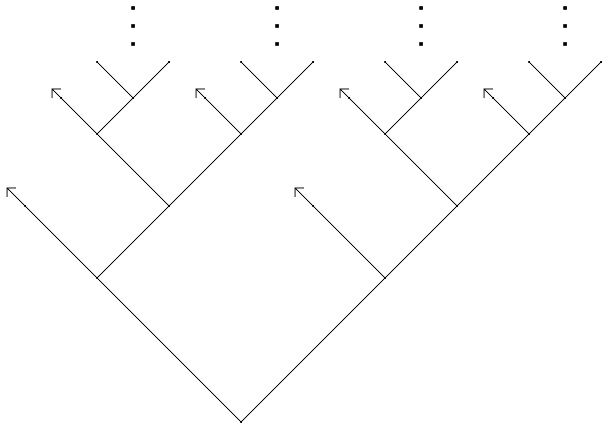
Definition: A *graceless* Π^0_1 class is an uncountable class with no perfect Π^0_1 subclasses.

Theorem (Solomon-Weber): There exists an uncountable Π^0_1 class which is thin and has densely-many isolated paths.

Corollary: There exists a graceless Π^0_1 class.
Proof: Let P be as in the theorem. We know P is uncountable. Since every subclass of P is relatively clopen and every interval of P contains an isolated path, no subclass of P is perfect. $\square$

Proof of Thm: The proof is similar to the construction of a perfect thin Π^0_1 class. To guarantee uncountability, the class minus its isolated paths will be perfect in the topological sense, though clearly not computable.

Creating a perfect thin class can be thought of as taking the complete tree and stretching it out, via successive homeomorphisms from 2^ω. This construction can be thought of as taking the tree in the figure and stretching it, although the way it is "stretched" may result in additional isolated paths between nodes of the perfect subtree. We cannot build the tree using homeomorphisms from 2^ω, ignoring the isolated paths, however, because that will result in a perfect Π^0_1 class skeleton for the class we are building, defeating the construction.



Since we cannot use the map from 2^ω, we lose our easy way to keep track of which nodes are working to ensure $[T_e]\bigcap[T]$ is relatively clopen in $[T]$; that is, the nodes which ensure that T_e is not a witness to the non-thinness of T. This is remedied by defining witness nodes as we go, which may or may not occur at the same level of the tree. However, we ensure that every extension of one requirement's witness node is either isolated or passes through a lower priority requirement's witness node. That way we keep the branching controlled.

We build our tree T in stages. At each stage $s \in \omega$ we will have a finite tree T_s such that $T_s \subseteq T_{s+1}$. Ultimately we will define $T = \bigcup_s T_s$. As usual, let $\{T_e\}_{e\in\omega}$ be a computable listing of computable trees.

To guarantee thinness, we must satisfy the requirements

R$_e$: $[T_e]\bigcap[T]$ is relatively clopen in $[T]$.

That is, $\exists$ clopen $C([T]\bigcap C = [T]\bigcap[T_e])$.

We split R_e into $\mathrm{R}_{<e,i>}$ where $0 \le i \le 2^e + 1$. Each $\mathrm{R}_{<e,i>}$ will be assigned a corresponding $\sigma_{<e,i>} \in T$, such that if $\sigma_{<e,i>}$ is a member of T_e, then all of $I(\sigma_{<e,i>})$ is in $[T_e]$, and if $\sigma_{<e,i>}$ is not in T_e, then $I(\sigma_{<e,i>}) \bigcap [T_e] = \emptyset$. The latter property is clear. For the former, we split R_e into the requirements

$\mathrm{R}_{<e,i>}$: $\sigma_{<e,i>} \in T_e \Rightarrow \forall \tau \succeq \sigma_{<e,i>} (\tau \in T \rightarrow \tau \in T_e)$.

We meet $\mathrm{R}_{<e,i>}$ in the usual way. Loosely speaking, if $\sigma_{<e,i>} \in T_e$, we wait for a node $\tau \succeq \sigma_{<e,i>}$ which is in T but not in T_e. If such a τ appears, we terminate all extensions of $\sigma_{<e,i>}$ which are incomparable with τ, so that all paths through $\sigma_{<e,i>}$ are routed through τ. That makes $I(\sigma_{<e,i>}) \bigcap [T]$ disjoint from $[T_e]$. We then redefine $\sigma_{<e,i>}$ to be τ. This explanation is inaccurate in that it does not account for the isolated paths we must build, or for keeping our alterations to the tree at the currently top level of the tree, but it has the correct intuition.

Before the construction begins we require some definitions:

Definition: $\mathrm{R}_{<e,i>}$ is of *higher priority* than $\mathrm{R}_{<e',i'>}$ if either

(1) $e < e'$, or
(2) $e = e'$ and $i < i'$.

Definition: $\mathrm{R}_{<e,i>}$ is *defined* if a node $\sigma_{<e,i>}$ has been assigned to it, and *undefined* otherwise.

Definition: $\mathrm{R}_{<e,i>}$ *requires attention at stage* s if either

(1) $\mathrm{R}_{<e,i>}$ is undefined, or
(2) $\mathrm{R}_{<e,i>}$ is defined, $\sigma_{<e,i>} \in T_e$, and there is some $\tau \succeq \sigma_{<e,i>}$ such that $\tau \in T_s$ but $\tau \notin T_e$.

We must keep track of which terminal nodes may be extended, and whether or not they are on what we currently believe to be isolated paths. To that end, let Iso_s be the set of nodes in T_s which are terminal and on currently-isolated paths, and let Ext_s be the remaining terminal extendible nodes of T_s.

The Construction:

Stage $s = 0$: $T_s = \{\lambda\}$, $\mathrm{Iso}_s = \emptyset$, $\mathrm{Ext}_s = \{\lambda\}$, all $\mathrm{R}_{<e,i>}$ undefined.

Stage $s + 1$: Choose the least $\mathrm{R}_{<e,i>}$ which requires attention.

Case 1: $\mathrm{R}_{<e,i>}$ is undefined.
Action: If $e = 0$, let $\sigma_{<e,i>} = i$. Otherwise, if $i = 2j$, let $\sigma_{<e,i>} = \sigma_{<e-1,j>}{}^\frown 10$. If $i = 2j + 1$, let $\sigma_{<e,i>} = \sigma_{<e-1,j>}{}^\frown 11$. Let $\mathrm{Ext}'_s = (\mathrm{Ext}_s \bigcup \{\sigma_{<e,i>}{}^\frown 1\}) - \{\sigma_{<e,i>}{}^\frown 0^\frown \tau : \tau \in 2^{<\omega}\}$, and let $\mathrm{Iso}'_s = \mathrm{Iso}_s \bigcup \{\sigma_{<e,i>}{}^\frown 0\}$. That is, isolate the leftward path extending $\sigma_{<e-1,j>}$, and choose a node on the appropriate rightward path for $\sigma_{<e,i>}$.

Case 2: $\sigma_{<e,i>} \in T_e$, and there is $\tau \succeq \sigma_{<e,i>}$ such that $\tau \in T_s$ but $\tau \notin T_e$.
Action: Choose some $\theta \in \mathrm{Ext}_s \bigcup \mathrm{Iso}_s$ such that $\theta \succeq \tau$, and let $\mathrm{Ext}'_s = (\mathrm{Ext}_s \bigcup \{\theta^\frown 1\}) - \{\nu \in \mathrm{Ext}_s : \nu \succeq \sigma_{<e,i>}\}$. Let $\mathrm{Iso}'_s = (\mathrm{Iso}_s \bigcup \{\theta^\frown 0\}) - \{\theta\}$. That is, choose a node extending τ and thus not in T_e, and route all but a finite number of paths in $I(\sigma_{<e,i>})$ through that node. We do not terminate any isolated paths since there will be only a finite number of them. Finally, let $\sigma_{<e,i>} = \theta$, and undefine all $\mathrm{R}_{<e,i>}$ of lower priority.

After taking action, let $\mathrm{Iso}_{s+1} = \{\sigma^\frown 0 : \sigma \in \mathrm{Iso}'_s\}$, $\mathrm{Ext}_{s+1} = \{\sigma^\frown i : \sigma \in \mathrm{Ext}'_s, i \in \{0,1\}\}$, and $T_{s+1} = T_s \bigcup \{\sigma : \exists \tau \in (\mathrm{Iso}_{s+1} \cup \mathrm{Ext}_{s+1}) \text{ st } \sigma \preceq \tau\}$.

Verification:

Lemma 1: Each requirement is permanently satistfied after a finite number of stages of the construction.

Proof: Suppose by induction that all requirements of lower priority than $\mathrm{R}_{<e,i>}$ have been satisfied. $\mathrm{R}_{<e,i>}$ will require definition, and then at most one additional action. If $I(\sigma_{<e,i>}) \bigcap T \subseteq T_e$ or $\sigma_{<e,i>} \notin T_e$, then no action is necessary to satisfy $\mathrm{R}_{<e,i>}$. Otherwise, at some finite stage s, we will find $\tau \succeq \sigma_{<e,i>}$ such that $\tau \in (\mathrm{Ext}_s \bigcup \mathrm{Iso}_s) - T_e$, terminate all non-isolated paths between $\sigma_{<e,i>}$ and τ, and redefine $\sigma_{<e,i>}$ to be τ. Since $\mathrm{R}_{<e,i>}$ cannot be injured by requirements of lower priority, after that action it is satisfied forever. □

Lemma 2: The tree $T = \bigcup_s T_s$ contains densely-many isolated paths.

Proof: We need to know that every basic open set $I(\sigma)$ in T, for $\sigma \in 2^\omega$, contains an isolated path. If $|I(\sigma)| = 1$, we are done. It is clear by the definition of the $\sigma_{<e,i>}$ that every branching node is below some $\sigma_{<e,i>}$. Every $\sigma_{<e,i>}$ is extended by an isolated path, so if $|I(\sigma)| > 1$, $I(\sigma)$ contains some $I(\sigma_{<e,i>})$ and thus an isolated path. □

Lemma 3: The tree $T = \bigcup_s T_s$ is thin.

Proof: Since all requirements $\mathrm{R}_{<e,i>}$ were satisfied, for each computable tree T_e, the intersection of T with T_e is composed exactly of some subset of $\{I(\sigma_{<e,i>}) : 0 \leq i \leq 2^e + 1\}$, together with a finite number of isolated paths branching off at points below the $\sigma_{<e,i>}$. Therefore, the intersection is clopen and T is thin. □

This completes the proof of the theorem. □