Harrington/Soare Glossary (Δ^0_3 automorphisms)

Note that α is always a node on the tree T. We consider the automorphism to be induced by a map from ω to $\hat{\omega}$. Everything is correctly hatted or unhatted in correspondence to which side it lives on, with exception of the V sets, which are reversed. When we take the dual, everything flips between being hatted or unhatted, except the U and V sets: U flips with V and $\hat{U}$ flips with $\hat{V}$, so we preserve under duality which player (RED or BLUE) is acting.

$<_L$: $\alpha <_L \beta$ means, for $\alpha \perp \beta$, that at the point of first difference α goes leftward of β: $(\exists a, b \in \omega)(\exists \gamma \in T)(\gamma^\frown a \subseteq \alpha \;\&\; \gamma^\frown b \subseteq \beta \;\&\; a < b)$. For comparing infinite strings, the formal definition is "there exists a finite initial segment such that..."

$\leq$: $\alpha \leq \beta$ if $\alpha <_L \beta$ or $\alpha \subseteq \beta$.

$\leq_B$: $\nu_0 \leq_B \nu_1$ if $\sigma_0 = \sigma_1$ and $\tau_0 \subseteq \tau_1$ (BLUE claims more $\hat{V}$ sets). $\hat{\nu}_0 \leq_B \hat{\nu}_1$ if $\hat{\sigma}_0 \subseteq \hat{\sigma}_1$ and $\hat{\tau}_0 = \hat{\tau}_1$ (BLUE claims more $\hat{U}$ sets).

$\leq_R$: $\nu_0 \leq_R \nu_1$ if $\sigma_0 \subseteq \sigma_1$ and $\tau_0 = \tau_1$ (RED claims more U sets). $\hat{\nu}_0 \leq_R \hat{\nu}_1$ if $\hat{\sigma}_0 = \hat{\sigma}_1$ and $\hat{\tau}_0 \subseteq \hat{\tau}_1$ (RED claims more V sets).

$\bigcup$: for states, $\bigcup\{\nu(\alpha, \sigma_i, \tau_i) : i \in I\} = \langle \alpha, \sigma, \tau \rangle$ for $\sigma = \cup\{\sigma_i : i \in I\}$ and $\tau = \cup\{\tau_i : i \in I\}$.

$\perp$: $\alpha \perp \beta$ means α and β are incomparable: neither extends the other.

$\preceq$: $\nu_1 \preceq \nu_0$ if $\exists \beta$ such that $\nu_0 \restriction \beta = \nu_1$.

$\restriction$: $\nu_0 \restriction \beta$ for an α-state ν_0 and $\beta \subseteq \alpha$ is $\langle \beta, \sigma_1, \tau_1 \rangle$, where $\sigma_1 = \sigma_0 \cap \{0, \ldots, e_\beta\}$ and $\tau_1 = \tau_0 \cap \{0, \ldots, \hat{e}_\beta\}$. For a set of α-states $\mathcal{C}_\alpha$, $\mathcal{C}_\alpha \restriction \beta = \{\nu \restriction \beta : \nu \in \mathcal{C}_\alpha\}$.

$\alpha(x, s)$: function giving location of ball x at end of stage s; $x \in S_{\alpha(x,s)}$. Dual: $\hat{x} \in \hat{S}_{\alpha(\hat{x},s)}$.

α-ineligible: x ($\hat{x}$) such that the true path at some stage $s \geq x$ ($s \geq \hat{x}$) lies to the left of α. Requires $x \notin S_{\alpha,t}$ ($\hat{x} \notin \hat{S}_{\alpha,t}$) for all $t \geq s$.

α-legal: an enumeration of x by BLUE such that $\nu(\alpha, x, s) \in \mathcal{M}_\alpha$, $x \in Y_{\alpha,s}$, and the resulting state $\nu(\alpha, x, s+1) \in \mathcal{M}_\alpha$ also. For $\alpha \subset f$, cofinitely many enumerations must be α-legal. This is guaranteed by making sure α is $\mathcal{M}$-consistent whenever $\alpha \subset f$.

α-marked: refers to $\mathcal{L}$ (or $\hat{\mathcal{L}}$), when it finishes a stage with all its entries with first element α marked.

α-region: R_α, the ω-balls in all pockets at and below α. $\hat{R}$ holds $\hat{\omega}$-balls.

α-section: S_α, the ω-balls in α's pocket. $\hat{S}$ holds $\hat{\omega}$-balls.

α-state: the state of x ($\hat{x}$), restricted by the length of α. Of the form $\langle \alpha, \sigma, \tau \rangle$, where $\sigma \subseteq \{0, \dots, e_\alpha\}$ and $\tau \subseteq \{0, \dots, \hat{e}_\alpha\}$. See $\nu(\alpha, x, s)$, $\hat{\nu}(\alpha, \hat{x}, s)$.

δ_t: for $t \leq s$ at stage s, defined in order to get the next approximation to the true path. Let $\delta_0 = \lambda$. Given δ_t, let $v \leq s$ be the largest v such that δ_t is a substring of f_v. If δ_t has never been on the true path, let $v = 0$. Let $\alpha \in T$, $\alpha^- = \delta_t$, be $\leq_L$-least (leftmost) such that $C_{\alpha,s} \neq C_{\alpha,v}$ (the chip set has increased since the last time the true path could have been through there). If such an α exists, let $\delta_{t+1} = \alpha$, and if not, let $\delta^{t+1} = \delta_t$.

λ-state: ν_{-1}, $\hat{\nu}_{-1}$.

$\nu_{-1} = \hat{\nu}_{-1} = \langle \lambda, \emptyset, \emptyset \rangle$.

$\nu(\alpha, x, s)$: the α-state of x at stage s. The triple $\langle \alpha, \sigma(\alpha, x, s), \tau(\alpha, x, s) \rangle$. Says what U and $\hat{V}$ sets x is in.

$\hat{\nu}(\alpha, \hat{x}, s)$: the α-state of $\hat{x}$ at stage s. The triple $\langle \alpha, \hat{\sigma}(\alpha, \hat{x}, s), \hat{\tau}(\alpha, \hat{x}, s) \rangle$. Says what $\hat{U}$ and V sets $\hat{x}$ is in.

$\nu^+(\alpha, x, s)$: another α-state of x at stage s. The same as $\nu(\alpha, x, s)$ except with the $U_{\alpha,s}$ in the definition of σ replaced by $Z_{e_\alpha, s}$. Note that only the final U is replaced by Z, so it is always the case that $\nu^+(\alpha, x, s) \upharpoonright \alpha^- = \nu(\alpha^-, x, s)$.

$\sigma(\alpha, x, s) = \{e_\beta : \beta \subseteq \alpha \ \& \ e_\beta > e_{\beta^-} \ \& \ x \in U_{\beta,s}\}$. Every e_β where we considered a new U set and x was in it.

$\hat{\sigma}(\alpha, \hat{x}, s) = \{e_\beta : \beta \subseteq \alpha \ \& \ e_\beta > e_{\beta^-} \ \& \ \hat{x} \in \hat{U}_{\beta,s}\}$. Every e_β where we considered a new $\hat{U}$ set and $\hat{x}$ was in it.

$\tau(\alpha, x, s) = \{\hat{e}_\beta : \beta \subseteq \alpha \ \& \ \hat{e}_\beta > \hat{e}_{\beta^-} \ \& \ x \in \hat{V}_{\beta,s}\}$. Every $\hat{e}_\beta$ where we considered a new $\hat{V}$ set and x was in it.

$\hat{\tau}(\alpha, \hat{x}, s) = \{\hat{e}_\beta : \beta \subseteq \alpha \ \& \ \hat{e}_\beta > \hat{e}_{\beta^-} \ \& \ \hat{x} \in V_{\beta,s}\}$. Every $\hat{e}_\beta$ where we considered a new V set and $\hat{x}$ was in it.

$\mathcal{B}_\alpha$: one piece of $\mathcal{N}_\alpha$, containing the states ν which α believes are being emptied by BLUE (that is, the x's with state ν are having their $\tau(\alpha, x, s)$ increased). Defined inductively: assume $\mathcal{R}_\gamma$, $\mathcal{B}_\gamma$, $\hat{\mathcal{R}}_\gamma$ and $\hat{\mathcal{B}}_\gamma$ have been defined for all $\gamma \subset \alpha$. Define $\mathcal{B}_\alpha^{<\alpha} = \{\nu : \nu \in \mathcal{M}_\alpha \ \& \ \nu \upharpoonright \alpha^- \in \mathcal{B}_{\alpha^-}\}$. Define $\mathcal{B}_\alpha^\alpha = \emptyset$ if $|\alpha| \not\equiv 4 \bmod 5$, and $\mathcal{B}_\alpha^\alpha = \{\nu : \hat{\nu} \in \hat{\mathcal{R}}_\alpha^\alpha\}$ otherwise (BLUE reacts to RED). Then $\mathcal{B}_\alpha = \mathcal{B}_\alpha^\alpha \sqcup \mathcal{B}_\alpha^{<\alpha}$.

$\hat{\mathcal{B}}_\alpha$: one piece of $\hat{\mathcal{N}}_\alpha$, containing the states $\hat{\nu}$ which α believes are being emptied by BLUE (that is, the $\hat{x}$'s with state $\hat{\nu}$ are having their $\hat{\sigma}(\alpha, \hat{x}, s)$ increased). Defined inductively: assume $\mathcal{R}_\gamma$, $\mathcal{B}_\gamma$, $\hat{\mathcal{R}}_\gamma$ and $\hat{\mathcal{B}}_\gamma$

have been defined for all $\gamma \subset \alpha$. Define $\hat{\mathcal{B}}_\alpha^{<\alpha} = \{\hat{\nu} : \hat{\nu} \in \hat{\mathcal{M}}_\alpha \ \&\ \hat{\nu} \upharpoonright \alpha^- \in \hat{\mathcal{B}}_{\alpha^-}\}$. Define $\hat{\mathcal{B}}_\alpha^\alpha = \emptyset$ if $|\alpha| \not\equiv 3 \bmod 5$, and $\hat{\mathcal{B}}_\alpha^\alpha = \{\hat{\nu} : \nu \in \mathcal{R}_\alpha^\alpha\}$ otherwise (BLUE reacts to RED). Then $\hat{\mathcal{B}}_\alpha = \hat{\mathcal{B}}_\alpha^\alpha \sqcup \hat{\mathcal{B}}_\alpha^{<\alpha}$.

BLUE: player 2. Controls $\hat{U}$ and $\hat{V}$ sets.

$\{C_\alpha\}_{\alpha \in T}$: "chip sets"; sets such that $\alpha \subset f$ iff $|C_\alpha| = \infty$ (recall we are working with all Π_2^0 predicates).

consistent: α which is $\mathcal{M}$-consistent and $\mathcal{R}$-consistent (and $\mathcal{C}$-consistent if including the section 6 material). If α is inconsistent, it is a terminal node on the tree T.

e_α: an index to spread out the U and $\hat{U}$ sets. Equals $e_{\alpha^-} + 1$ if $|\alpha| \equiv 1 \bmod 5$, e_{α^-} otherwise.

$\hat{e}_\alpha$: an index to spread out the V and $\hat{V}$ sets. Equals $\hat{e}_{\alpha^-} + 1$ if $|\alpha| \equiv 2 \bmod 5$, $\hat{e}_{\alpha^-}$ otherwise.

$e_\lambda = \hat{e}_\lambda = -1$.

$\mathcal{E}_\alpha = \{\nu : (\exists^\infty x)(\exists s)[x \in S_{\alpha,s} - \bigcup\{S_{\alpha,t} : t < s\} \ \&\ \nu(\alpha, x, s) = \nu]\}$. The list of states well-visited by elements x when they first enter R_α. Since elements must enter R_α via S_α, this is the collection of states that an infinite number of x's have when they first appear in S_α. $\mathcal{E}_\alpha \subseteq \mathcal{F}_\alpha$.

$\mathcal{E}_\alpha^k = \{\nu : (\exists^\infty x)(\exists s)[x \in S_{\alpha,s}^k - \bigcup\{S_{\alpha,t}^k : t < s\} \ \&\ \nu(\alpha, x, s) = \nu]\}$. Defined for $k \in \{0, 1\}$.

f: the true path on the tree. Proved equal to $\liminf_s f_s$, but defined by induction on n as follows. If $\beta = f \upharpoonright (n-1)$ has been defined and is consistent, then $f \upharpoonright n$ is the $<_L$-least length-n extension α of β such that the following hold:

(i) $n \equiv 1 \bmod 5 \Rightarrow \mathcal{M}_\alpha = \mathcal{F}_\beta^+$ and $k_\alpha = k_\beta^+$

(ii) $n \equiv 2 \bmod 5 \Rightarrow \hat{\mathcal{M}}_\alpha = \hat{\mathcal{F}}_\beta^+$ and $k_\alpha = k_\beta^+$

(iii) $n \equiv 3 \bmod 5 \Rightarrow \mathcal{R}_\alpha^\alpha = \{\nu : \nu \in \mathcal{M}_\alpha - (\mathcal{R}_\alpha^{<\alpha} \cup \mathcal{B}_\alpha^{<\alpha}) \ \&\ F(\beta, \nu)\}$ and $\hat{\mathcal{B}}_\alpha^\alpha = \{\hat{\nu} : \nu \in \mathcal{R}_\alpha^\alpha\}$

(iv) $n \equiv 4 \bmod 5 \Rightarrow \hat{\mathcal{R}}_\alpha^\alpha = \{\hat{\nu} : \hat{\nu} \in \hat{\mathcal{M}}_\alpha - (\hat{\mathcal{R}}_\alpha^{<\alpha} \cup \hat{\mathcal{B}}_\alpha^{<\alpha}) \ \&\ \hat{F}(\beta, \nu)\}$ and $\mathcal{B}_\alpha^\alpha = \{\nu : \hat{\nu} \in \hat{\mathcal{R}}_\alpha^\alpha\}$

(v) unless specified above, $\mathcal{M}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$, and k_α have the values $\mathcal{M}_\beta$, $\mathcal{R}_\beta$, $\mathcal{B}_\beta$, and k_β, respectively.

f_s: the stage s approximation to the true path. Defined to be δ_s, where δ_t, $t \leq s$, have been defined inductively from scratch at stage s.

$\mathcal{F}_\alpha = \{\nu : (\exists^\infty x)(\exists s)[x \in R_{\alpha,s} \ \&\ \nu(\alpha, x, s) = \nu]\}$. The list of states well-visited by elements x while they remain in R_α. That is, all the states an infinite number of elements x have at some time while they are in R_α, not just when they first appear. $\mathcal{E}_\alpha \subseteq \mathcal{F}_\alpha$. Furthermore, we wait to increase the α-state of elements x until given permission by $\mathcal{M}_\alpha$, so (after some proof) $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$.

$\mathcal{F}^+_{\alpha^-} = \{\nu : (\exists^\infty x)(\exists s)[x \in Y^1_{\alpha^-,s} \ \&\ \nu^+(\alpha, x, s) = \nu]\}$, when $e_\alpha > e_{\alpha^-}$. If $\hat{e}_\alpha > \hat{e}_{\alpha^-}$, $\mathcal{F}^+_{\alpha^-} = \{\nu : \hat{\nu} \in \hat{\mathcal{F}}^+_{\alpha^-}\}$. If $e_\alpha = e_{\alpha^-}$ and $\hat{e}_\alpha = \hat{e}_{\alpha^-}$, $\mathcal{F}^+_{\alpha^-} = \mathcal{F}_{\alpha^-}$. Note that $\mathcal{F}^+_{\alpha^-}$, although it is relevant to the node α, depends only on sets built above α.

$\hat{\mathcal{F}}^+_{\alpha^-} = \{\hat{\nu} : (\exists^\infty x)(\exists s)[x \in \hat{Y}^1_{\alpha^-,s} \ \&\ \hat{\nu}^+(\alpha, x, s) = \hat{\nu}]\}$, when $\hat{e}_\alpha > \hat{e}_{\alpha^-}$. If $e_\alpha > e_{\alpha^-}$, $\hat{\mathcal{F}}^+_{\alpha^-} = \{\hat{\nu} : \nu \in \mathcal{F}^+_{\alpha^-}\}$. If $e_\alpha = e_{\alpha^-}$ and $\hat{e}_\alpha = \hat{e}_{\alpha^-}$, $\hat{\mathcal{F}}^+_{\alpha^-} = \hat{\mathcal{F}}_{\alpha^-}$.

$F(\alpha^-, \nu)$: the Π^0_2 predicate given by $(\forall x)[[x > |\alpha^-| \ \&\ x \in Y_{\alpha^-}] \Rightarrow \nu(\alpha, x) \neq \nu]$. Defined only for $|\alpha| \equiv 3 \bmod 5$. Is true or false depending on whether $|\alpha^-|$ witnesses to the non-well-residedness of ν. Controls $\mathcal{R}^\alpha_\alpha$. Π^0_2 instead of Σ^0_3 because we have fixed α^-.

$\hat{F}(\alpha^-, \hat{\nu})$: the Π^0_2 predicate given by $(\forall \hat{x})[[\hat{x} > |\alpha^-| \ \&\ \hat{x} \in \hat{Y}_{\alpha^-}] \Rightarrow \hat{\nu}(\alpha, \hat{x}) \neq \hat{\nu}]$. Defined only for $|\alpha| \equiv 4 \bmod 5$. Is true or false depending on whether $|\alpha^-|$ witnesses to the non-well-residedness of $\hat{\nu}$. Controls $\hat{\mathcal{R}}^\alpha_\alpha$. Π^0_2 instead of Σ^0_3 because we have fixed α^-.

h_α: a target function taking states to states. In the construction, it must satisfy $h_\alpha : \mathcal{B}_\alpha \to (\mathcal{M}_\alpha - \mathcal{B}_\alpha)$ and $(\forall \nu \in \mathcal{B}_\alpha)[\nu <_B h_\alpha(\nu)]$. Motivation is that we know x can leave any non-well-resided state (e.g., one in $\mathcal{B}_\alpha$) for $\alpha \subset f$; h gives a possible destination.

$\hat{h}_\alpha$: a target function taking states to states. In the construction, it must satisfy $\hat{h}_\alpha : \hat{\mathcal{B}}_\alpha \to (\hat{\mathcal{M}}_\alpha - \hat{\mathcal{B}}_\alpha)$ and $(\forall \hat{\nu} \in \hat{\mathcal{B}}_\alpha)[\hat{\nu} <_B \hat{h}_\alpha(\hat{\nu})]$. Motivation is that we know $\hat{x}$ can leave any non-well-resided state (e.g., one in $\hat{\mathcal{B}}_\alpha$) for $\alpha \subset f$; $\hat{h}$ gives a possible destination.

initialize a node α: remove every $x \in S_{\alpha,s}$ ($\hat{x} \in \hat{S}_{\alpha,s}$), and put x into $S^1_{\beta,s}$ ($\hat{x}$ into $\hat{S}^1_{\beta,s}$) for $\beta = \alpha \cap f_{s+1}$. That is, yank all the balls up into the pocket where their location joins back to the true path.

k_α: a marker for α such that only $x > k_\alpha$ are allowed to enter Y_α. We allow any k_α to be put on the tree and then eliminate from contention for the true path any α whose k_α doesn't match our criterion: To be on the true path, α must have $k_\alpha = k^+_{\alpha^-}$ (only an issue if $e_\alpha > e_{\alpha^-}$ or $\hat{e}_\alpha > \hat{e}_{\alpha^-}$; else $k^+_{\alpha^-}$ is defined that way).

$k^+_{\alpha^-} = \min\{y : (\forall x > y)(\forall s)[[x \in Y^1_{\alpha^-,s} \ \&\ \nu^+(\alpha, x, s) = \nu_1] \Rightarrow \nu_1 \in \mathcal{F}^+_{\alpha^-}]\}$ if $e_\alpha > e_{\alpha^-}$. If $\hat{e}_\alpha > \hat{e}_{\alpha^-}$, the dual definition. If $e_\alpha = e_{\alpha^-}$ and $\hat{e}_\alpha = \hat{e}_{\alpha^-}$, $k^+_{\alpha^-} = k_{\alpha^-}$. Does not have a hatted version.

$\mathcal{K}_\alpha = \{\nu_1 : \neg(\exists^\infty x)[x \in Y_\alpha \ \&\ \nu(\alpha, x) = \nu_1]\}$. The set of non-well-resided α-states. Σ^0_3, so approximated by $\mathcal{N}_\alpha$.

$\mathcal{L}$: a list of elements of the form $\langle \alpha, \nu_1 \rangle$, some of which are marked. $\mathcal{L}$ is augmented with new elements beginning with α whenever it and $\hat{\mathcal{L}}$ finish a stage with all their entries with first element α marked (that is, the lists are both α-marked).

$\hat{\mathcal{L}}$: a list of elements of the form $\langle \alpha, \hat{\nu}_1 \rangle$, some of which are marked. $\hat{\mathcal{L}}$ is augmented in the same manner as $\mathcal{L}$.

$m(\alpha, s)$: how many times $\mathcal{L}$ and $\hat{\mathcal{L}}$ have been α-marked by the end of stage s. Does not have a hatted version.

$\mathcal{M}_\alpha$: almost α's "guess" at the true $\mathcal{F}_\alpha$ (really, α's guess at $\mathcal{F}^+_{\alpha^-}$, so that all children of α^- are guessing about the same set). Consists of well-visited states. On the tree we will see as $\mathcal{M}_\alpha$'s all sets of states which cannot be immediately discarded as incorrect, and any α such that $\mathcal{M}_\alpha \neq \mathcal{F}^+_{\alpha^-}$ will be eliminated from contention for f. Note that by definition, for all nodes α, $\hat{\mathcal{M}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{M}_\alpha\}$.

$\mathcal{M}$-inconsistent: α such that $e_\alpha > e_{\alpha^-}$ and there exist α-states $\nu_0 <_B \nu_1$ such that $\nu_0 \in \mathcal{M}_\alpha$, $\nu_1 \upharpoonright \alpha^- \in \mathcal{M}_{\alpha^-}$, but $\nu_1 \notin \mathcal{M}_\alpha$. That is, $\mathcal{M}_\alpha$ is not large enough with respect to $<_B$: there is a state which could be reached in one BLUE move, but it is not available in $\mathcal{M}_\alpha$. $\mathcal{M}$-consistency ensures cofinitely many BLUE enumerations at α will be α-legal, a necessary condition for $\alpha \subset f$.

$\mathcal{N}_\alpha$: a Π^0_2 approximation to the set $\mathcal{K}_\alpha$ of non-well-resided states. Divided into the disjoint union of $\mathcal{R}_\alpha$ and $\mathcal{B}_\alpha$.

non-well-resided: an infinite number of balls (elements) enter the state, but a cofinite number then leave again. Σ^0_3.

provably incorrect: α such that $(\exists x)(\exists s)[x \in Y_{\alpha,s} \ \&\ \nu(\alpha, x, s) \notin \mathcal{M}_\alpha]$. If that statement is true, α is provably incorrect for all stages $t \geq s$, and kept off the true path (basically works by showing we have a witness that k_α is wrong).

$\mathcal{R}_\alpha$: one piece of $\mathcal{N}_\alpha$, containing the states ν which α believes are being emptied by RED (that is, the x's with state ν are having their $\sigma(\alpha, x, s)$ increased). Defined inductively: assume $\mathcal{R}_\gamma$, $\mathcal{B}_\gamma$, $\hat{\mathcal{R}}_\gamma$ and $\hat{\mathcal{B}}_\gamma$ have been defined for all $\gamma \subset \alpha$. Define $\mathcal{R}^{<\alpha}_\alpha = \{\nu : \nu \in \mathcal{M}_\alpha \ \&\ \nu \upharpoonright \alpha^- \in \mathcal{R}_{\alpha^-}\}$ (note this depends only on nodes up to α^-). Define $\mathcal{R}^\alpha_\alpha = \emptyset$ if $|\alpha| \not\equiv 3$ mod 5; it might be nonempty otherwise. In particular, for $\alpha \subset f$, $\mathcal{R}^\alpha_\alpha = \{\nu : \nu \in \mathcal{M}_\alpha - (\mathcal{R}^{<\alpha}_\alpha \cup \mathcal{B}^{<\alpha}_\alpha) \ \&\ F(\alpha^-, \nu)\}$. Then $\mathcal{R}_\alpha = \mathcal{R}^\alpha_\alpha \sqcup \mathcal{R}^{<\alpha}_\alpha$.

$\hat{\mathcal{R}}_\alpha$: one piece of $\hat{\mathcal{N}}_\alpha$, containing the states $\hat{\nu}$ which α believes are being emptied by RED (that is, the $\hat{x}$'s with state $\hat{\nu}$ are having their $\hat{\tau}(\alpha, \hat{x}, s)$ increased). Defined inductively: assume $\mathcal{R}_\gamma$, $\mathcal{B}_\gamma$, $\hat{\mathcal{R}}_\gamma$ and $\hat{\mathcal{B}}_\gamma$ have been defined for all $\gamma \subset \alpha$. Define $\hat{\mathcal{R}}^{<\alpha}_\alpha = \{\hat{\nu} : \hat{\nu} \in \hat{\mathcal{M}}_\alpha \ \&\ \hat{\nu} \upharpoonright \alpha^- \in \hat{\mathcal{R}}_{\alpha^-}\}$

(note this depends only on nodes up to α^-). Define $\hat{\mathcal{R}}^\alpha_\alpha = \emptyset$ if $|\alpha| \not\equiv 4$ mod 5; it might be nonempty otherwise. In particular, for $\alpha \subset f$, $\hat{\mathcal{R}}^\alpha_\alpha = \{\hat{\nu} : \hat{\nu} \in \hat{\mathcal{M}}_\alpha - (\hat{\mathcal{R}}^{<\alpha}_\alpha \cup \hat{\mathcal{B}}^{<\alpha}_\alpha) \ \& \ \hat{F}(\alpha^-, \hat{\nu})\}$. Then $\hat{\mathcal{R}}_\alpha = \hat{\mathcal{R}}^\alpha_\alpha \sqcup \hat{\mathcal{R}}^{<\alpha}_\alpha$.

$\mathcal{R}$-consistent: α such that $(\forall \nu_0 \in \mathcal{R}_\alpha)(\exists \nu_1)[\nu_0 <_R \nu_1 \ \& \ \nu_1 \in \mathcal{M}_\alpha]$. Another closure property of $\mathcal{M}_\alpha$: if α thinks ν_0 is non-well-resided and emptied by RED, it must also see a state the elements with state ν_0 could get to via RED moves.

$R_{\alpha,s} = \{x : \alpha(x,s) \supseteq \alpha\}$; R_α is all the ω-balls at and below node α. Is emptied at the end of stage s for any α such that $f_s <_L \alpha$.

$\hat{R}_{\alpha,s} = \{\hat{x} : \alpha(\hat{x},s) \supseteq \alpha\}$; $\hat{R}_\alpha$ is all the $\hat{\omega}$-balls at and below node α. Is emptied at the end of stage s for any α such that $f_s <_L \alpha$.

$R^1_{\alpha,s} = \{x : x \in S^1_{\alpha,s} \text{ or } (\exists \gamma \supset \alpha)[x \in S_{\gamma,s}]\}$.

RED: player 1. Controls U and V sets.

$S_{\alpha,s} = \{x : \alpha(x,s) = \alpha\}$; S_α is the pocket at node α and holds finitely-many ω-balls at one time.

$\hat{S}_{\alpha,s} = \{\hat{x} : \alpha(\hat{x},s) = \alpha\}$; $\hat{S}_\alpha$ is the pocket at node α and holds finitely-many $\hat{\omega}$-balls at one time.

$S^0_{\alpha,s}$: a subsection of $S_{\alpha,s}$, consisting of elements which may be appointed as α-witnesses.

$S^1_{\alpha,s}$: a subsection of $S_{\alpha,s}$, consisting of elements which are available to be moved into S_γ for any $\gamma \supset \alpha$.

T: the tree used in the construction, considered by coding to be a subset of $\omega^{<\omega}$. Defined inductively: $\lambda \in T$; we have $\mathcal{M}_\lambda = \mathcal{R}_\lambda = \mathcal{B}_\lambda = \emptyset$ and $k_\lambda = e_\lambda = \hat{e}_\lambda = -1$. If $\beta \in T$, put $\alpha = \beta^\frown\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha\rangle$ in T provided it meets the following conditions:
(i) β is consistent.
(ii) $\mathcal{M}_\alpha$ is a set of α-states; $\mathcal{R}_\alpha, \mathcal{B}_\alpha \subseteq \mathcal{M}_\alpha$; $\mathcal{R}_\alpha \cap \mathcal{B}_\alpha = \emptyset$.
(iii) $\mathcal{M}_\alpha \restriction \beta = \mathcal{M}_\beta$.
(iv) $(e_\alpha = e_{\alpha^-} \ \& \ \hat{e}_\alpha = \hat{e}_{\alpha^-}) \Rightarrow \mathcal{M}_\alpha = \mathcal{M}_\beta$.
(v) $\mathcal{R}^{<\alpha}_\alpha \subseteq \mathcal{R}_\alpha$; $\mathcal{B}^{<\alpha}_\alpha \subseteq \mathcal{B}_\alpha$.
(vi) $\mathcal{R}^\alpha_\alpha \neq \emptyset \Rightarrow |\alpha| \equiv 3$ mod 5; $\mathcal{B}^\alpha_\alpha \neq \emptyset \Rightarrow |\alpha| \equiv 4$ mod 5.

target function: see h_α.

v_α: for $\alpha \subset f$, a stage such that any x which is in R_α after stage v_α never leaves. Also has the properties that for $s > v_\alpha$, $f_s \not<_L \alpha$, and no $\beta <_L \alpha$ acts at stage s, so $Y_{<\alpha,s} = Y_{<\alpha}$. Defined in Lemma 5.4(v).

well-visited: an infinite number of balls (elements) enter the state, but do not necessarily stay. Π^0_2.

$Y_{\alpha,s} = \bigcup\{R_{\alpha,t} : t \leq s\}$; Y_α is every ω-ball which ever enters a pocket at or below node α (so Y_λ is every ball on the machine). Finite for all $\alpha <_L f$.

$Y^1_{\alpha,s} = \bigcup\{R^1_{\alpha,t} : t \leq s\}$.

$Y_{<\alpha} = \bigcup\{Y_\delta : \delta <_L \alpha\}$

$Z_{e_\alpha} = \bigcup_s Z_{e_\alpha,s}$. A slowed-down approximation to U_α. Defined only for α such that $e_\alpha > e_{\alpha^-}$.

$\hat{Z}_{\hat{e}_\alpha} = \bigcup_s \hat{Z}_{\hat{e}_\alpha,s}$. A slowed-down approximation to V_α. Defined only for α such that $\hat{e}_\alpha > \hat{e}_{\alpha^-}$.

$Z_{e_\alpha,s+1} = \{x : x \in U_{e_\alpha,s+1} \;\&\; x \in Y^1_{\alpha^-,s}\}$.

$Z_{\hat{e}_\alpha,s+1} = \{x : x \in \hat{V}_{\hat{e}_\alpha,s+1} \;\&\; x \in Y^1_{\alpha^-,s}\}$.

$\hat{Z}_{e_\alpha,s+1} = \{\hat{x} : \hat{x} \in \hat{U}_{e_\alpha,s+1} \;\&\; \hat{x} \in \hat{Y}^1_{\alpha^-,s}\}$.

$\hat{Z}_{\hat{e}_\alpha,s+1} = \{\hat{x} : \hat{x} \in V_{\hat{e}_\alpha,s+1} \;\&\; \hat{x} \in \hat{Y}^1_{\alpha^-,s}\}$.

Appendix 1: How do x's move?

On the tree:

- **Step 1:** can move x from $R^1_{\alpha^-,s}$ into $S^{k \bmod 2}_\alpha$
- **Step 2:** can move x from $S^1_{\alpha^-,s}$ to S^1_α
- **Step 11C:** can pull x up from the right into S^1_β where β is the juncture of the original path with the true path
- **Step 11E:** can put x into Y_λ if x has never been on the machine before

In the c.e. sets:

- **Step 1:** can enumerate x into $U_{\alpha,s+1}$ if $e_\alpha > e_{\alpha^-}$, or into $\hat{V}_{\alpha,s+1}$ if $\hat{e}_\alpha > \hat{e}_{\alpha^-}$ (RED or BLUE enumeration)
- **Step 3:** can enumerate x into $\hat{V}_{\delta,s+1}$ for $\delta \subset \alpha$ such that $e_\delta \in \tau_1$ (BLUE enumeration)
- **Step 4:** can enumerate x into $U_{\alpha,s+1}$ (RED enumeration)
- **Step 5:** can enumerate x into $\hat{V}_{\delta,s+1}$ for $\delta \subseteq \alpha$ such that $\hat{e}_\delta > \hat{e}_{\delta^-}$ and $e_\delta \in \tau_1 - \tau_0$ (BLUE enumeration)

Appendix 2: What happens at different $|\alpha|$'s (mod 5)?

(1) consider one new U set (make sure $\alpha \subset f \Rightarrow U_\alpha =^* U_n$)
    V_α, $\hat{V}_\alpha$ undefined
    $e_\alpha = e_{\alpha^-} + 1$, $\hat{e}_\alpha = \hat{e}_{\alpha^-}$
    $\mathcal{R}^\alpha_\alpha = \hat{\mathcal{B}}^\alpha_\alpha = \emptyset$, $\hat{\mathcal{R}}^\alpha_\alpha = \mathcal{B}^\alpha_\alpha = \emptyset$

(2) consider one new V set (make sure $\alpha \subset f \Rightarrow V_\alpha =^* V_n$)
    U_α, $\hat{U}_\alpha$ undefined
    $e_\alpha = e_{\alpha^-}$, $\hat{e}_\alpha = \hat{e}_{\alpha^-} + 1$
    $\mathcal{R}^\alpha_\alpha = \hat{\mathcal{B}}^\alpha_\alpha = \emptyset$, $\hat{\mathcal{R}}^\alpha_\alpha = \mathcal{B}^\alpha_\alpha = \emptyset$

(3) consider new α-states ν which might be non-well-resided on Y_α
    U_α, $\hat{U}_\alpha$, V_α, $\hat{V}_\alpha$ undefined
    $e_\alpha = e_{\alpha^-}$, $\hat{e}_\alpha = \hat{e}_{\alpha^-}$
    $\mathcal{M}_\alpha = \mathcal{M}_{\alpha^-}$, define $F(\alpha^-, \nu)$, allow $\mathcal{R}^\alpha_\alpha \neq \emptyset$, define $\hat{\mathcal{B}}^\alpha_\alpha = \{\hat{\nu} : \nu \in \mathcal{R}^\alpha_\alpha\}$
    $\hat{\mathcal{R}}^\alpha_\alpha = \mathcal{B}^\alpha_\alpha = \emptyset$

(4) consider new α-states $\hat{\nu}$ which might be non-well-resided on $\hat{Y}_\alpha$
    U_α, $\hat{U}_\alpha$, V_α, $\hat{V}_\alpha$ undefined
    $e_\alpha = e_{\alpha^-}$, $\hat{e}_\alpha = \hat{e}_{\alpha^-}$
    $\mathcal{R}^\alpha_\alpha = \hat{\mathcal{B}}^\alpha_\alpha = \emptyset$
    allow $\hat{\mathcal{R}}^\alpha_\alpha \neq \emptyset$, define $\mathcal{B}^\alpha_\alpha = \{\nu : \hat{\nu} \in \hat{\mathcal{R}}^\alpha_\alpha\}$

(5) U_α, $\hat{U}_\alpha$, V_α, $\hat{V}_\alpha$ undefined
    $e_\alpha = e_{\alpha^-}$, $\hat{e}_\alpha = \hat{e}_{\alpha^-}$
    $\mathcal{R}^\alpha_\alpha = \hat{\mathcal{B}}^\alpha_\alpha = \emptyset$, $\hat{\mathcal{R}}^\alpha_\alpha = \mathcal{B}^\alpha_\alpha = \emptyset$