \frac{1}{1-v}C(\frac{qt}{v}). \tag{11}$$

Finally, for H_3 we get that

$$H_3(x,q,z,y) = \tag{12}$$

$$= G(x,q,z,y) + G(\frac{x}{q},\frac{1}{q},qz,\frac{1}{y}) - G(x,q,z,0) - \frac{y}{1-y}C(zy) + \frac{1}{1-y}C(\frac{qz}{y}) - 1.$$

Substituting these expressions for H_1, H_2 and H_3 in (6) we obtain an equation for G. Note that the common factor t in $H_1(x,q,t,v)$ guarantees that this equation will express the coefficients of the series expansion in z of $H_3(x,q,z,y)$ in terms of coefficients of G of smaller order in the series expansion in t of $H_1(x,q,t,v)H_2(x,q,t,v)$, so it uniquely determines G as a GF. The final step of the proof is to guess an expression for G and check that it satisfies this equation.

Proposition 4.2. *We have*

$$G(x,q,t,v)=\frac{\dfrac{1-v+(q-1)tvC(tv)}{1-v+(q-1)tvF_{321}(1,q,t)}-(x-1)tvC(tv)}{[1-qt(F_{321}(1,q,t)-1)-xt](1-v)}. \tag{13}$$

Before proving this proposition, we observe that it implies theorem 2.1. Indeed, we have by definition

$$G(x,q,t,0)=\sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^{\mathrm{ct}_0(D)}q^{\mathrm{lt}_0(D)}t^n=F_{132}(x,q,t).$$

But if 4.2 holds, then

$$G(x,q,t,0)=\frac{1}{1-qt(F_{321}(1,q,t)-1)-xt}=F_{321}(x,q,t),$$

where the last equality follows from (1). So, all that remains is to prove Proposition 4.2.

Proof. The computations that follow have been done using *Maple*. Let $\widetilde{H}_1$, $\widetilde{H}_2$ and $\widetilde{H}_3$ be the expressions obtained respectively from (10), (11) and (12) when G is substituted with the expression given in (13). All we have to check is that

$$\mathrm{diag}^z_{v,t}\ \widetilde{H}_1(x,q,t,\frac{v}{y})\widetilde{H}_2(x,q,ty,v)=\widetilde{H}_3(x,q,z,y).$$

Let $\widetilde{P}(x,q,t,v,y):=\widetilde{H}_1(x,q,t,\frac{v}{y})\widetilde{H}_2(x,q,ty,v)$. We want to compute $\mathrm{diag}^z_{v,t}\ \widetilde{P}$. In [15, chapter 6], a general method is described for obtaining diagonals of rational functions. This theory does not apply to our function $\widetilde{P}$, because it is not rational. However, we will show that in this particular case we can modify the technique to obtain $\mathrm{diag}^z_{v,t}\ \widetilde{P}$.

The series expansion of $\widetilde{P}$ in v and t,

$$\widetilde{P}(x,q,t,v,y)=\sum_{\substack{n\geq 0\\ j\geq -n}}\widetilde{P}_{j,n}(x,q,y)v^jt^n=\sum_{n,i\geq 0}\widetilde{P}_{i-n,n}(x,q,y)v^i\left(\frac{t}{v}\right)^n,$$

converges for $|v|<\beta$, $|\frac{t}{v}|<\alpha$, if $\alpha,\beta>0$ are taken sufficiently small. Similarly,

$$\mathrm{diag}^z_{v,t}\ \widetilde{P}=\sum_{n\geq 0}\widetilde{P}_{n,n}(x,q,y)z^n$$

converges for $|z|$ sufficiently small. Fix such a small z with $|z|<\alpha\beta^2$. The series

$$\widetilde{P}(x,q,t,\frac{z}{t},y)=\sum_{\substack{n\geq 0\\ j\geq -n}}\widetilde{P}_{j,n}(x,q,y)z^jt^{n-j}$$

will converge for $|\frac{z}{t}|<\beta$ and $|\frac{t^2}{z}|<\alpha$. Regarded as a function of t, it will converge for $|t|$ in the annulus $\frac{|z|}{\beta}<|t|<\sqrt{\alpha|z|}$, which is non-empty because $|z|<\alpha\beta^2$. In particular, it converges on some circle $|t|=\rho$ in the annulus. By [6, Theorem 1],

$$\mathrm{diag}^z_{v,t}\ \widetilde{P}=\frac{1}{2\pi i}\int_{|t|=\rho}\widetilde{P}(x,q,t,\frac{z}{t},y)\frac{dt}{t}.$$

It can be checked that the singularities of $\widetilde{P}(x,q,t,\frac{z}{t},y)/t$ (as a function of t) that lie inside the circle $|t|=\rho$ are all simple poles. These poles are

$$t_1=0,\ t_2=z,\ t_3=\frac{z}{y},\ t_{4,5}=\frac{(1+q)y\pm(1-q)\sqrt{y(y-4qz)}}{2y(y+z(1-q)^2)}z,$$

$$t_{6,7}=\frac{1+q\pm(1-q)\sqrt{1-4zy}}{2(q+zy(1-q)^2)}z.$$

There are also branch points for $t=\pm\frac{1}{2}\sqrt{\frac{z}{y}}$ and $t=\pm\frac{1}{2}\sqrt{\frac{z}{qy}}$, but they lie outside the circle for an appropriate choice of ρ in the annulus $\frac{|z|}{\beta}<\rho<\sqrt{\alpha|z|}$. The remaining singularities do not depend on z and lie outside the circle.

So, by the residue theorem, the integral can be obtained by summing up the residues at the poles inside $|t|=\rho$. Computing them in *Maple*, we see that all the residues are 0 except for those in t_2 and t_3. Thus,

$$\mathrm{diag}^z_{v,t}\ \widetilde{P}=\mathrm{Res}_{t=z}\ \widetilde{P}(x,q,t,\frac{z}{t},y)\frac{1}{t}+\mathrm{Res}_{t=\frac{z}{y}}\ \widetilde{P}(x,q,t,\frac{z}{t},y)\frac{1}{t},$$

and this turns out to be precisely $\widetilde{H}_3(x,q,z,y)$. □

5. Some other bijections involving $\mathcal{S}_n(321)$ and $\mathcal{D}_n$

Looking at permutations as arrays of crosses, as we did to define Φ_{L}, some other known bijections between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$ can easily be viewed in a systematic way, as paths with down and right steps from the upper-left corner to the lower-right corner of the $n\times n$ array. For each of these bijections, our canonical example is $\pi=23147586$. One such bijection was established by Billey, Jockusch and Stanley in [1, p. 361]. Denote it by Ψ_{L}. Consider the path that leaves the crosses corresponding to excedances to the right, and stays always as far from the main diagonal as possible (Figure 8). Then $\Psi_{\text{L}}(\pi)$ can be obtained from it just by reading an up-step every time the path moves to the right and a down-step every time the path moves down.

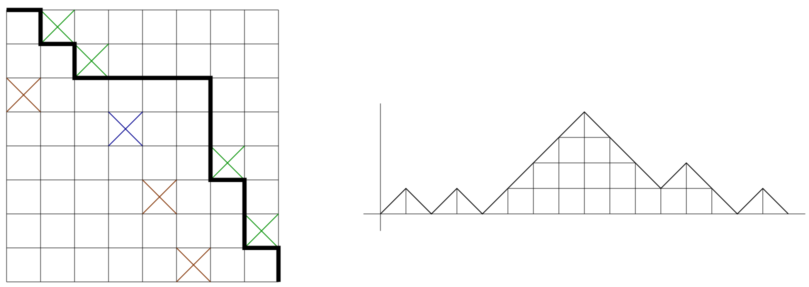

FIGURE 8. The bijection Ψ_{L}.

In [8], Krattenthaler describes a bijection from $\mathcal{S}_n(123)$ to $\mathcal{D}_n$. If we omit the last step, consisting of reflecting the path over a vertical line, and compose the bijection with the reversal operation, that maps a permutation $\pi_1\pi_2\cdots\pi_n$ into $\pi_n\cdots\pi_2\pi_1$, we get a bijection from $\mathcal{S}_n(321)$ to $\mathcal{D}_n$. Denote it by $\Phi_{\neg}$. In the array representation, $\Phi_{\neg}(\pi)$ corresponds (by the same trivial transformation as before) to the path that leaves all the crosses to the left and remains always as close to the main diagonal as possible (Figure 9).

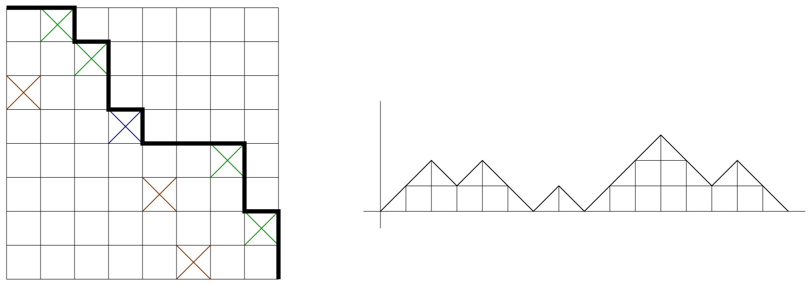

FIGURE 9. The bijection $\Phi_{\neg}$.

Our first bijection is related to this last one by $\Phi_{\neg}(\pi) = \Phi_{\llcorner}(\pi^{-1})$. In a similar way, we could still define a fourth bijection $\Psi_{\neg} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ by $\Psi_{\neg}(\pi) := \Psi_{\llcorner}(\pi^{-1})$ (Figure 10).

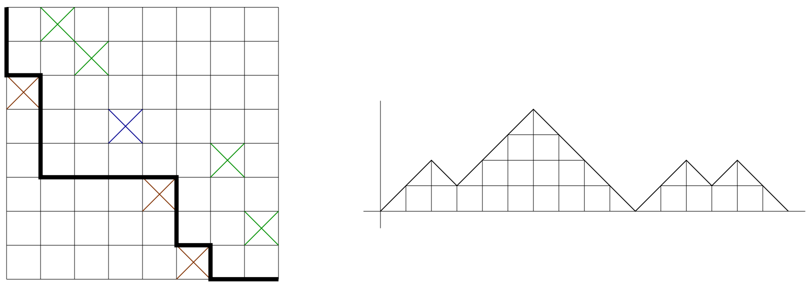

FIGURE 10. The bijection $\Psi_{\neg}$.

Combining these bijections and their inverses, one can get some automorphisms on Dyck paths and on 321-avoiding permutations with interesting properties. Recall that a *valley* of a Dyck path D is a down-step followed by an up-step (du in the Dyck word). Denote by $\mathrm{va}(D)$ the number of valleys of D. Denote by $p_2(D)$ the number of peaks of D of height at least 2. Clearly, both $p_2(D) + h(D)$ and $\mathrm{va}(D) + 1$ equal the total number of peaks of D.

It can be checked that $\Phi_{\llcorner} \circ \Psi_{\llcorner}^{-1}$ is an involution on $\mathcal{D}_n$ with the property that $\mathrm{va}(\Phi_{\llcorner} \circ \Psi_{\llcorner}^{-1}(D)) = \mathrm{dr}(D)$ and $\mathrm{dr}(\Phi_{\llcorner} \circ \Psi_{\llcorner}^{-1}(D)) = \mathrm{va}(D)$. Indeed, this follows from the fact that excedances are sent to valleys by $\Psi_{\llcorner}$ and to double rises by $\Phi_{\llcorner}$. This bijection gives yet another proof of the symmetry of the bivariate distribution of the pair $(\mathrm{va}, \mathrm{dr})$ of statistics in Dyck paths. A different involution with this property was introduced in [2].

Another involution on $\mathcal{D}_n$ is given by $\Phi_{\llcorner} \circ \Phi_{\neg}^{-1}$. This one shows the symmetry of the distribution of the pair (dr, p_2), because $\mathrm{dr}(\Phi_{\llcorner} \circ \Phi_{\neg}^{-1}(D)) = p_2(D)$ and $p_2(\Phi_{\llcorner} \circ \Phi_{\neg}^{-1}(D)) = \mathrm{dr}(D)$. In addition, it preserves the number of hills, i.e., $h(\Phi_{\llcorner} \circ \Phi_{\neg}^{-1}(D)) = h(D)$. To see this, just note that both $\Phi_{\neg}$ and $\Phi_{\llcorner}$ send fixed points to hills, whereas excedances are sent to peaks of height at least 2 by $\Phi_{\neg}$ and to double rises by $\Phi_{\llcorner}$.

On the other hand, the involution on $\mathcal{S}_n(321)$ that maps π to $(\Psi_{\llcorner}^{-1}(\Phi_{\llcorner}(\pi)))^{-1}$ gives a combinatorial proof of the fact that the number of 321-avoiding permutations with k excedances equals the number of 321-avoiding permutations with with $k+1$ weak excedances (recall that i is a *weak excedance* of π if $\pi_i \geq i$). The analogous result for general permutations is well known. An implication of Theorem 2.1 is that this result is also true for 132-avoiding permutations.

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Note. A bijective proof of Theorem 2.1 has recently been found by the author and Igor Pak. A preprint of this result appears in [4].

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