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Measuring symmetry in lattice paths and partitions

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Abstract. We introduce the notion of *degree of symmetry* for lattice paths and related combinatorial objects. The degree of symmetry measures how symmetric an object is, usually ranging from zero (completely asymmetric) to its size (completely symmetric). We study the behavior of this statistic on Dyck paths and grand Dyck paths, where the symmetry is given by reflection along a vertical line through their midpoint; partitions, where the symmetry is given by conjugation; and certain compositions interpreted as bargraphs. We find expressions for the generating functions for these objects with respect to their degree of symmetry, and their semilength or semiperimeter. The generating functions are algebraic in most cases, with the notable exception of Dyck paths, for which we apply techniques from walks in the plane to find a functional equation for the generating function, and conjecture that it is D-finite but not algebraic.

Keywords: symmetry, lattice path, Dyck path, partition, composition, D-finite

1 Introduction

For combinatorial objects with a standard reflection operation, it is natural to study the subset of those that are *symmetric*, that is, invariant under such reflection. Examples of symmetric combinatorial objects include symmetric Dyck paths [4] and grand Dyck paths, self-conjugate partitions [14, Prop. 1.8.4], palindromic compositions [8], and symmetric binary trees, all of which have been considered.

In this abstract we refine the concept of symmetric objects by introducing a type of combinatorial statistic that we call the *degree of symmetry*, which measures how close the object is to being symmetric. The notion of degree of symmetry, suggested by Emeric Deutsch (personal communication, March 2018) appears to be new. In some instances, it is related to other statistics studied in the literature, such as the number of *centered tunnels* in Dyck paths (introduced in [6]) or the number of transpositions in permutations.

Let us start by defining the degree of symmetry of certain lattice paths. Let $\mathcal{GD}_n$ be the set of all lattice paths in the plane with up-steps $U = (1,1)$ and down-steps $D = (1,-1)$ from $(0,0)$ to $(2n,0)$. These are called *grand Dyck paths* (also referred to as

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free Dyck paths or *bridges* in the literature), and n is called the *semilength*. Let $\mathcal{D}_n$ be the subset of those that do not go below the x-axis. These are called *Dyck paths*. We use the notation $[n] = \{1, 2, \dots, n\}$.

Given a path $P \in \mathcal{GD}_n$, we view its steps as segments in the plane, which we denote by $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_{2n}$ from left to right. For example, $\bar{p}_1$ has endpoints $(0, 0)$ and $(1, \pm 1)$, and $\bar{p}_{2n}$ has endpoints $(2n - 1, \pm 1)$ and $(2n, 0)$. For $i \in [n]$, we say that P is symmetric in position i (or that $\bar{p}_i$ is a *symmetric step*) if $\bar{p}_i$ and $\bar{p}_{2n+1-i}$ are mirror images of each other with respect to the reflection along the vertical line $x = n$. We define the *degree of symmetry* of P, denoted by $\mathrm{ds}(P)$, as the number of $i \in [n]$ such that P is symmetric in position i. See Figure 1 for an example.

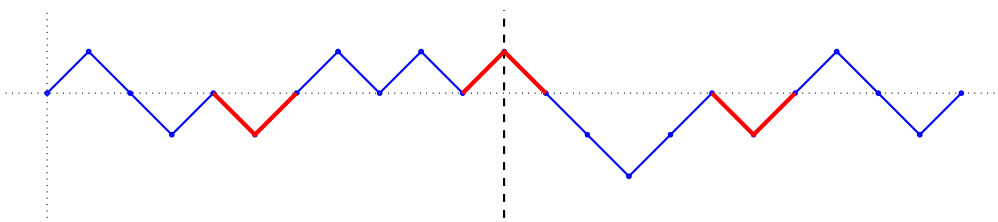

Figure 1: A grand Dyck path with degree of symmetry 3. The symmetric steps and their mirror images are highlighted in red.

In Section 2 we find an expression for the generating function of grand Dyck paths with respect to semilength and degree of symmetry. Section 3 deals with the analogous problem for Dyck paths, which is significantly harder. Using bijections for walks in the quarter plane, we derive a functional equation for the corresponding generating function. Section 4 focuses on the degree of symmetry of partitions and certain unimodal compositions, giving generating functions with respect to a few possible definitions of size and degree of symmetry. The proofs of most results in this extended abstract are omitted or only sketched.

2 Grand Dyck paths

For grand Dyck paths we obtain a surprisingly simple generating function.

Theorem 2.1. *The generating function for grand Dyck paths with respect to their degree of symmetry is*

$$\sum_{n\ge 0}\sum_{P\in\mathcal{GD}_n} s^{\mathrm{ds}(P)} z^n = \frac{1}{2(1-s)z + \sqrt{1-4z}}.$$

The proof of this result relies on a bijection to another kind of lattice paths. A *bicolored grand Motzkin path* starts at the origin, ends on the x-axis, and can have steps $U = (1, 1)$,

$D = (1,-1)$, and horizontal steps $(1,0)$ of two kinds (or colors), denoted by H_1 and H_2. If such a path does not go below the x-axis, it is called a *bicolored Motzkin path*. Denote by $\mathcal{GM}^2$ the set of bicolored grand Motzkin paths, and by $\mathcal{M}^2$ the set of bicolored Motzkin paths. We often identify a path with its sequence of steps. For a path $M \in \mathcal{GM}^2$, let $u(M)$ denote its number of U steps (which also equals its number $d(M)$ of D steps), and define $h_1(M)$ and $h_2(M)$ analogously. Additionally, let $h_1^0(M)$ denote the number of H_1 steps of M on the x-axis, and define $h_2^0(M)$ similarly.

Define the *length* of a path M to be its total number of steps, which we denote by $|M|$. Let $\mathcal{M}_n^2 \subset \mathcal{M}^2$ and $\mathcal{GM}_n^2 \subset \mathcal{GM}^2$ denote the subsets consisting of paths of length n in each case. Let $G(x,y,s_1,s_2) = \sum_{M\in\mathcal{GM}^2} x^{d(M)+h_1(M)} y^{u(M)+h_2(M)} s_1^{h_1^0(M)} s_2^{h_2^0(M)}$.

Lemma 2.2. *We have*

$$G(x,y,s_1,s_2) = \frac{1}{(1-s_1)x + (1-s_2)y + \sqrt{(1-x-y)^2 - 4xy}}.$$

To prove Theorem 2.1, given $P \in \mathcal{GD}_n$, construct two paths as follows. Let P_L denote the left half of P, and let P_R be the path obtained by reflecting the right half of P along the vertical line $x = n$. Note that P_L and P_R are paths with steps U and D from $(0,0)$ to some common endpoint on the line $x = n$. Denote the ith step of P_L by $\bar{\ell}_i$ when viewed as a segment in the plane, and let $\ell_i \in \{U, D\}$ be the direction of this step. Define $\bar{r}_i$ and r_i similarly for the path P_R.

Next we describe a bijection ϕ from $\mathcal{GD}_n$ to $\mathcal{GM}_n^2$; see Figure 2 for an example. For $P \in \mathcal{GD}_n$ with the above notation, let $\phi(P) \in \mathcal{GM}_n^2$ be the path whose ith step is equal to

$$\begin{cases} U & \text{if } \ell_i = U \text{ and } r_i = D, \\ D & \text{if } \ell_i = D \text{ and } r_i = U, \\ H_1 & \text{if } \ell_i = r_i = D, \\ H_2 & \text{if } \ell_i = r_i = U. \end{cases}$$

Lemma 2.3. *The map $\phi : \mathcal{GD}_n \to \mathcal{GM}_n^2$ is a bijection with the property that, if $M = \phi(P)$, then* $\mathrm{ds}(P) = h_1^0(M) + h_2^0(M)$.

Proof of Theorem 2.1. Combining Lemmas 2.3 and 2.2,

$$\sum_{n\geq 0} \sum_{P\in\mathcal{GD}_n} s^{\mathrm{ds}(P)} z^n = \sum_{M\in\mathcal{GM}^2} s^{h_1^0(M)+h_2^0(M)} z^{|M|} = G(z,z,s,s) = \frac{1}{2(1-s)z + \sqrt{1-4z}}.$$

□

When P_L lies strictly above P_R (except at their common endpoints), the pair (P_L, P_R) is called a *parallelogram polyomino* [11], and its semiperimeter is defined to be the length of

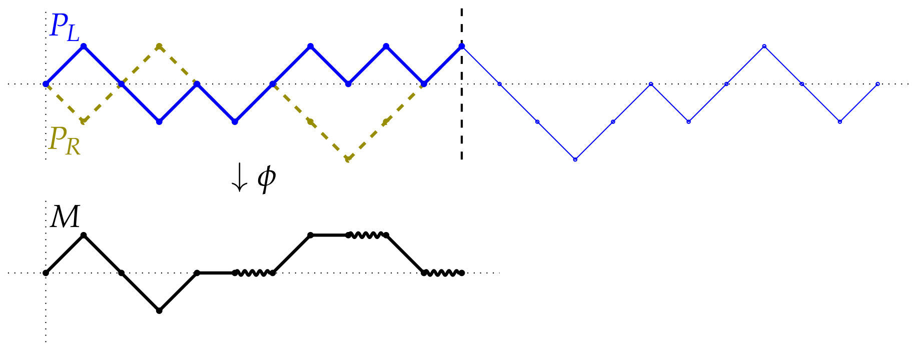

Figure 2: The bijection $\phi : \mathcal{GD}_n \to \mathcal{GM}_n^2$. The path $P \in \mathcal{GD}_n$ is drawn in blue, and its reflected right half P_R is drawn in olive color with dahes. The steps H_2 in $\phi(P)$ are drawn with wavy lines.

either of the two paths. It is well known that parallelogram polyominos of semiperimeter n are counted by the Catalan number C_{n-1}. This also follows follows from our bijection and Lemma 2.2.

An alternative measure of the symmetry of a grand Dyck path is its *number of symmetric vertices*, that is, vertices in the first half of the path that are mirror images of vertices in the second half. We do not consider the midpoint itself as a symmetric vertex. Denote by $\mathrm{sv}(P)$ the number of symmetric vertices of the path $P \in \mathcal{GD}_n$, and let

$$C(z) = \sum_{n\geq 0} C_n z^n = \frac{1-\sqrt{1-4z}}{2z}.$$

Theorem 2.4. *The generating function for grand Dyck paths with respect to their number of symmetric vertices is*

$$\sum_{n\geq 0} \sum_{P\in\mathcal{GD}_n} v^{\mathrm{sv}(P)} z^n = \frac{1}{1-2vzC(z)} = \frac{1}{1-v+v\sqrt{1-4z}}.$$

For $P \in \mathcal{GD}_n$, denote by $\mathrm{ret}(P)$ the number of returns of P to the x-axis. It is easy to see that the generating function in Theorem 2.4 also enumerates grand Dyck paths with respect to this statistic; see [13, A108747]. The following result also has a bijective proof.

Corollary 2.5. *The statistics* sv *and* ret *are equidistributed on* $\mathcal{GD}_n$*; that is, for all* $n, k \geq 0$,

$$|\{P \in \mathcal{GD}_n : \mathrm{sv}(P) = k\}| = |\{P \in \mathcal{GD}_n : \mathrm{ret}(P) = k\}|.$$

3 Dyck paths

Let

$$D(s,z) = \sum_{n\geq 0} \sum_{P\in\mathcal{D}_n} s^{\mathrm{ds}(P)} z^n$$

denote the generating function for Dyck paths with respect to their degree of symmetry. In contrast to the simplicity of the generating function in Theorem 2.1 for grand Dyck paths, the generating function $D(s,z)$ is unwieldy. We will first rephrase the problem in terms of walks in the plane, and then apply some transformations on the walks that will allow us to obtain a functional equation for a refinement of $D(s,z)$.

Let $\mathcal{W}_n^1$ denote the set of walks in the first quadrant $\{(x,y) : x,y \geq 0\}$ starting at the origin, ending on the diagonal $y = x$, and having n steps in $\{\mathrm{NE}, \mathrm{NW}, \mathrm{SE}, \mathrm{SW}\}$, where we use the notation $\mathrm{NE} = (1,1)$, $\mathrm{NW} = (-1,1)$, $\mathrm{SE} = (1,-1)$, $\mathrm{SW} = (-1,-1)$.

We start by describing a standard bijection $\omega : \mathcal{D}_n \to \mathcal{W}_n^1$, which, in a similar form, has been used in [5, 7, 2]. Given $P \in \mathcal{D}_n$, first define two paths as in Section 2: P_L is the left half of P, and P_R is the path obtained by reflecting the right half of P along the vertical line $x = n$. Denote the ith step of P_L, P_R by $\ell_i, r_i \in \{U, D\}$, respectively. Now let $\omega(P) \in \mathcal{W}_n^1$ be the walk whose ith step is equal to

$$\begin{cases} \mathrm{NE} & \text{if } \ell_i = r_i = U, \\ \mathrm{NW} & \text{if } \ell_i = U \text{ and } r_i = D, \\ \mathrm{SE} & \text{if } \ell_i = D \text{ and } r_i = U, \\ \mathrm{SW} & \text{if } \ell_i = r_i = D. \end{cases}$$

Under this bijection, symmetric steps of P, which correspond to common steps of P_L and P_R, become steps of $\omega(P)$ lying entirely on the diagonal $y = x$. An equation for the generating function of walks with a variable keeping track of the number of such steps would be troublesome, since it would contain a term corresponding to walks ending on the diagonal $y = x$, which would involve taking diagonals of generating functions.

To circumvent this problem, we modify the walks so that the steps that we need to keep track of lie on the boundary of the region. Folding walks in $\mathcal{W}_n^1$ along the diagonal $y = x$ (reflecting the steps above the diagonal onto steps below it), we obtain walks in the first octant $\{(x,y) : x \geq y \geq 0\}$. In order not to lose information while folding, we allow the resulting walks in the octant to use two colors for steps SE leaving the diagonal. These colors keep track of whether the portion of the walk between the colored step and the next return to the diagonal was above or below the diagonal on the original quadrant walk. We obtain a bijection between $\mathcal{W}_n^1$ and the set $\mathcal{W}_n^2$ of walks in the first octant starting at the origin, ending on the diagonal $y = x$, having n steps in $\{\mathrm{NE}, \mathrm{NW}, \mathrm{SE}, \mathrm{SW}\}$, and where steps SE leaving the diagonal $y = x$ can have two colors.

Next we apply the linear transformation $(x,y) \mapsto (y, \frac{x-y}{2})$ from the first octant to the first quadrant. This transformation gives a bijection between $\mathcal{W}_n^2$ and the set $\mathcal{W}_n^3$ of walks in the first quadrant starting at the origin, ending on the x-axis, having n steps in $\{E, W, \text{NW}, \text{SE}\}$ (denoting $E = (1,0)$ and $W = (-1,0)$), and where steps NW leaving the x-axis can have 2 colors. Under this bijection, steps of walks in $\mathcal{W}_n^2$ lying on the diagonal become steps of walks in $\mathcal{W}_n^3$ lying on the x-axis. Table 1 summarizes the above sequence of bijections from $\mathcal{D}_n$ to $\mathcal{W}_n^3$. It follows that $D(s,z)$ is the generating function for walks in $\mathcal{W}_n^3$ where z marks the length and s marks the number of steps on the x-axis.

	$\mathcal{D}_n$	$\mathcal{W}_n^1$	$\mathcal{W}_n^2$	$\mathcal{W}_n^3$
walks in	first octant	first quadrant	first octant	first quadrant
allowed steps	$U = \text{NE}, D = \text{SE}$	NE, NW, SE, SW	NE, NW, SE, SW	E, W, NW, SE
length	$2n$	n	n	n
ending on	x-axis	diagonal	diagonal	x-axis
2 colors for			SE leaving diagonal	NW leaving x-axis
ds counts	symmetric steps	steps on diagonal	steps on diagonal	steps on x-axis

Table 1: A summary of the bijections between Dyck paths and walks.

To enumerate walks in $\mathcal{W}_n^3$, we consider more general walks where any endpoint is allowed. Let $W(x,y,s,z)$ be the generating function where the coefficient of $x^i y^j s^k z^n$ is the number of walks in the first quadrant with n steps in $\{E, W, \text{NW}, \text{SE}\}$, starting at the origin, ending at (i,j), having k steps entirely on the x-axis, and where steps NW leaving the x-axis can have 2 colors. By considering the different possibilities for the last step of the walk, we obtain the following functional equation for $W(x,y) := W(x,y,s,z)$.

$$\begin{aligned} W(x,y) = 1 + z\left(x + \frac{1}{x} + \frac{x}{y} + \frac{y}{x}\right) W(x,y) - z\left(\frac{1}{x} + \frac{y}{x}\right) W(0,y) - z\,\frac{x}{y} W(x,0) \\ + z\,\frac{y}{x}\left(W(x,0) - W(0,0)\right) + z\,(s-1)\left(x + \frac{1}{x}\right) W(x,0) - z\,(s-1)\frac{1}{x} W(0,0). \end{aligned}$$

Theorem 3.1. *The generating function for Dyck paths with respect to the statistic* ds *is* $D(s,z) = W(1,0,s,z)$, *where* $W(x,y) := W(x,y,s,z)$ *satisfies the functional equation*

$$\begin{aligned} \left(xy - z(y+x^2)(1+y)\right) W(x,y) = xy - zy(1+y)W(0,y) \\ + z\left(y^2 - x^2 + (s-1)y(x^2+1)\right) W(x,0) - zy(y+s-1)W(0,0). \end{aligned}$$

Even though we have been unable to solve this functional equation using the kernel method, the equation suggests that the generating function $D(s,z) = W(1,0,s,z)$ is D-finite. Computations by Alin Bostan (personal communication, July 2019) using Theorem 3.1 have led to the following conjecture.

Conjecture 3.2. *The generating function* $D(s,z)$ *is D-finite in* z *but not algebraic. Specifically, it satisfies a linear differential equation of order 5 with polynomial coefficients of maximum degree* 27.

The coefficients of $s^k z^n$ for small values of k and n in the generating functions from Theorems 2.1 and 3.1 are given in Table 2.

$|\{P \in \mathcal{GD}_n : \mathrm{ds}(P) = k\}|$

$n \setminus k$	0	1	2	3	4	5	6
1	0	2					
2	2	0	4				
3	4	8	0	8			
4	14	16	24	0	16		
5	44	64	48	64	0	32	
6	148	208	216	128	160	0	64

$|\{P \in \mathcal{D}_n : \mathrm{ds}(P) = k\}|$

$n \setminus k$	1	2	3	4	5	6
1	1					
2	0	2				
3	2	0	3			
4	2	6	0	6		
5	8	8	16	0	10	
6	16	32	24	40	0	20

Table 2: The number of grand Dyck paths (left, see Theorem 2.1) and Dyck paths (right, see Theorem 3.1) of semilength $n \leq 6$ with a given degree of symmetry.

Note the surprising contrast between the simple algebraic generating function for grand Dyck paths in Theorem 2.1 and the complicated one for Dyck paths in Theorem 3.1.

4 Partitions

Let $\mathcal{P}$ denote the set of integer partitions, that is, sequences $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ where $k \geq 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. We draw the Young diagram of λ in English notation, by arranging boxes (unit squares) into k left-justified rows, where the ith row from the top has λ_i boxes for each i. The *conjugate* of λ, denoted by λ', is the partition defined by $\lambda'_i = |\{j : \lambda_j \geq i\}|$ for $1 \leq i \leq \lambda_1$. The Young diagram of λ' is obtained by transposing the Young diagram of λ.

Each one of the following subsections considers a different measure of the symmetry of a partition. The first one views partitions inside a square and relates them to grand Dyck paths. The second one is perhaps the most natural measure of symmetry: the number of parts that equal the corresponding part in the conjugate partition. The third measure involves a decomposition of partitions into diagonal hooks.

4.1 Partitions inside a square

Let $\mathcal{P}_n^{\square} \subset \mathcal{P}$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with $k \leq n$ and $\lambda_1 \leq n$. These can be thought of as partitions whose Young diagram fits inside an $n \times n$ square. For such $\lambda \in \mathcal{P}_n^{\square}$, let $\tilde{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_k, 0, \dots, 0)$ denote the sequence of length n obtained by appending $n-k$ zeros to λ. Let $\tilde{\lambda}'$ be the sequence of length n obtained by conjugating $\tilde{\lambda}$, that is, $\tilde{\lambda}'_i = |\{j : \tilde{\lambda}_j \geq i\}|$ for $1 \leq i \leq n$.

Viewing λ as a partition inside a square, one can define the following measure of symmetry, where the zeros in $\tilde{\lambda}$ are allowed to contribute. Let

$$\mathrm{ds}_n^{\square}(\lambda) = |\{i \in [n] : \tilde{\lambda}_i = \tilde{\lambda}'_i\}|.$$

For example, if $\lambda = (5,4,4,2,1,1)$, then $\mathrm{ds}_6^{\square}(\lambda) = 2$ but $\mathrm{ds}_7^{\square}(\lambda) = 3$, since in the second case, $\tilde{\lambda} = (5,4,4,2,1,1,0)$ and $\tilde{\lambda}' = (6,4,3,3,1,0,0)$ coincide in positions 2, 5, and 7.

To relate partitions inside a square and grand Dyck paths, we define a straightforward bijection $\partial_n : \mathcal{P}_n^{\square} \to \mathcal{GD}_n$ by $\partial_n(\lambda) = D^{\tilde{\lambda}_n} U D^{\tilde{\lambda}_{n-1}-\tilde{\lambda}_n} U D^{\tilde{\lambda}_{n-2}-\tilde{\lambda}_{n-1}} U \ldots D^{\tilde{\lambda}_1-\tilde{\lambda}_2} U D^{n-\tilde{\lambda}_1}$.

This bijection can be visualized by placing the Young diagram of λ inside an $n \times n$ square (aligned with the top and left edges), reading the south-east boundary of the diagram from the south-west corner of the square to the north-east corner, and then translating north steps to U steps and east steps to D steps. The following is a consequence of Theorem 2.1.

Corollary 4.1. *The generating function for partitions whose Young diagram fits inside a square with respect to the side length of the square and the statistic* $\mathrm{ds}_n^{\square}$ *is*

$$\sum_{n\geq 0} \sum_{\lambda \in \mathcal{P}_n^{\square}} s^{\mathrm{ds}_n^{\square}(\lambda)} z^n = \frac{1}{2(1-s)z + \sqrt{1-4z}}.$$

4.2 Symmetry by self-conjugate parts

Another notion of symmetry, which we call simply the degree of symmetry of $\lambda \in \mathcal{P}$, is defined as

$$\mathrm{ds}(\lambda) = |\{i : \lambda_i = \lambda'_i\}|,$$

that is, the number of parts of λ that equal the corresponding parts of its conjugate. For example, if $\lambda = (5,4,4,2,1,1)$, then $\mathrm{ds}(\lambda) = 2$ because $\lambda_2 = \lambda'_2 = 4$ and $\lambda_5 = \lambda'_5 = 1$, but $\lambda_i \neq \lambda'_i$ for every other i for which these quantities are defined.

The following is a consequence of Corollary 4.1 and the fact that $\mathrm{ds}_m^{\square}(\lambda) = \mathrm{ds}(\lambda)$ when $m = \max\{\lambda_1, \lambda'_1\}$.

Corollary 4.2. *The generating function for partitions with respect to the side length of the smallest square containing their Young diagram and their degree of symmetry is*

$$\sum_{\lambda \in \mathcal{P}} s^{\mathrm{ds}(\lambda)} z^{\max\{\lambda_1, \lambda'_1\}} = \frac{1-sz}{2(1-s)z + \sqrt{1-4z}}.$$

For $\lambda \in \mathcal{P}$, let $\mathrm{sp}(\lambda) = \lambda_1 + \lambda'_1$ denote the semiperimeter of its Young diagram.

Theorem 4.3. *The generating function for partitions with respect to their semiperimeter and their degree of symmetry is*

$$\sum_{\lambda\in\mathcal{P}} s^{\mathrm{ds}(\lambda)} z^{\mathrm{sp}(\lambda)} = 1 + \frac{z^2\left((1-s)(1-2z)-\sqrt{1-4z^2}\right)}{(2z-1)\left(2(1-s)z^2+\sqrt{1-4z^2}\right)}. \tag{4.1}$$

It is interesting to note that, while the generating function for partitions by semiperimeter is rational, namely $\frac{z^2}{1-2z}$ (setting $s=1$ in Equation (4.1)), the generating function by semiperimeter and degree of symmetry is not.

4.3 Symmetry by self-conjugate hooks

Next we consider a third notion of symmetry for partitions. As in [1], the boxes in the Young diagram of $\lambda \in \mathcal{P}$ can be decomposed into *diagonal hooks* as follows: the first hook is the largest hook, consisting of the first row and the first column; the second hook is the largest hook after the first hook has been removed, and so on. The number of hooks in this decomposition equals the largest δ such that $\lambda_\delta \geq \delta$ (also known as the side length of Durfee square of λ). Let $\mathrm{ds}^\ulcorner(\lambda)$ be the number of diagonal hooks in the Young diagram of λ that are self-conjugate, that is, they have the same number of boxes in the row than in the column. For example, the partition $\lambda = (4,4,3,2,1)$ in Figure 3 has two symmetric diagonal hooks, and so $\mathrm{ds}^\ulcorner(\lambda) = 2$, whereas $\mathrm{ds}(\lambda) = 3$.

Proposition 4.4. *The generating functions for partitions with respect to the statistic* $\mathrm{ds}^\ulcorner$ *and the side length of any (in the first formula) or the smallest (in the second formula) square containing their Young diagram are*

$$\sum_{n\geq 0}\sum_{\lambda\in\mathcal{P}_n^\square} s^{\mathrm{ds}^\ulcorner(\lambda)} z^n = \frac{1}{(1-s)z+\sqrt{1-4z}},$$

$$\sum_{\lambda\in\mathcal{P}} s^{\mathrm{ds}^\ulcorner(\lambda)} z^{\max\{\lambda_1,\lambda_1'\}} = \frac{1-z}{(1-s)z+\sqrt{1-4z}}.$$

For $P \in \mathcal{GD}_n$, denote by $\mathrm{ph}_1(P)$ the number of peaks of P at height 1, that is, occurrences of UD whose middle vertex has y-coordinate equal to 1.

Corollary 4.5. *For* $n,k \geq 0$,

$$|\{\lambda\in\mathcal{P}_n^\square : \mathrm{ds}^\ulcorner(P)=k\}| = |\{P\in\mathcal{GD}_n : \mathrm{ph}_1(P)=k\}|.$$

Proof. Consider the following bijection $\psi : \mathcal{P}_n^\square \to \mathcal{GD}_n$, which is also used in [1, Lemma 3.5]. Given $\lambda \in \mathcal{P}_n^\square$, let δ be the number of hooks in its diagonal hook decomposition described above. For $1 \leq i \leq \delta$, let a_i (resp. ℓ_i) denote the arm length (resp. leg length) of the i-th diagonal hook, defined as the number of boxes in the top row (resp. left column) of the hook, not including the corner box. Let

$$\psi(\lambda) = D^{a_\delta} U^{\ell_\delta+1} D^{a_{\delta-1}-a_\delta} U^{\ell_{\delta-1}-\ell_\delta} \dots D^{a_1-a_2} U^{\ell_1-\ell_2} D^{n-a_1} U^{n-1-\ell_1}.$$

Then $\mathrm{ds}^{\ulcorner}(\lambda) = \mathrm{ph}_1(\psi(\lambda))$. Indeed, the peaks of $\psi(\lambda)$ occur at heights $1+\ell_\delta - a_\delta, 1+\ell_{\delta-1} - a_{\delta-1}, \dots, 1+\ell_1 - a_1$, and so $\mathrm{ph}_1(\psi(\lambda)) = |\{i : a_i = \ell_i\}| = \mathrm{ds}^{\ulcorner}(\lambda)$. Figure 3 shows an example of this construction. □

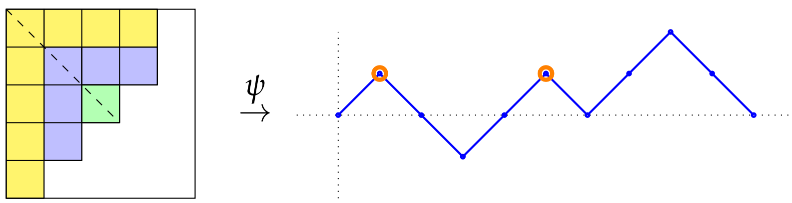

Figure 3: The bijection ψ applied to $\lambda = (4,4,3,2,1) \in \mathcal{P}_5^{\square}$. Here $\delta = 3$, $a_1 = 3$, $\ell_1 = 4$, $a_2 = \ell_2 = 2$, $a_3 = \ell_3 = 0$. The peaks at height 1 in $\phi(\lambda)$ are highlighted in orange.

4.4 Unimodal compositions

A composition is a sequence of positive integers $(a_1, a_2, \dots, a_k)$ for some $k \geq 1$. Its degree of symmetry is the number of indices $i \leq k/2$ such that $a_i = a_{k+1-i}$. Similarly to how partitions are represented as Young diagrams, compositions can be represented as *bargraphs*, by arranging boxes (unit squares) into k bottom-justified columns, where column i from the left has a_i boxes for each i; see Figure 4 for an example. Bargraphs have been studied in the literature as a special case of column-convex polyominoes (see e.g. [3, 12]), and they are used in statistical physics to model polymers.

A bargraph can be identified with the lattice path determined by its upper boundary, namely, a self-avoiding path with steps $N = (0,1)$, $E = (1,0)$ and $S = (0,-1)$ starting at the origin and returning to the x-axis only at the end. For a bargraph B, let $e(B)$ and $n(b)$ denote its number of E and N steps, respectively. The semiperimeter of B is defined as $\mathrm{sp}(B) = e(B) + n(B)$, and its degree of symmetry is defined as the degree of symmetry of the composition determined by its column heights, and denoted by $\mathrm{ds}(B)$.

Interpreting partitions as weakly decreasing compositions, it is natural to consider the related notion of unimodal compositions; equivalently, unimodal bargraphs. Let $\mathcal{U}$ denote the set of *unimodal bargraphs* with a centered maximum, defined as those whose column heights satisfy $1 \leq a_1 \leq \cdots \leq a_{\lfloor (k+1)/2 \rfloor}$ and $a_{\lceil (k+1)/2 \rceil} \geq \cdots \geq a_{k-1} \geq 1$.

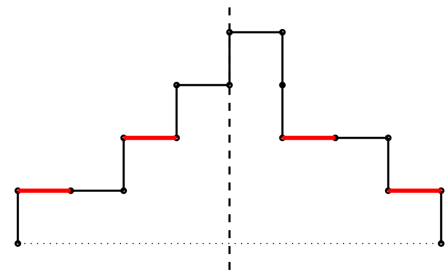

Figure 4: A unimodal bargraph B with $\mathrm{ds}(B) = 2$, $e(B) = 8$ and $n(B) = 4$, corresponding to the composition $(1, 1, 2, 3, 4, 2, 2, 1)$.

Theorem 4.6. *The generating function for unimodal bargraphs with a centered maximum with respect to their degree of symmetry is*

$$\sum_{B \in \mathcal{U}} s^{\mathrm{ds}(B)} x^{e(B)} y^{n(B)} = \frac{y(1 + x - y)}{(1 - s)x^2 + \sqrt{((x + 1)^2 - y)((x - 1)^2 - y)}} - y.$$

For partitions and compositions (represented as Young diagrams and bargraphs, respectively), a natural measure of size other than the semiperimeter is the sum of the entries (equivalently, the area). Whereas the generating function for compositions by area and degree of symmetry is straightforward, the corresponding generating functions for partitions and unimodal compositions are not known.

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