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Tutorial on Π_1^0 Classes

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Plan:

- Basic definitions; examples
- Basis and antibasis theorems
- Connections to randomness
- Enumeration and index sets
- Lattice intervals and invariance
- Lattice embeddings and theories

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Definitions

ω : natural numbers, beginning at 0 (k, n, m)

$2^{<\omega}$: set of all finite binary sequences (a.k.a. $\{0, 1\}^*$);
complete binary-branching tree (σ, τ, ρ)

2^ω : set of all infinite binary sequences; Cantor space (X, Y, Z)

subtree of $2^{<\omega}$: subset closed under initial segment; dead ends allowed

Π_1^0 class: set of infinite paths through a computable subtree of $2^{<\omega}$

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Definitions

Empty string denoted λ (often also written $\langle \rangle$)

Length of σ is $|\sigma|$

$0^n, 1^n, 0^\omega, 1^\omega$: string of all 0s or all 1s of length in superscript

Concatenation of σ and τ indicated by $\sigma\tau$ or $\sigma\hat{\ } \tau$.

If τ extends σ ($\exists \rho(\sigma\rho = \tau)$), write $\sigma \subseteq \tau$. If $\sigma \not\subseteq \tau$ and $\tau \not\subseteq \sigma$, $\sigma \perp \tau$.

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Definitions

$X \upharpoonright i$ is the length- i initial segment of X

If $|\sigma| = n$, $\sigma(0)$ is the first bit of σ and $\sigma(n-1)$ the last.

$[\cdot]$ means “infinite strings associated with” for us: If T is a tree, $[T]$ is the associated Π_1^0 class; if σ is a finite string, $[\sigma]$ is the set of all infinite strings that extend σ (*interval* around σ)

Lattice of all Π_1^0 classes: denoted \mathcal{E}_Π .

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Working within 2^ω

Topology: basic clopen sets are intervals.

Measure: size of interval $[\sigma]$ is $2^{-|\sigma|}$ (coin-toss probability measure).

Metric: distance between X and Y is measure of least interval containing both; i.e., $2^{-|\sigma|}$ for σ the longest initial segment common to X and Y .

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Solution sets

Perhaps the most significant use of Π_1^0 classes is as representations of solution sets to problems of finding examples of something (e.g. separating sets, ideals, zeros of a function). When the problem is presented as a computably enumerable sequence of computable requirements we can often build a Π_1^0 class with paths corresponding exactly to solutions of the problem.

Note these proofs need not be effective – there may be no solution that is computable. If we collect the solutions into a Π_1^0 class, we may be able to make other complexity-related statements about them, though.

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Template for building a Π_1^0 class

- Start with λ , the empty node.
- Unless otherwise instructed, at stage $s + 1$ enumerate both children of every length- s node in the tree.
- Concurrently enumerate a list of properties the infinite paths must have.
- Cease extending any node when you see all sequences in its interval will fail some property.
- To survive at all levels, a path must satisfy all properties.
- Why computable? Only put nodes in, never take them out, and all length- s nodes are in at stage s .

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Example 1: Separating sets (the canonical example)

Have disjoint c.e. A and B . Associate elements of ω with levels of the tree, starting at level 1; paths are interpreted as characteristic functions.

Requirement on paths X : if $n \in A$, $X(n) = 1$. If $n \in B$, $X(n) = 0$. Enumeration of A and B gives enumeration of desired properties.

Pruning method: If n enters A at stage s , cease extending any living length- s node σ such that $\sigma(n) = 0$. Likewise for B and $\sigma(n) = 1$.

Correct Π_1^0 class: Since any number entering A or B must enter at some finite stage, at that stage all paths containing the wrong level- n value will be killed.

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Example 2: Zeros of a computable function

Have computable $f : 2^\omega \rightarrow 2^\omega$, presented by enumeration of pairs of intervals $\langle [\sigma_n], J_n \rangle$ ($\{\sigma_n\}$ enumeration of all finite strings in lexicographical order) such that $f[[\sigma_n]] \subseteq J_n$ and if $\{X\} = \lim_i \sigma_i$, $\{f(X)\} = \bigcap_i J_i$. Paths of the tree interpreted as elements of 2^ω .

Requirement on paths X : $f(X) = 0^\omega$.

Pruning method: If $\langle [\sigma_s], J_s \rangle$ is such that $0^\omega \notin J_s$, cease extending any living length- s node extending σ_s .

Correct Π_1^0 class: If $f(X) \neq 0^\omega$, then $f(X) = Y$ for some Y a nonzero distance from 0^ω . At some finite stage the sequence of intervals intersecting to $\{Y\}$ will be small enough to exclude 0^ω .

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Likewise:

- Fixed points of a computable function (prune when the preimage and image intervals are disjoint)
- Points at which the computable function attains a maximum [minimum] (when you see $\langle I_1, J_1 \rangle, \langle I_2, J_2 \rangle$ such that all elements of J_1 are strictly less [greater] than all elements of J_2 , prune I_1)

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Other examples

- Complete consistent extensions of an axiomatizable first-order theory (i.e., one whose true sentences form a c.e. set): levels of tree correspond to all sentences in language; prune when you see inconsistency.
- Prime ideals of a c.e. commutative ring with unity: levels of tree correspond to all elements of ring; prune when you see 1s at levels a, b and 0 at level $a + b$, or 1 at level a and 0 level ab for some b , or 0s at levels a, b and 1 at level ab (need commutativity for that characterization of primality).

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Completions of PA

PA, or Peano Arithmetic, is a first-order formalization of arithmetic consisting of $=, +, \cdot, 0$, successor, and induction.

PA is axiomatizable so its completions form a Π_1^0 class.

Solovay & Scott proved the degrees of consistent extensions of PA and completions of PA coincide with each other and with the degrees \mathbf{a} such that every Π_1^0 class contains a path of degree $\leq_T \mathbf{a}$ (these are the PA degrees, denoted $\mathbf{a} \gg \mathbf{0}$).

DNR₂

Let $\{\varphi_e\}_{e \in \omega}$ be an enumeration of partial computable functions. $X \in k^\omega$ is *diagonally non-recursive* (DNR_k) if $(\forall n)[X(n) \neq \varphi_n(n)]$.

The DNR₂ sets form a Π_1^0 class (whenever you see convergence of a new computation of $\varphi_n(n)$, prune paths that agree with it at level n).

The Turing degrees of paths of DNR₂ are the same as those of PA, but DNR₂ is a separating class.

Steve will tell more about DNR₂ in the context of Medvedev and Muchnik degrees.

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Representability theorems

Every Π_1^0 class represents

- the set of zeros of some computable function (can build the function out of the tree).
- the set of fixed points of some computable function.
- the set of points at which some computable function attains its minimum [maximum].
- the set of complete consistent extensions of some axiomatizable theory.
- the set of prime ideals of some c.e. commutative ring with unity.

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We should note that in some cases *every* Π_1^0 class represents a solution set for some instantiation of a given problem, and in some cases not.

For example:

Not every Π_1^0 class is a separating class, clearly: need only two length- n nodes with different branching properties.

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A few basis theorems (Jockusch and Soare, 1972)

Every nonempty Π_1^0 class $P \subseteq 2^\omega$ contains

- (a) a path of low Turing degree;
- (b) a path of c.e. Turing degree;
- (c) a computable path or two paths with degree infimum zero;
- (d) a path of hyperimmune-free degree.

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Consequences of basis theorems

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A computable function need not have a computable zero, but it must have a zero of low degree and one of c.e. degree. If it has no computable zeros it has two zeros which form a minimal pair in the Turing degrees.

Likewise a pair of c.e. sets must have a c.e. separating set (this is clear anyway) and a low separating set.

Antibasis theorems

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- A nonempty Π_1^0 class need not have a computable member (Kreisel 1953)
 - The Π_1^0 class with no computable member may even have positive measure, though its measure cannot be a computable real
 - The low and c.e. paths need not be the same (Arslanov 1981)
 - The minimal pair need not both be Δ_2^0 (Kučera 1988)
- PA, and hence DNR_2 , satisfies all but the second of these.

One proof: Low basis theorem (forcing with Π_1^0 classes)

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Given $P = [T]$ for computable T , define a sequence of computable subtrees $T = T_0 \supseteq T_1 \supseteq \dots$ so $\bigcap_e [T_e]$ is nonempty and contains only low paths.

By induction, assume T_e is defined and infinite. Let $U_e = \{\sigma : \Phi_{e,|\sigma|}^\sigma(e) \uparrow\}$ (standard enumeration of functionals Φ_e); U_e is a computable tree. Using $\mathbf{0}'$, choose $T_{e+1} = T_e$ if $U_e \cap T_e$ is finite, and $T_{e+1} = U_e \cap T_e$ otherwise. Hence in T_{e+1} either all paths X give $\Phi_e^X(e) \uparrow$ or all give $\Phi_e^X(e) \downarrow$, and all T_e are infinite so $\bigcap_e [T_e] \neq \emptyset$ by compactness. The construction is computable in $\mathbf{0}'$, so $X' \leq_T \mathbf{0}'$ for all $X \in \bigcap_e [T_e] \subseteq [T]$.

Consequences of antibasis theorems

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We can't *a priori* say anything about separating sets, since not all Π_1^0 classes are separating classes (though many antibasis theorems hold for separating classes as well – as DNR_2 witnesses).

However, we can say there is a computable function with no computable zeros, even one that has a set of zeros of positive measure but still no computable one.

More membership theorems

We have a lot of degree control (Jockusch and Soare, 1972):

- There is a nonempty Π_1^0 class such that the only c.e. degree \geq_T any path of the class is $\mathbf{0}'$.
- For any c.e. degree \mathbf{c} there is a Π_1^0 class such that the degrees of its c.e. paths are exactly those $\geq_T \mathbf{c}$.
- For any degree \mathbf{a} there is a nonempty Π_1^0 class with no members of degree $\mathbf{0}$ or \mathbf{a} .
- There is a nonempty Π_1^0 class all of whose members are Turing incomparable.

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Connections to randomness

We take on faith that the random reals are exactly those that pass the *universal Martin-Löf test*. That is, there is a computable sequence of Σ_1^0 classes (subsets of 2^ω) such that the nonrandom reals are exactly those reals in the intersection of the sequence. Furthermore the n^{th} class in the sequence has measure bounded by 2^{-n} (Denis will elaborate).

As the complement of a Σ_1^0 class is a Π_1^0 class, there are Π_1^0 classes all of whose elements are random; in fact with measure arbitrarily close to 1.

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A version of that last one for separating classes (JS '72):

General: There is a nonempty Π_1^0 class all of whose members are Turing incomparable.

Specific: There are disjoint c.e. sets A and B that are computably inseparable such that any two separating sets of A and B either have finite difference or are Turing incomparable.

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The Π_1^0 classes of positive measure are exactly those containing a random real (observation/Kurtz).

Every Π_1^0 class of positive measure has an element of every 1-random degree (Kučera).

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Downey and Miller jump inversion (2006):

If P is a Π_1^0 class of positive measure, then for every Σ_2^0 set $S \geq_T \mathbf{0}'$, there is a Δ_2^0 real $A \in P$ such that $A' \equiv_T S$.

Taking P to be one of the Π_1^0 classes containing only random reals, we get a Δ_2^0 random real A that jumps to S .

Enumerations

Before constructing an enumeration of all Π_1^0 classes, we show the complexity of tree representation is flexible:

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Proposition. For any $P \subseteq 2^\omega$, TFAE:

- (a) $P = [T]$ for some Π_1^0 tree $T \subseteq 2^{<\omega}$;
- (b) $P = [T]$ for some computable tree $T \subseteq 2^{<\omega}$;
- (c) $P = [T]$ for some primitive recursive tree $T \subseteq 2^{<\omega}$.

Enumerating the Π_1^0 classes via primitive recursive trees

For $\{W_e\}_{e \in \omega}$ an enumeration of all c.e. sets and $\{\sigma_e\}_{e \in \omega}$ an enumeration of $2^{<\omega}$ (lexicographically, say), define the tree T_e by

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$$\sigma \in T_e \Leftrightarrow (\forall n < |\sigma|) [\sigma_n \subseteq \sigma \rightarrow n \notin W_{e,|\sigma|}].$$

Then $P_e = [T_e]$ enumerates all Π_1^0 classes.

Another proof:

(c) \Rightarrow (b) \Rightarrow (a) is clear.

(a) \Rightarrow (b): From Π_1^0 T given by computable relation R such that $\sigma \in T \Leftrightarrow (\forall n)R(n, \sigma)$, build computable tree $S \supseteq T$:

$$\sigma \in S \Leftrightarrow (\forall m, n \leq |\sigma|)R(m, \sigma \upharpoonright n).$$

(b) \Rightarrow (c): From computable T given by total computable $\{0, 1\}$ -valued function φ such that $\sigma \in T \Leftrightarrow \varphi(\sigma) = 1$, build primitive recursive tree $S \subseteq T$:

$$\sigma \in S \Leftrightarrow (\forall n < |\sigma|) \neg \varphi_{|\sigma|}(\sigma \upharpoonright n) = 0.$$

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Note that neither the proposition about equivalence of representations nor the construction of the enumeration of Π_1^0 classes is dependent on using Π_1^0 subclasses of 2^ω ; both will go through if we use ω^ω . We will stick to the former but there are many additional index set results for ω^ω .

Recall that given an enumeration $\{\xi_e\}_{e \in \omega}$ (of anything) an *index set* \mathcal{I} is any subset of ω such that if $a \in \mathcal{I}$ and $\xi_a = \xi_b$, then $b \in \mathcal{I}$.

A set $A \subseteq \omega$ is H_n^m -complete (for $H = \Pi, \Sigma, \Delta$) if it is H_n^m and every other H_n^m set B is 1-reducible to A .

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In our setting, the index sets will often be properties of trees, but sets of indices of Π_1^0 classes. That is, many of the sets will be of the form

$$\mathcal{I} = \{e : P_e \text{ has a tree representation with property } \alpha\},$$

and all indices i of P_e will be in the set if at least one of them corresponds to a tree T_i with property α .

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Why should we care?

We can transfer these results to statements about index sets of computable mathematical problems. For instance, the index set of primitive recursive graphs with a 4-coloring is Π_1^0 -complete, but the index set of those with a *computable* 4-coloring is Σ_3^0 -complete; this strengthens the result that there is a computable 4-colorable graph with no computable 4-coloring.

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Let $\mathcal{I}(\mathcal{P})$ be the index set of classes with property \mathcal{P} .

- $\mathcal{I}(\text{nonempty})$ is Π_1^0 -complete.
- $\mathcal{I}(\text{no more than } c \text{ paths})$ is Π_2^0 -complete for fixed $c \geq 1$.
- $\mathcal{I}(\text{exactly } c \text{ paths})$ is Π_2^0 -complete for $c = 1$ and D_2^0 -complete for $c > 1$.
- $\mathcal{I}(\text{finite})$ is Σ_3^0 -complete.
- $\mathcal{I}(\text{countable})$ is Π_1^1 -complete.

D_n^m sets are those expressible as the difference of two Σ_n^m sets.

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A few more, as we are often interested in the existence of computable solutions to problems:

- $\mathcal{I}(\text{no comp. paths})$ and $\mathcal{I}(\text{nonempty; no comp. paths})$ are Σ_3^0 -complete.
- $\mathcal{I}(\text{more than } c \text{ comp. paths})$ is Σ_3^0 -complete.
- $\mathcal{I}(\text{exactly } c \text{ comp. paths})$ is D_3^0 -complete.
- $\mathcal{I}(\text{infinitely many comp. paths})$ is Π_4^0 -complete.

Again: $D_n^m = \text{difference of two } \Sigma_n^m \text{ sets.}$

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Center and Remmel (CR):

There exist computable functions taking indices for computably continuous functions (CCFs) on 2^ω to indices for Π_1^0 classes representing their set of zeroes and conversely.

This allows us to transfer index set results. For example,

- The index set of CCFs which have exactly c zeros for any fixed $c \geq 1$ is D_2^0 -complete.
- The index set of CCFs which have exactly c computable zeros for any fixed $c \geq 1$ is D_3^0 -complete.
- The index set of CCFs which have more than c zeros for any fixed $c \geq 1$ is Σ_2^0 -complete.
- The index set of CCFs which have more than c computable zeros for any fixed $c \geq 1$ is Π_3^0 -complete.

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One more theorem about tree representations

In fact, polynomial-time computable trees suffice to represent all Π_1^0 classes.

Of course, must say what we mean by polynomial-time computable tree. In 2^ω it is straightforward; if we were dealing with Π_1^0 classes in a different space we would have to do some work.

Given a computable function φ for a tree T , we approximate T by T_s , where

$$\sigma \in T_s \Leftrightarrow \varphi_s(\sigma) \uparrow \text{ or } \downarrow = 1.$$

The p-time tree P is defined by

$$\sigma \in P \Leftrightarrow (\forall \tau \subset \sigma)[\tau \in T_{|\sigma|}].$$

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Lattice Structure

The collection of all Π_1^0 classes ordered by inclusion forms a distributive lattice, denoted \mathcal{E}_Π .

Top and bottom: 2^ω and \emptyset

Meet and join: \cap and \cup

Atoms (minimal elements): singletons (computable paths)

Complemented elements: clopen sets (finite unions of intervals)

Intervals in the lattice: $[P, P'] = \{Q \in \mathcal{E}_\Pi : P \subseteq Q \subseteq P'\}$

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Once we have a lattice, we can look at intervals of and embeddings into the lattice, as well as definability. There are several computably isomorphic (though order-reversing) settings we can work in to obtain these results.

- \mathcal{E}_Π itself;
- the lattice of c.e. ideals/filters of the countable atomless Boolean algebra Q ;
- the lattice of c.e. ideals/filters of $2^{<\omega}$;

[Isomorphisms laid out in CCDH and W]

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The isomorphic setting we will use is the c.e. ideals of $2^{<\omega}$:

A string $\sigma \in 2^{<\omega}$ is a *nonextendible node* of the Π_1^0 class P if $[\sigma] \cap P = \emptyset$.

If $[T] = P$ for a computable tree T , $\sigma \notin T$ is nonextendible, and $\sigma \in T$ such that all extensions of σ dead-end is also nonextendible.

The nonextendible nodes of P for any Π_1^0 P form a c.e. ideal of $2^{<\omega}$; can see the isomorphism is order-reversing.

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The computable isomorphisms are easiest to see in the setting of c.e. ideals, where we are looking at an *initial* segment (interval of all ideals contained in the given ideal).

The *root set* of an ideal \mathcal{I} is the minimal generating set: $\{\sigma_i\}_{i \in I}$ such that $\mathcal{I} = \{\tau : (\exists i \in I)(\tau \supseteq \sigma_i)\}$ and $i \neq j \Rightarrow \sigma_i \perp \sigma_j$.

A clopen Π_1^0 class corresponds to an ideal with a finite root set of size $k + 1$, say; we may map the i^{th} element to $1^i 0$, $0 \leq i < k$, with the final element mapping to 1^k . (If the root set has size 1 map it to the empty node.) Fill in $2^{<\omega}$ in the natural way; this generates an isomorphism on ideals.

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Intervals

There are exactly two isomorphism types for nontrivial end segments of \mathcal{E}_Π .

– Cholak, Coles, Downey, Herrmann (CCDH):

If $P \subsetneq 2^\omega$ is a clopen Π_1^0 class, then $[P, 2^\omega] \cong \mathcal{E}_\Pi$ computably.

If $P, Q \in \mathcal{E}_\Pi$ are nonclopen, then $[P, 2^\omega] \cong [Q, 2^\omega]$ computably.

– Cenzer and Nies (CN2):

If $P \in \mathcal{E}_\Pi$ is nonclopen, then $[P, 2^\omega] \not\cong \mathcal{E}_\Pi$.

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A *basis* of an ideal is a set B that generates the ideal such that any two elements of B are incomparable.

For the nonclopen isomorphism we need a lemma:

Any c.e. ideal has a c.e. basis.

Given a nonclopen Π_1^0 class, let $\{\sigma_i\}_{i \in \omega}$ be a c.e. basis for the associated c.e. ideal. We'll map it to one standard nonclopen ideal: the one with root set $\{1^j 0 : j \in \omega\}$. Map the basis to the root set (in order of enumeration) and fill in $2^{<\omega}$ in the natural way; the map generated is an isomorphism between the two initial segments of ideals.

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The nonisomorphism between end segments starting with clopen or nonclopen Π_1^0 classes is harder to prove.

(CN2) is a contradiction argument. Nies later found a Σ_3^0 -definable difference in the setting of c.e. ideals of the countable atomless Boolean algebra Q .

For two ideals $A, E \in I(Q)$, A is *small in* E ($A \subset_s E$) if $A \subset E$, E is noncomplemented in $I(Q)$, A is noncomplemented in $[0, E]$, and if $Y \subseteq A$ is complemented in $[0, E]$, then Y is also complemented in $I(Q)$.

Let β be the statement $\exists E \exists A (A \subset_s E)$. $I(Q) \models \beta$ but for nonprincipal (corresponding to nonclopen Π_1^0 class) ideal M , $[0, M] \not\models \beta$.

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Thin classes

A Π_1^0 class P is *thin* if every Π_1^0 subclass of P is relatively clopen; that is, for each $Q \subseteq P$ there is clopen $C \subseteq 2^\omega$ such that $Q = P \cap C$.

The thin Π_1^0 classes are exactly those P such that $[\emptyset, P]$ is a Boolean algebra (i.e. distributive, complemented lattice) - hence thinness (including finite) is definable in \mathcal{E}_Π .

The index set of thin classes is Π_4^0 -complete.

Perfect thin classes

A Π_1^0 class is *perfect* if it has no isolated paths. In other words, every extendible node of its representative tree has at least two incomparable extensions.

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Perfect thin classes P are exactly those such that $[0, P]$ is an atomless Boolean algebra, because a computable path must be isolated in a thin class; hence they are definable in \mathcal{E}_Π .

CCDH show that perfect thin classes witness degree invariance of the array noncomputable degrees. That is, each anc degree is represented by a perfect thin class and all perfect thin classes have anc degree; definability implies invariance under automorphisms. In fact they form an orbit.

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Minimal classes

A Π_1^0 class P is *minimal* if every Π_1^0 subclass of P is finite or cofinite in P .

The minimal Π_1^0 classes are the atoms when working with \mathcal{E}_Π^* ($:= \mathcal{E}_\Pi$ modulo finite difference).

The minimal classes with noncomputable paths are exactly the thin classes with exactly one non-isolated point.

The index set of minimal classes is Π_4^0 -complete.

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Comparisons between \mathcal{E}_Π and \mathcal{E} , the lattice of c.e. sets, are often fruitful lines of research.

Nies proved that if an interval of \mathcal{E} is not a Boolean algebra, then it has an undecidable theory (in fact its theory interprets true arithmetic).

Cenzer and Nies (CN1) proved that there are intervals of \mathcal{E}_Π that are not Boolean algebras but have decidable theories.

The proof is via \mathcal{E}_Π^* .

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Given a lattice (L, \leq) we denote join (l.u.b.) by \vee and meet (g.l.b.) by \wedge ; the greatest and least elements are 1 and 0.

(L, \leq) satisfies the *dual reduction property* if for any $a, b \in L$, there exist $a_1 \geq a$ and $b_1 \geq b$ such that $a_1 \vee b_1 = 1$ and $a_1 \wedge b_1 = a \wedge b$.

(CN1) step 1: For any finite distributive lattice L that satisfies the dual reduction property, there is a Π_1^0 class P such that $[\emptyset, P]^* \cong L$.

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Small pieces of proof:

For one-element L , any finite P will do.

For two-element L , P must be minimal, so $[\emptyset, P]^*$ will have two elements.

For larger L , the construction of a minimal Π_1^0 class is generalized: make P with subclasses that are “minimal over” each other (aligned with the structure of L).

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(CN1) step 2: For the P constructed, $[\emptyset, P]$ is isomorphic to a sublattice of the $\mathcal{P}(\mathbb{N})$ that is closed under finite differences.

Lachlan: If a lattice $L \subset \mathcal{P}(\mathbb{N})$ is closed under finite differences, then the theory of L is many-one reducible to the theory of L^* .

(CN1) step 3: The theory of $[\emptyset, P]$ is many-one reducible to the theory of L (the original finite lattice) and is hence decidable.

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So in \mathcal{E} , “not a Boolean algebra” \Rightarrow “interprets true arithmetic”.

In \mathcal{E}_Π , “not a Boolean algebra” doesn’t even imply “undecidable”.

However, if $P \in \mathcal{E}_\Pi$ is *decidable* and $[\emptyset, P]$ is not a Boolean algebra, then the theory of $[\emptyset, P]$ interprets true arithmetic.

Decidability for a Π_1^0 class P means the tree T with no dead ends such that $[T] = P$ is computable.

Some topics we didn’t cover

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- Π_1^0 classes in ω^ω , \mathbb{R} , or $[0, 1]$
- The Cantor-Bendixson derivative and rank
- Reverse mathematics and Ramsey theory
- The structure of the lattice $[P, 2^\omega]$ for P nonclopen
- More examples and applications: graph theory and combinatorics, orderings, nonmonotonic logic

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