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A Simple and Unusual Bijection for Dyck Paths and its Consequences

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Abstract. In this paper we introduce a new bijection from the set of Dyck paths to itself. This bijection has the property that it maps statistics that appeared recently in the study of pattern-avoiding permutations into classical statistics on Dyck paths, whose distribution is easy to obtain. We also present a generalization of the bijection, as well as several applications of it to enumeration problems of statistics in restricted permutations.

Keywords: Dyck paths, bijections, restricted permutations

1. Introduction

Motivated by the study of statistics on pattern-avoiding permutations, new statistics on Dyck paths have recently been introduced in [2, 3]. These statistics, which are based on the notion of tunnel of a Dyck path, have important applications in the study of pattern-avoiding permutations for patterns of length 3. Several enumeration problems regarding permutation statistics can be solved more easily considering their counterpart in terms of Dyck paths.

In Section 3 we present a new bijection Φ from the set of Dyck paths to itself, and in Section 4 we study its properties. The interesting properties are that the statistics mentioned above involving tunnels are mapped by Φ into other known statistics such as hills, returns and even rises, which have been widely studied in the literature. This bijection gives new ways to derive generating functions enumerating those statistics.

In Section 5 we give a family of bijections depending on an integer parameter r, from which the main bijection Φ is the particular case $r = 0$. These bijections give correspondences involving new statistics of Dyck paths, which generalize the above ones. We give multivariate generating functions for them.

Finally, Section 6 discusses several applications of our bijections to enumeration of statistics in 321- and 132-avoiding permutations. In particular, we generalize a recent theorem about fixed points in restricted permutations, and we find a multivariate generating function for fixed points, excedances and descents in 132-avoiding permutations.

2. Preliminaries

Recall that a *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ which never goes below the x-axis. Sometimes it will be convenient to encode each up-step by a letter **u** and each down-step by **d**, obtaining an encoding of the Dyck path as a *Dyck word*. We shall denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$, and by $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$ the class of all Dyck paths. It is well-known that $|\mathcal{D}_n| = \mathbf{C}_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number. If $D \in \mathcal{D}_n$, we will write $|D| = n$ to indicate the semilength of D. The generating function that enumerates Dyck paths according to their semilength is $\sum_{D\in\mathcal{D}} z^{|D|} = \sum_{n\geq 0}\mathbf{C}_n z^n = \frac{1-\sqrt{1-4z}}{2z}$, which we denote by $\mathbf{C}(z)$.

We will use D to refer indistinctively to the Dyck path D or to the Dyck word associated to it. In particular, given $D_1 \in \mathcal{D}_{n_1}$, $\mathcal{D}_2 \in \mathcal{D}_{n_2}$, we will write D_1D_2 to denote the concatenation of D_1 and D_2 (note that, as seen in terms of lattice paths, D_2 has to be shifted $2n_1$ units to the right).

A *peak* of a Dyck path $D \in \mathcal{D}$ is an up-step followed by a down-step (i.e., an occurrence of **ud** in the associated Dyck word). The x-coordinate of a peak is given by the point at the top of it. A *hill* is a peak at height 1, where the height is the y-coordinate of the top of the peak. Denote by $h(D)$ the number of hills of D. A *valley* of a Dyck path $D \in \mathcal{D}$ is a down-step followed by an up-step (i.e., an occurrence of **du** in the associated Dyck word). An *odd rise* is an up-step in an odd position when the steps are numbered from left to right starting with 1 (or, equivalently, it is an up-step at odd level when the steps leaving the x-axis are considered to be at level 1). Denote by $\mathrm{or}(D)$ the number of odd rises of D. *Even rises* and $\mathrm{er}(D)$ are defined analogously. The x-coordinate of an odd or even rise is given by the rightmost end of the corresponding up-step.

A *return* of a Dyck path is a down-step landing on the x-axis. An *arch* is a part of the path joining two consecutive points on the x-axis. Clearly for any $D \in \mathcal{D}_n$ the number of returns equals the number of arches. Denote it by $\mathrm{ret}(D)$. Define the x-coordinate of an arch as the x-coordinate of its leftmost point.

We will use the concept of tunnel introduced by the first author in [2]. For any $D \in \mathcal{D}$, define a *tunnel* of D to be a horizontal segment between two lattice points of D that intersects D only in these two points, and stays always below D. Tunnels are in obvious one-to-one correspondence with decompositions of the Dyck word $D = A\mathbf{u}B\mathbf{d}C$, where $B \in \mathcal{D}$ (no restrictions on A and C). In the decomposition, the tunnel is the segment that goes from the beginning of the **u** to the end of the **d**. If $D \in \mathcal{D}_n$, then D has exactly n tunnels, since such a decomposition can be given for each up-step **u** of D.

A tunnel of $D \in \mathcal{D}_n$ is called a *centered tunnel* if the x-coordinate of its midpoint is n, that is, the tunnel is centered with respect to the vertical line through the middle of D. In terms of the decomposition $D = A\mathbf{u}B\mathbf{d}C$, this is equivalent to saying that A and C have the same length. Denote by $\mathrm{ct}(D)$ the number of centered tunnels of D.

A tunnel of $D \in \mathcal{D}_n$ is called a *right tunnel* if the x-coordinate of its midpoint is strictly greater than n, that is, the midpoint of the tunnel is to the right of the vertical line through the middle of D. Clearly, in terms of the decomposition $D = A\mathbf{u}B\mathbf{d}C$, this is equivalent to saying that the length of A is strictly bigger than the length of C. Denote by $\mathrm{rt}(D)$ the number of right tunnels of D. In Figure 1, there is one centered tunnel drawn with a solid line, and four right tunnels drawn with dotted lines. Similarly, a tunnel is called a *left tunnel* if the x-coordinate of its midpoint is strictly less than n. Denote by $\mathrm{lt}(D)$ the number of left tunnels of D. Clearly, $\mathrm{lt}(D)+\mathrm{rt}(D)+\mathrm{ct}(D)=n$ for any $D \in \mathcal{D}_n$.

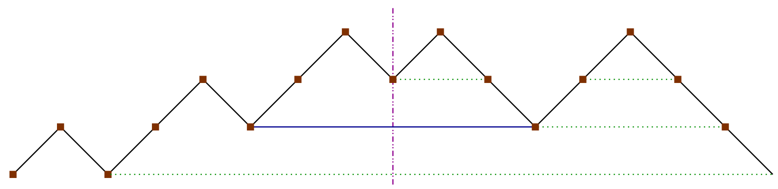

Figure 1: One centered and four right tunnels.

For any $D \in \mathcal{D}$, we define a *multitunnel* of D to be a horizontal segment between two lattice points of D such that D never goes below it. In other words, a multitunnel is just a concatenation of tunnels, so that each tunnel starts at the point where the previous one ends. Similarly to the case of tunnels, multitunnels are in obvious one-to-one correspondence with decompositions of the Dyck word $D = ABC$, where $B \in \mathcal{D}$ is not empty. In the decomposition, the multitunnel is the segment that connects the initial and final points of B.

A multitunnel of $D \in \mathcal{D}_n$ is called a *centered multitunnel* if the x-coordinate of its midpoint (as a segment) is n, that is, the tunnel is centered with respect to the vertical line through the middle of D. In terms of the decomposition $D = ABC$, this is equivalent to saying that A and C have the same length. Denote by $\mathrm{cmt}(D)$ the number of centered multitunnels of D.

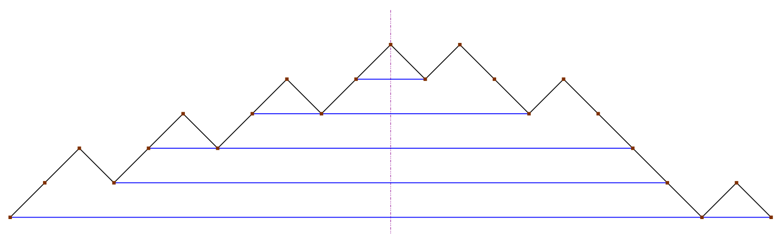

Figure 2: Five centered multitunnels, two of which are centered tunnels.

3. The Bijection

In this section we describe a bijection Φ from $\mathcal{D}_n$ to itself. Let $D \in \mathcal{D}_n$. Each up-step of D has a corresponding down-step together with which it determines a tunnel. Match each such pair of steps. Let $\sigma \in \mathcal{S}_{2n}$ be the permutation defined by

$$\sigma_i = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ 2n+1-\frac{i}{2}, & \text{if } i \text{ is even.} \end{cases}$$

In two-line notation,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 2n-3 & 2n-2 & 2n-1 & 2n \\ 1 & 2n & 2 & 2n-1 & 3 & 2n-2 & \cdots & n-1 & n+2 & n & n+1 \end{pmatrix}.$$

Then $\Phi(D)$ is created as follows. For i from 1 to $2n$, consider the σ_i-th step of D (i.e., D is read in zigzag). If its corresponding matching step has not yet been read, define the i-th step of $\Phi(D)$ to be an up-step, otherwise let it be a down-step. In the first case, we say that the σ_i-th step of D *opens* a tunnel, in the second we say that it *closes* a tunnel.

The bijection Φ applied to the Dyck paths of semilength at most 3 is shown in Figure 3.

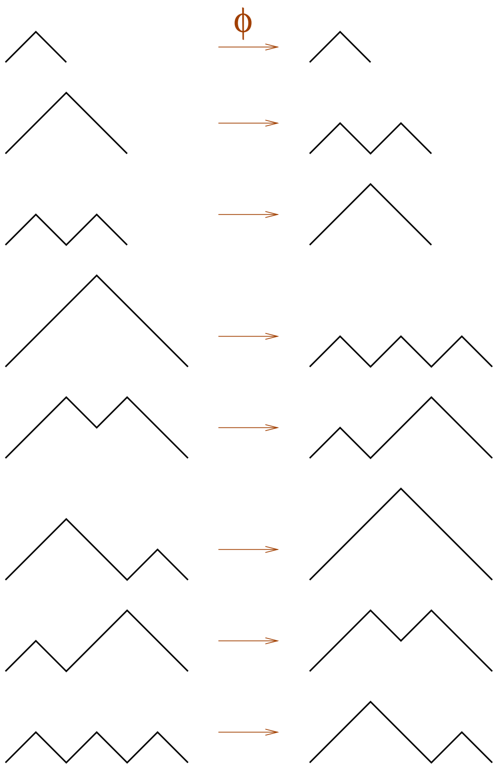

Figure 3: The bijection Φ for paths of length at most 3.

Figure 4 shows Φ applied to the example of the Dyck path $D =$ **uuduudududdddud**.

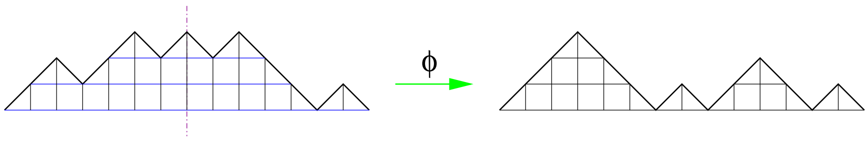

Figure 4: An example of Φ.

It is clear from the definition that $\Phi(D)$ is a Dyck path. Indeed, it never goes below the x-axis because at any point the number of down-steps drawn so far can never exceed the number of up-steps, since each down-step is drawn when the second step of a matching pair in D is read, and in that case the first step of the pair has already produced an up-step in $\Phi(D)$. Also, $\Phi(D)$ ends in $(2n,0)$ because each of the matched pairs of D produces an up-step and a down-step in $\Phi(D)$.

To show that Φ is indeed a bijection, we will describe the inverse map Φ^{-1}. Given $D' \in \mathcal{D}_n$, the following procedure recovers the $D \in \mathcal{D}_n$ such that $\Phi(D) = D'$. Consider the permutation σ defined above, and let $W = w_1 w_2 \cdots w_{2n}$ be the word obtained from D' as follows. For i from 1 to $2n$, if the i-th step of D' is an up-step, let $w_{\sigma_i} = o$, otherwise let $w_{\sigma_i} = c$. W contains the same information as D', with the advantage that the o's are located in the positions of D in which a tunnel is opened when D is read in zigzag, and the c's are located in the positions where a tunnel is closed. Equivalently, the o's are located in the positions of the left walls of the left and centered tunnels of D, and in the positions of the right walls of the right tunnels. For an example see Figure 5.

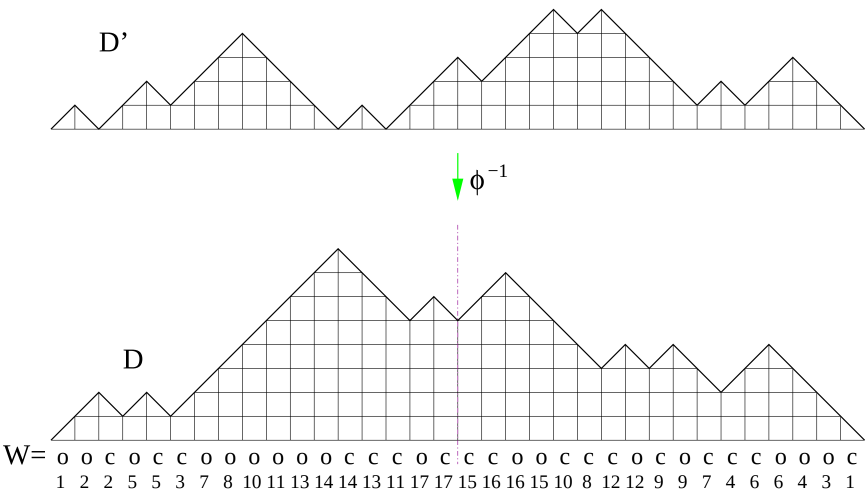

Figure 5: The inverse of Φ.

Now we define a matching between the o's and the c's in W, so that each matched pair will give a tunnel in D. We will label the o's with $1, 2, \ldots, n$ and similarly the c's, to indicate that an o and a c with the same label are matched. By left (resp. right) half of W we mean the symbols w_i with $i \leq n$ (resp. $i > n$). For i from 1 to $2n$, if $w_{\sigma_i} = o$, place in it the smallest label that has not been used yet. If $w_{\sigma_i} = c$, match it with the unmatched o in the same half of W as w_{σ_i} with largest label, if such an o exists. If it does not, match w_{σ_i} with the unmatched o in the opposite half of W with smallest label. Note that since D' was a Dyck path, at any time the number of c's read so far does not exceed the number of o's, so each c has some o to be paired up with.

Once the symbols in W have been labelled, D can be recovered by reading the labels from left to right, drawing an up-step for each label that is read for the first time, and a down-step for each label that appears the second time. In Figure 5 the labelling is shown under W.

4. Properties of Φ

Lemma 4.1. *Let $D = ABC$ be a decomposition of a Dyck path D, where B is a Dyck path, and A and C have the same length. Then $\Phi(ABC) = \Phi(AC)\Phi(B)$. In particular, $\Phi(\mathbf{u}B\mathbf{d}) = \mathbf{ud}\Phi(B)$.*

Proof. It follows immediately from the definition of Φ, since the path D is read in zigzag while $\Phi(D)$ is built from left to right. ∎

Theorem 4.2. *Let D be any Dyck path, and let $D' = \Phi(D)$. We have the following correspondences*:

(1) $\mathrm{ct}(D) = h(D')$,

(2) $\mathrm{rt}(D) = \mathrm{er}(D')$,

(3) $\mathrm{lt}(D) + \mathrm{ct}(D) = \mathrm{or}(D')$,

(4) $\mathrm{cmt}(D) = \mathrm{ret}(D')$.

Proof. First we show (1). Consider a centered tunnel given by the decomposition $D = A\mathbf{u}B\mathbf{d}C$. Applying Lemma 4.1 twice,

$$D' = \Phi(A\mathbf{u}B\mathbf{d}C) = \Phi(AC)\Phi(\mathbf{u}B\mathbf{d}) = \Phi(AC)\mathbf{ud}\Phi(B),$$

so we have a hill $\mathbf{ud}$ in D'. Reciprocally, any hill in D', say $D' = X\mathbf{ud}Y$, where $X, Y \in \mathcal{D}$, comes from a centered tunnel $D = Z_1\mathbf{u}\Phi^{-1}(Y)\mathbf{d}Z_2$, where Z_1 and Z_2 are respectively the first and second halves of $\Phi^{-1}(X)$.

The proof of (4) is very similar. Recall that $\mathrm{ret}(D')$ equals the number of arches of D'. Given a centered multitunnel corresponding to the decomposition $D = ABC$, we have $\Phi(D) = \Phi(AC)\Phi(B)$, so D' has an arch starting at the first step of $\Phi(B)$, which is nonempty.

To show (2), consider a right tunnel given by the decomposition $D = A\mathbf{u}B\mathbf{d}C$, where $\mathrm{length}(A) > \mathrm{length}(C)$. Of the two steps $\mathbf{u}$ and $\mathbf{d}$ delimiting the tunnel, $\mathbf{d}$ will be encountered before $\mathbf{u}$ when D is read in zigzag, since $\mathrm{length}(A) > \mathrm{length}(C)$. So $\mathbf{d}$ will open a tunnel, producing an up-step in D'. Besides, this up-step will be at an even position, since $\mathbf{d}$ was in the right half of D. Reciprocally, an even rise of D' corresponds to a step in the right half of D that opens a tunnel when D is read in zigzag, so it is necessarily a right tunnel.

Relation (3) follows from (2) and the fact that the total number of tunnels of D is $\mathrm{lt}(D)+\mathrm{ct}(D)+\mathrm{rt}(D)=n$, and the total number of up-steps of D' is $\mathrm{or}(D')+\mathrm{er}(D')=n$. ∎

One of the most interesting applications of this bijection is that it can be used to enumerate Dyck paths according to the number of centered, left, and right tunnels, and number of centered multitunnels. We are looking for a multivariate generating function for these four statistics, namely

$$\widetilde{G}(x,u,v,w,z)=\sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}u^{\mathrm{lt}(D)}v^{\mathrm{rt}(D)}w^{\mathrm{cmt}(D)}z^{|D|}.$$

By Theorem 4.2, this generating function can be expressed as

$$\widetilde{G}(x,u,v,w,z)=G\left(\frac{x}{u},u,v,w,z\right),$$

where

$$G(t,u,v,w,z)=\sum_{D\in\mathcal{D}} t^{h(D)}u^{\mathrm{or}(D)}v^{\mathrm{er}(D)}w^{\mathrm{ret}(D)}z^{|D|}.$$

We can derive an equation for G using the symbolic method described in [4] and [8]. A recursive definition for the class $\mathcal{D}$ is given by the fact that every nonempty Dyck path D can be decomposed in a unique way as $D=\mathbf{u}A\mathbf{d}B$, where $A,B\in\mathcal{D}$. The number of hills of $\mathbf{u}A\mathbf{d}B$ is $h(B)+1$ if A is empty, and $h(B)$ otherwise. The odd rises of A become even rises of $\mathbf{u}A\mathbf{d}B$, and the even rises of A become odd rises of $\mathbf{u}A\mathbf{d}B$. Thus, we have $\mathrm{er}(\mathbf{u}A\mathbf{d}B)=\mathrm{or}(A)+\mathrm{er}(B)$, and $\mathrm{or}(\mathbf{u}A\mathbf{d}B)=\mathrm{er}(A)+\mathrm{or}(B)+1$, where the extra odd rise comes from the first step $\mathbf{u}$. We also have $\mathrm{ret}(\mathbf{u}A\mathbf{d}B)=\mathrm{ret}(B)+1$. Hence, we obtain the following equation for G:

$$G(t,u,v,w,z)=1+uzw(G(1,v,u,1,z)-1+t)G(t,u,v,w,z). \tag{4.1}$$

Denote $G_1:=G(1,u,v,1,z)$, $H_1:=G(1,v,u,1,z)$. Substituting $t=w=1$ in (4.1), we obtain

$$G_1=1+uzH_1G_1, \tag{4.2}$$

and interchanging u and v,

$$H_1=1+vzG_1H_1. \tag{4.3}$$

Solving (4.2) and (4.3) for H_1, gives

$$H_1=\frac{1+(u-v)z-\sqrt{1-2(v+u)z+(v-u)^2z^2}}{2uz}.$$

Thus, from (4.1),

$$\begin{aligned}G(t,u,v,w,z)&=\frac{1}{1-uzw(H_1-1+t)}\\&=\frac{2}{2-w+(v+u-2tu)wz+w\sqrt{1-2(v+u)z+(v-u)^2z^2}}.\end{aligned}$$

Now, switching to $\widetilde{G}$, we obtain the following theorem.

Theorem 4.3. *The multivariate generating function for Dyck paths according to centered, left, and right tunnels, centered multitunnels, and semilength is*

$$\sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}u^{\mathrm{lt}(D)}v^{\mathrm{rt}(D)}w^{\mathrm{cmt}(D)}z^{|D|}$$
$$=\frac{2}{2-w+(v+u-2x)wz+w\sqrt{1-2(v+u)z+(v-u)^2z^2}}.$$

5. Generalizations

Here we present a generalization Φ_r of the bijection Φ, which depends on a nonnegative integer parameter r. Given $D\in\mathcal{D}_n$, copy the first $2r$ steps of D into the first $2r$ steps of $\Phi_r(D)$. Now, read the remaining steps of D in zigzag in the following order: $2r+1$, $2n$, $2r+2$, $2n-1$, $2r+3$, $2n-2$, and so on. For each of these steps, if its corresponding matching step in D has not yet been encountered, draw an up-step in $\Phi_r(D)$, otherwise draw a down-step. Note that for $r=0$ we get the same bijection Φ as before.

Note that Φ_r can be defined exactly as Φ with the difference that instead of σ, the permutation that gives the order in which the steps of D are read is $\sigma^{(r)}\in\mathcal{S}_{2n}$, defined as

$$\sigma_i^{(r)}=\begin{cases} i, & \text{if } i\le 2r,\\ \frac{i+1}{2}+r, & \text{if } i>2r \text{ and } i \text{ is odd},\\ 2n+1-\frac{i}{2}-r, & \text{if } i>2r \text{ and } i \text{ is even}.\end{cases}$$

Figure 6 shows an example of Φ_r for $r=2$ applied to the path $D=$ **uduuduuduudu dddudd**.

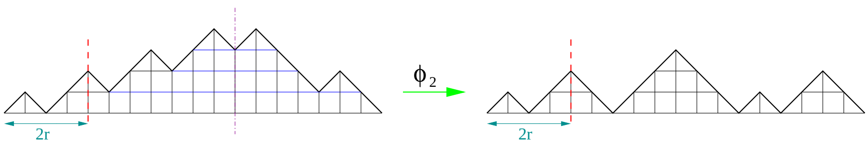

Figure 6: An example of Φ_2.

It is clear from the definition that $\Phi_r(D)$ is a Dyck path. A reasoning similar to the one used for Φ shows that Φ_r is indeed a bijection.

The properties of Φ given in Theorem 4.2 generalize to analogous properties of Φ_r. We will prove them using the following lemma, which follows immediately from the definition of Φ_r.

Lemma 5.1. *Let $r\ge 0$, and let $D=ABC$ be a decomposition of a Dyck path D, where B is a Dyck path, and* $\mathrm{length}(A)=\mathrm{length}(C)+2r$. *Then $\Phi_r(ABC)=\Phi_r(AC)\Phi(B)$.*

Theorem 5.2. *Let $r\ge 0$, let D be any Dyck path, and let $D'=\Phi_r(D)$. We have the following correspondences*:

(1) $\#\{$*tunnels of D with midpoint at $x=n+r$*$\}=\#\{$*hills of D' in $x>2r$*$\}$,

(2) $\#\{$*tunnels of D with midpoint in* $x > n+r\} = \#\{$*even rises of* D' *in* $x > 2r\}$,

(3) $\#\{$*tunnels of D with midpoint in* $x \le n+r\} = \#\{$*odd rises of* D' *in* $x > 2r\}$
$+\#\{$*up-steps of* D' *in* $x \le 2r\}$,

(4) $\#\{$*multitunnels of D with midpoint at* $x = n+r\} = \#\{$*arches of* D' *in* $x \ge 2r\}$.

Proof. Fist we show (1). A tunnel given by the decomposition $D = A\mathbf{u}B\mathbf{d}C$ has its midpoint at $x = n+r$ exactly when $\text{length}(A) = \text{length}(C)+2r$. Applying Lemmas 5.1 and 4.1, $D' = \Phi_r(A\mathbf{u}B\mathbf{d}C) = \Phi_r(AC)\Phi(\mathbf{u}B\mathbf{d}) = \Phi_r(AC)\Phi(\mathbf{ud})\Phi(B) = \Phi_r(AC)\mathbf{ud}\Phi(B)$, and $\mathbf{ud}$ is a hill of D' in $x > 2r$, since $\text{length}(\Phi_r(AC)) \ge 2r$. Reciprocally, any hill of D' in $x > 2r$, say $D' = X\mathbf{ud}Y$, where $X,Y \in \mathcal{D}$ and $\text{length}(X) \ge 2r$, comes from a tunnel with midpoint at $x = n+r$, namely $D = Z_1\mathbf{u}\Phi^{-1}(Y)\mathbf{d}Z_2$, where $Z_1Z_2 = \Phi_r^{-1}(X)$ and $\text{length}(Z_1) = \text{length}(Z_2)+2r$.

The proof of (4) is very similar. A multitunnel given by $D = ABC$ has its midpoint at $x = n+r$ exactly when $\text{length}(A) = \text{length}(C)+2r$. In this case, $\Phi_r(D) = \Phi_r(AC)\Phi(B)$ by Lemma 5.1, so D' has an arch starting at the first step of $\Phi(B)$. Notice that this arch is in $x \ge 2r$ because $\text{length}(\Phi_r(AC)) \ge 2r$.

To show (2), consider a tunnel in D with midpoint in $x > n+r$. This is equivalent to saying that it is given by a decomposition $D = A\mathbf{u}B\mathbf{d}C$ with $\text{length}(A) > \text{length}(C)+2r$. In particular, the tunnel is contained in the halfspace $x \ge 2r$, so the two steps $\mathbf{u}$ and $\mathbf{d}$ delimiting the tunnel are in the part of D that is read in zigzag in the process to obtain $\Phi_r(D)$, and $\mathbf{d}$ will be encountered before $\mathbf{u}$, since $\text{length}(A)-2r > \text{length}(C)$. So $\mathbf{d}$ will open a tunnel, producing an up-step of D' in $x > 2r$. Besides, this up-step will be at an even position, since $\mathbf{d}$ is in $x > n+r$, that is, in the right half of the part of D that is read in zigzag. Reciprocally, an even rise of D' in $x > 2r$ corresponds to a step of D in $x > n+r$ that opens a tunnel when D is read according to $\sigma^{(r)}$, so it is necessarily tunnel with midpoint to the right of $x = n+r$.

Relation (3) follows from (2) and the fact that the total number of tunnels of D is $\#\{\text{tunnels of } D \text{ with midpoint in } x > n+r\} + \#\{\text{tunnels of } D \text{ with midpoint in } x \le n+r\} = n$, and the total number of up-steps of D' is $\#\{\text{even rises of } D' \text{ in } x > 2r\} + \#\{\text{odd rises of } D' \text{ in } x > 2r\} + \#\{\text{up-steps of } D' \text{ in } x \le 2r\} = n$. ∎

Similarly to how we used the properties of Φ to prove Theorem 4.3, we can use the properties of Φ_r to prove a more general theorem. Our goal is to enumerate Dyck paths according to the number tunnels with midpoint on, to the right of, and to the left of an arbitrary vertical line $x = n+r$, and multitunnels with midpoint on that line. In generating function terms, we are looking for an expression for

$$\begin{aligned} &F(t,u,v,w,y,z) \\ &:= \sum_{\substack{n\ge 0\\ 0\le r\le n}} \sum_{D\in\mathcal{D}_n} t^{\#\{\text{tun. of } D \text{ w/ midp. at } x=n+r\}} u^{\#\{\text{tun. of } D \text{ w/ midp. in } x\le n+r\}} \\ &\qquad v^{\#\{\text{tun. of } D \text{ w/ midp. in } x>n+r\}} w^{\#\{\text{multitun. of } D \text{ w/ midp. at } x=n+r\}} y^r z^n. \end{aligned}$$

Note that the variable y marks the position of the vertical line $x = n+r$ with respect to which the tunnels are classified. The following theorem gives an expression for F.

Theorem 5.3. *Let F, G and $\mathbf{C}$ be defined as above. Then,*

$$F(t,u,v,w,y,z) = \frac{\mathbf{C}(uyz)G(t,u,v,w,z)}{1-yu^2z^2\mathbf{C}^2(uyz)G(1,u,v,1,z)G(1,v,u,1,z)}$$
$$= \frac{2B(2+(v-u)z+A)}{[2+(u+v-2tu)wz+wA][(A+(v-u)z)B-4uyz]},$$

where $A := \sqrt{1-2(u+v)z+(u-v)^2z^2}-1$, $B := \sqrt{1-4uyz}-1$.

Proof. By Theorem 5.2, the generating function F can be expressed as

$$F(t,u,v,w,y,z) = \sum_{\substack{n\geq 0\\ 0\leq r\leq n}} \sum_{D\in\mathcal{D}_n} t^{\#\{\text{hills of } D \text{ in } x>2r\}} u^{\#\{\text{odd rises of } D \text{ in } x>2r\}+\#\{\text{up-steps of } D \text{ in } x\leq 2r\}}$$
$$v^{\#\{\text{even rises of } D \text{ in } x>2r\}} w^{\#\{\text{arches of } D \text{ in } x\geq 2r\}} y^r z^n. \tag{5.1}$$

For each path D in this summation, the y-coordinate of its intersection with the vertical line $x=2r$ has to be even. Fix $h\geq 0$. We will now focus only on the paths $D\in\mathcal{D}$ for which this intersection has y-coordinate equal to $2h$. Let $D=AB$, where A and B are the parts of the path respectively to the left and to the right of $x=2r$. Then, $\#\{\text{hills of } D \text{ in } x>2r\} = \#\{\text{hills of } B\}$, $\#\{\text{odd rises of } D \text{ in } x>2r\} = \#\{\text{odd rises of } B\}$, $\#\{\text{up-steps of } D \text{ in } x\leq 2r\} = \#\{\text{up-steps of } A\}$, and $\#\{\text{arches of } D \text{ in } x\geq 2r\} = \#\{\text{arches of } B\}$.

B can be any path starting at height $2h$ and landing on the x-axis, never going below it. If $h>0$, consider the first down-step of B that lands at height $2h-1$. Then B can be decomposed as $B=B_1dB'$, where B_1 is any Dyck path, and B' is any path starting at height $2h-1$ and landing on the x-axis, never going below it. Applying this decomposition recursively, B can be written uniquely as $B=B_1\mathbf{d}B_2\mathbf{d}\cdots B_{2h}\mathbf{d}B_{2h+1}$, where the B_i's for $1\leq i\leq 2h+1$ are arbitrary Dyck paths. The number of hills and number of arches of B are given by those of B_{2h+1}. The odd rises of B are the odd rises of the B_i's with odd subindex plus the even rises of those with even subindex. In a similar way one can describe the even rises of B. The semilength of B is the sum of semilengths of the B_i's plus h, which comes from the $2h$ additional down-steps. Thus, the generating function for all paths B of this form, where t, u, v, and z mark respectively number of hills, number of odd rises, number of even rises, and semilength, is

$$z^hG^h(1,u,v,1,z)G^h(1,v,u,1,z)G(t,u,v,w,z). \tag{5.2}$$

Similarly, A can be decomposed uniquely as $A=A_1\mathbf{u}A_2\mathbf{u}\cdots A_{2h}\mathbf{u}A_{2h+1}$. The number of up-steps of A is the sum of the number of up-steps of each A_i, plus a $2h$ term that comes from the additional up-steps. The generating function for paths A of this form, where u marks the number of up-steps, and y and z mark both the semilength, is

$$z^hy^hu^{2h}\mathbf{C}^{2h+1}(uyz). \tag{5.3}$$

The product of (5.2) and (5.3) gives the generating function for paths $D = AB$ where the height of the intersection point of D with the vertical line between A and B is $2h$, where the variables mark the same statistics as in (5.1). Note that the exponent of y is half the distance between the origin of D and this vertical line. Summing over h, we obtain

$$\begin{aligned}
&F(t,u,v,w,y,z)\\
&\quad = \sum_{h\geq 0} z^{2h}y^h u^{2h}\mathbf{C}^{2h+1}(uyz)G^h(1,u,v,1,z)G^h(1,v,u,1,z)G(t,u,v,w,z)\\
&\quad = \frac{\mathbf{C}(uyz)G(t,u,v,w,z)}{1-yu^2z^2\mathbf{C}^2(uyz)G(1,u,v,1,z)G(1,v,u,1,z)}.
\end{aligned}$$

∎

6. Connection to Pattern-Avoiding Permutations

The bijection Φ has applications to the subject of enumeration of statistics in pattern-avoiding permutations.

First we review some basic definitions in this subject. Given two permutations $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_m \in \mathcal{S}_m$, with $m \leq n$, we say that π *contains* σ if there exist indices $i_1 < i_2 < \ldots < i_m$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ is in the same relative order as $\sigma_1\sigma_2\cdots\sigma_m$. If π does not contain σ, we say that π is σ-*avoiding*. For example, if $\sigma = 132$, then $\pi = 24531$ contains σ, because $\pi_1\pi_3\pi_4 = 253$. However, $\pi = 42351$ is σ-avoiding.

We say that i is a *fixed point* of a permutation π if $\pi_i = i$, and that i is an *excedance* of π if $\pi_i > i$. We say that $i \leq n-1$ is a *descent* of $\pi \in \mathcal{S}_n$ if $\pi_i > \pi_{i+1}$. Denote by $\mathrm{fp}(\pi)$, $\mathrm{exc}(\pi)$, and $\mathrm{des}(\pi)$ the number of fixed points, the number of excedances, and the number of descents of π respectively. Denote by $\mathcal{S}_n(\sigma)$ the set of σ-avoiding permutations in $\mathcal{S}_n$.

The distribution of fixed points in pattern-avoiding permutations was considered for the first time in [7]. There appears the following result.

Theorem 6.1. [7] *The number of* 321*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = k$ *equals the number of* 132*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = k$*, for any* $0 \leq k \leq n$.

This theorem can be refined considering not only fixed points but also excedances, as shown in [2].

Theorem 6.2. [2] *The number of* 321*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = k$ *and* $\mathrm{exc}(\pi) = l$ *equals the number of* 132*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = k$ *and* $\mathrm{exc}(\pi) = l$*, for any* $0 \leq k, l \leq n$.

The proofs given in [2,7] are not bijective. The first bijective proof of these theorems appears in [3]. Our bijection Φ_r can be used to give a bijective proof of the following generalization of Theorem 6.1. Note that the particular case $r = 0$ gives a new bijective proof of such theorem.

Theorem 6.3. *Fix $r, n \geq 0$. For any $\pi \in \mathcal{S}_n$, define $\alpha_r(\pi) = \#\{i\colon \pi_i = i+r\}$, $\beta_r(\pi) = \#\{i\colon i > r, \pi_i = i\}$. Then, the number of* 321*-avoiding permutations $\pi \in \mathcal{S}_n$ with $\beta_r(\pi) = k$ equals the number of* 132*-avoiding permutations $\pi \in \mathcal{S}_n$ with $\alpha_r(\pi) = k$, for any $0 \leq k \leq n$.*

Proof. We will use two bijections given in [2], one from 321-avoiding permutations to Dyck paths and one from 132-avoiding permutations to Dyck paths.

The first bijection, which we denote $\Psi\colon \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$, can be defined as follows. Any permutation $\pi \in \mathcal{S}_n$ can be represented as an $n \times n$ array with a cross on the squares (i, π_i). It is known that π is 321-avoiding if and only if both the subsequence determined by its excedances and the one determined by the remaining elements are increasing. Given the array of $\pi \in \mathcal{S}_n(321)$, consider the path with *down* and *right* steps along the edges of the squares that goes from the upper-left corner to the lower-right corner of the array leaving all the crosses to the right and remaining always as close to the main diagonal as possible. Define $\Psi(\pi)$ to be the Dyck path obtained from this path by reading an up-step every time the path moves down, and a down-step every time the path moves to the right. Figure 7 shows a picture of this bijection for $\pi = 23147586$.

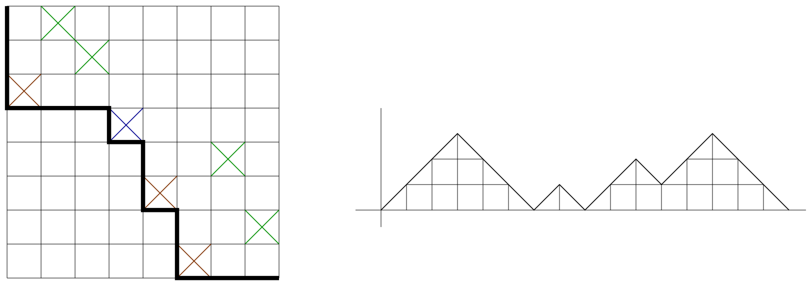

Figure 7: The bijection Ψ.

It can easily be checked that Ψ maps fixed points to hills. More precisely, i is a fixed point of π if and only if $\Psi(\pi)$ has a hill with x-coordinate $2i-1$. This implies that $\beta_r(\pi) = \#\{\text{hills of } \Psi(\pi) \text{ in } x > 2r\}$.

The second bijection, which we denote $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$, was given by Krattenthaler in [5], up to reflection of the path over a vertical line. It can be described as follows. Given $\pi \in \mathcal{S}_n(132)$ represented by an $n \times n$ array as before, consider the path with *up* and *right* steps that goes from the lower-left corner to the upper-right corner, leaving all the crosses to the right, and staying always as close to the diagonal connecting these two corners as possible. Then $\varphi(\pi)$ is the Dyck path obtained from this path by reading an up-step every time the path goes up and a down-step every time it goes right. Figure 8 shows an example when $\pi = 67435281$.

The property of φ that is useful here is that an element i with $\pi_i = i+r$ corresponds to a tunnel of $\varphi(\pi)$ with midpoint at $x = n+r$. To show this, we repeat the reasoning in [2, 3]. There is an easy way to recover a permutation $\pi \in \mathcal{S}_n(132)$ from $\varphi(\pi)$. Consider the path from the lower-left corner to the upper-right corner or the array, used to define $\varphi(\pi)$. Now, row by row, put a cross in the leftmost square to the

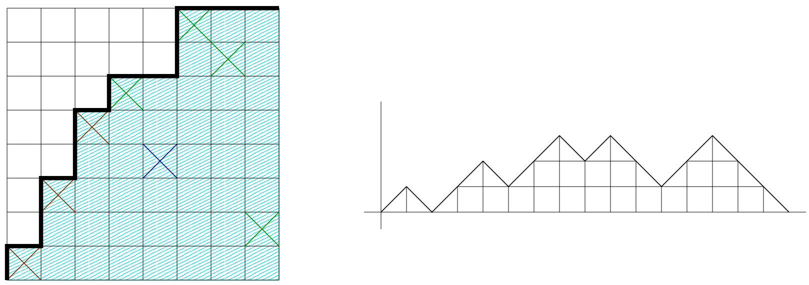

Figure 8: The bijection φ.

right of this path such that there is exactly one cross in each column. This gives us π back. To each cross we can associate a tunnel in a natural way. Indeed, each cross produces a decomposition $\varphi(\sigma) = A\mathbf{u}B\mathbf{d}C$ where B corresponds to the part of the path above and to the left of the cross. Here $\mathbf{u}$ corresponds to the vertical step directly to the left of the cross, and $\mathbf{d}$ to the horizontal step directly above the cross. Thus, fixed points, which correspond to crosses on the main diagonal, give centered tunnels. More generally, crosses $(i, i+r)$ give tunnels with midpoint r units to the right of the center, that is, tunnels of $\varphi(\pi)$ with midpoint at $x = n+r$. Therefore, we have that $\alpha_r(\pi) = \#\{\text{tunnels of } \varphi(\pi) \text{ with midpoint at } x = n+r\}$.

Now all we need to do is use Φ_r and property (1) given in Theorem 5.2. From this it follows that the bijection $\Psi^{-1} \circ \Phi_r \circ \varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{S}_n(321)$ has the property that $\beta_r(\Psi^{-1} \circ \Phi_r \circ \varphi(\pi)) = \#\{\text{hills of } \Phi_r \circ \varphi(\pi) \text{ in } x > 2r\} = \#\{\text{tunnels of } \varphi(\pi) \text{ with mid-point at } x = n+r\} = \alpha_r(\pi)$. ∎

The proof of Theorem 6.2 given in [2] describes a simple way to enumerate 321-avoiding permutations with respect to the statistics fp and exc. However, the analogous enumeration for 132-avoiding permutations is done in a more intricate way. As an application of the properties of Φ, we give a more direct derivation of the multivariate generating function for 132-avoiding permutations according to number of fixed points and number of excedances.

Corollary 6.4 of Theorem 4.3. *We have*

$$\sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)} v^{\mathrm{exc}(\pi)} z^n = \frac{2}{1+(1+v-2x)z+\sqrt{1-2(1+v)z+(1-v)^2z^2}}. \tag{6.1}$$

Proof. Let φ be the bijection between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$ described above. It follows from the above reasoning (and is also shown in [2]) that φ maps fixed points to centered tunnels, and excedances to right tunnels, i.e., $\mathrm{fp}(\pi) = \mathrm{ct}(\varphi(\pi))$ and $\mathrm{exc}(\pi) = \mathrm{rt}(\varphi(\pi))$. Therefore, the left hand side of (6.1) equals $\sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)} v^{\mathrm{rt}(D)} z^{|D|}$. The result now is obtained just applying Theorem 4.3 for $u = w = 1$. ∎

Compare this expression (6.1) with Equation (2) in [2]. Note that Theorem 6.2 follows from these two expressions.

As a further application, we can use the bijection Φ to give the following refinement of Corollary 6.4, which gives an expression for the multivariate generating function for number of fixed points, number of excedances, and number of descents in 132-avoiding permutations. An analogous result for 321-avoiding permutations is given in [2, Section 3].

Theorem 6.5. *Let*

$$L(x,v,p,z) := 1 + \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)} v^{\mathrm{exc}(\pi)} p^{\mathrm{des}(\pi)+1} z^n.$$

Then

$$L(x,v,p,z) = \frac{2(1+xz(p-1))}{1+(1+v-2x)z - vz^2(p-1)^2 + \sqrt{f_1(v,z)}}, \tag{6.2}$$

where $f_1(v,z) = 1-2(1+v)z+[(1-v)^2-2v(p-1)(p+3)]z^2-2v(1+v)(p-1)^2z^3+v^2(p-1)^4z^4$.

Proof. We use again that φ maps fixed points to centered tunnels, and excedances to right tunnels. It can easily be checked that it also maps descents of the permutation to valleys of the corresponding Dyck path. Clearly, the number of valleys of any nonempty Dyck path equals the number of peaks minus one (in the empty path both numbers are 0). Thus, L can be expressed as

$$L(x,v,p,z) = \sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)} v^{\mathrm{rt}(D)} p^{\#\{\text{peaks of } D\}} z^{|D|}.$$

By Theorem 4.2, Φ maps centered tunnels into hills and right tunnels into even rises. Let us see what peaks are mapped to by Φ. Given a peak **ud** in $D\in\mathcal{D}$, D can be written as $D = A\mathbf{ud}C$, where A and C are the parts of the path before and after the peak respectively. This decomposition corresponds to a tunnel of D that goes from the beginning of the **u** to the end of the **d**. Assume first that the peak occurs in the left half (i.e., length$(A) <$ length(C)). When D is read in zigzag, the **u** opens a tunnel that is closed by the **d** two steps later. This produces in $\Phi(D)$ an up-step followed by a down-step two positions ahead, that is, an occurrence of $\mathbf{u}\star\mathbf{d}$ in the Dyck word of $\Phi(D)$, where $\star$ stands for any symbol (either a **u** or a **d**).

If the peak occurs in the right half of D (i.e., length$(A) >$ length(C)), the reasoning is analogous, with the difference that the **d** opens a tunnel that is closed by the **u** two steps ahead. So, such a peak produces also an occurrence of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$. Reciprocally, we claim that an occurrence of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$ can come only from a peak of D either in the left or in the right half. Indeed, using the notation from the procedure above describing the inverse of Φ, an occurrence of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$ corresponds to either an occurrence of *oc* in the left half of W or an occurrence of *co* in the right half of W. In both cases, the algorithm given above will match these two letters c and o with each other, so they correspond to an occurrence of **ud** in D.

If the peak occurs in the middle (i.e., length$(A) =$ length(C)), then by Lemma 4.1, $\Phi(A\mathbf{ud}C) = \Phi(AC)\mathbf{ud}$, so it is mapped to an occurrence of **ud** at the end of $\Phi(D)$. Clearly we have such an occurrence only when D has a peak in the middle.

Thus, we have shown that peaks in D are mapped by Φ to occurrences of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$ and occurrences of **ud** at the end of $\Phi(D)$, or, equivalently, to occurrences of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)\mathbf{d}$ (here $\Phi(D)\mathbf{d}$ is a Dyck path followed by a down-step). Denote by $\lambda(D)$ the number of occurrences of $\mathbf{u}\star\mathbf{d}$ in $D\mathbf{d}$. This implies that L can be written as

$$L(x,v,p,z)=\sum_{D\in\mathcal{D}}x^{h(D)}v^{\mathrm{er}(D)}p^{\lambda(D)}z^{|D|}.$$

We are left with a Dyck path enumeration problem, which is solved in the following lemma. Let J be defined in Lemma 6.6. It is easy to see that we have $L(x,v,p,z)=1+J(x,1,p,1,v,p,z)$. Making use of (6.3) and (6.4), it follows at once that

$$L(x,v,p,z)=\frac{1-xz+xpz}{1-xz-zK_1},$$

where K_1 is given by

$$zK_1^2-[1-z-vz+v(1-p)^2z^2]K_1+p^2vz=0.$$

From these equations we obtain (6.2). ∎

Lemma 6.6. *Denote by* $\mathrm{ih}(D)$ $(\mathrm{fh}(D))$ *the number of initial (final) hills in D (obviously, their only possible values are* 0 *and* 1*). Denote by $\mu(D)$ the number of occurrences of* $\mathbf{u}\star\mathbf{d}$ *in D. Then the generating function*

$$J(x,t,s,u,v,q,z):=\sum x^{h(D)}t^{\mathrm{ih}(D)}s^{\mathrm{fh}(D)}u^{\mathrm{or}(D)}v^{\mathrm{er}(D)}q^{\mu(D)}z^{|D|},$$

where the summation is over all nonempty Dyck paths, is given by

$$J(x,t,s,u,v,q,z)=\frac{uz[xts+(1-xu(1-t)(1-s)z)K]}{1-xuz-uzK},\tag{6.3}$$

where K is given by

$$uzK^2-[1-(u+v)z+uv(1-q)^2z^2]K+q^2vz=0.\tag{6.4}$$

Proof. Every nonempty Dyck path has one of the following four forms: **ud**, **ud**B, **u**A**d**, or **u**A**d**B, where A and B are nonempty Dyck paths. The generating functions of these four pairwise disjoint sets of Dyck paths are

(i) $xtsuz$,

(ii) $xtuzJ(x,1,s,u,v,q,z)$,

(iii) $uzJ(1,q,q,v,u,q,z)$,

(iv) $uzJ(1,q,q,v,u,q,z)J(x,1,s,u,v,q,z)$,

respectively. Only the third factor in (iii) and (iv) needs an explanation: the hills of A are not hills in **u**A**d**; an initial (final) hill in A gives a **uud** (**udd**) in **u**A**d**; an odd (even) rise in A becomes an even (odd) rise in **u**A**d**.

Consequently, the generating function J satisfies the functional equation

$$\begin{aligned} J(x,t,s,u,v,q,z) &= xtsuz + xtuzJ(x,1,s,u,v,q,z) + uzJ(1,q,q,v,u,q,z) \\ &\quad + uzJ(1,q,q,v,u,q,z)J(x,1,s,u,v,q,z). \end{aligned} \tag{6.5}$$

From Equation (6.5) it is clear that, whether interested or not in the statistics "number of initial (final) hills", we had to introduce them for the sake of the statistic marked by the variable q. Also, without any additional effort we could use two separate variables to mark the number of **uud**'s and the number of **udd**'s, and obtain a slightly more general generating function, although we do not need it here.

Denoting $H = J(x,1,s,u,v,q,z)$, $K = J(1,q,q,v,u,q,z)$, Equation (6.5) becomes

$$J = xtsuz + xtuzH + uzK + uzHK. \tag{6.6}$$

Setting here $t = 1$, we obtain

$$H = xsuz + xuzH + uzK + uzHK. \tag{6.7}$$

Solving (6.7) for H and introducing it into (6.6), we obtain (6.3).

It remains to show that K satisfies the quadratic Equation (6.4). Setting $x = 1$, $t = q$, $s = q$ in (6.6), and interchanging u and v, we get

$$K = q^2vz + qvzM + vz\widehat{K} + vzM\widehat{K}, \tag{6.8}$$

where $M = J(1,1,q,v,u,q,z)$ and $\widehat{K}$ is K with u and v interchanged, i.e. $\widehat{K} = J(1,q,q,u,v,q,z)$.

Now in (6.6) we set $x = 1$, $t = 1$, $s = q$, and we interchange u and v, to get

$$M = qvz + vzM + vz\widehat{K} + vzM\widehat{K}. \tag{6.9}$$

Eliminating M from (6.8) and (6.9), we obtain

$$vz(2qvz - q^2vz + 1 - vz)\widehat{K} + (vz-1)K + vzK\widehat{K} + q^2vz = 0. \tag{6.10}$$

Finally, eliminating $\widehat{K}$ from (6.10) and the equation obtained from (6.10) by interchanging u and v, we obtain Equation (6.4). Note that, as expected, J is symmetric in the variables t and s and linear in each of these two variables. ∎

From Theorem 6.5 one can see that the first terms of $L(x,v,p,z)$ are

$$1 + xpz + (vp^2 + x^2p)z^2 + (v^2p^2 + vp^2 + xvp^3 + xvp^2 + x^3p)z^3 + \cdots,$$

corresponding to Dyck paths of semilength at most 3 (or equivalently, to 321-avoiding permutations of length at most 3).

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