AN ELEMENTARY PROOF OF DIRICHLET'S THEOREM IN THE POLYNOMIAL SETTING

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Abstract. We show that H.N. Shapiro's 1950 proof of Dirichlet's theorem can be adapted to prove the analogue in $\mathbf{F}_q[t]$, where $\mathbf{F}_q$ is the finite field of q elements.

1. Introduction

Dirichlet's famous 1837 theorem on primes in progressions states that if $m > 0$ and a are integers with $(a, m) = 1$, then there are infinitely many primes p with $p \equiv a \bmod m$. In fact, Dirichlet established much more:

$$\lim_{s \downarrow 1} \left(\sum_{p \equiv a \bmod m} p^{-s} \Big/ \log \frac{1}{(s-1)} \right) = \frac{1}{\varphi(m)}. \tag{1}$$

Since $\log 1/(s-1) = \log \zeta(s) + O(1) = \sum_p p^{-s} + O(1)$ as $s \downarrow 1$, this shows that in some sense the primes are equally distributed into progressions.

Unfortunately, Dirichlet's original proof cannot be considered entirely elementary, for it uses the Taylor expansion of the complex logarithm about $s = 1$. In addition, Dirichlet's proof of the non-vanishing of certain infinite sums (the L-series corresponding to the real nonprincipal characters at $s = 1$) depends on difficult investigations in the theory of binary quadratic forms.

In 1950, H.N. Shapiro published an *elementary* proof of Dirichlet's theorem. [1] It is elementary in that it avoids the use of the complex logarithm and other tools from function theory entirely, establishes the non-vanishing of the L-functions at $s = 1$ with a minimum of technical machinery, and with few exceptions (e.g., the definition of $L(1, \chi)$) works only with finite sums.

Shapiro's proof actually yields an estimate for $\sum \frac{\log p}{p}$ when $(a, m) = 1$, $m > 0$:

$$\sum_{\substack{p \equiv a \bmod m \\ p \leq x}} \frac{\log p}{p} = \frac{1}{\varphi(m)} \log x + O(1). \tag{2}$$

The analog of Dirichlet's theorem in the case of a polynomial ring over $\mathbf{F}_q$ was first proven by Kornblum [Kor19]. [2] The structure of his proof is largely the same as in the classical case and culminates in the analog of Eq. 1. (For a more modern proof also along classical lines, see [Ros02].) It is the purpose of this note to show that Shapiro's proof and his estimate can be adapted to the case of $\mathbf{F}_q[t]$. In fact,

[1] In addition to Shapiro's original paper [Sha50], see §9.4 of [Sha83] or chapter 7 of [Apo98] for readable expositions. Our treatment most closely follows [GL66].

[2] Kornblum's paper actually only deals with the case when q is prime, but the arguments translate without difficulty to the general case.

over $\mathbf{F}_q[t]$, the argument can be considered slightly more elementary, for as we shall see the L-series we need shall all turn out to be finite sums.

Before proceeding, we introduce a bit of notation:

For $p \in \mathbf{F}_q[t]$, define $|p| = q^{\deg p}$, so that $|p_1 p_2| = |p_1|\,|p_2|$, and define $\varphi(p)$ as the size of the units group of $\mathbf{F}_q[t]/(p)$.

Throughout this paper, π always denotes a monic irreducible of $\mathbf{F}_q[t]$, d, f always denote monic polynomials, n always denotes a nonnegative integer. Sums over polynomials are always understood to be taken only over monic polynomials.

With these conventions, we can state our main result as:

Theorem 1 (Dirichlet's Theorem in $\mathbf{F}_q[t]$, strong form)**.** *Let $\mathbf{F}_q$ be a finite field, $a, m \in \mathbf{F}_q[t]$ with $(a, m) = 1$, $m \neq 0$. Then for $n \geq 0$,*

$$\sum_{\substack{\pi \equiv a \bmod m \\ \deg \pi \leq n}} \frac{\log |\pi|}{|\pi|} = \frac{1}{\varphi(m)} \log (q^n) + O(1).$$

We can rewrite this in a way that closely parallels (2):

Corollary 1. *Under the assumptions of Theorem 1, we have for $x \geq 1$,*

$$\sum_{\substack{\pi \equiv a \bmod m \\ |\pi| \leq x}} \frac{\log |\pi|}{|\pi|} = \frac{1}{\varphi(m)} \log x + O(1)$$

Proof. For $x = q^n$, where $0 \leq n \in \mathbf{Z}$ this is immediate from Theorem 1. The result follows for all $x \geq 1$ from the fact that for $q^n \leq x < q^{n+1}$, $\log x = \log (q^n) + O(1)$. $\square$

2. Preliminaries

We first prove an analog of the estimate $\log[x]! = x \log x - x + O(\log x)$:

Lemma 1. *For $n \geq 0$,*

$$\sum_{\deg f \leq n} \log |f| = \frac{q^{n+1}}{q-1} \log (q^n) - \log q \frac{q}{q-1} \frac{q^n - 1}{q-1}$$

Proof. Let S denote the sum in question. Then

$$S = \sum_{k=0}^{n} k \log q \sum_{\deg f = k} 1 = \log q \sum_{k=0}^{n} k q^k,$$

so

$$S(1-q) = \log q \left(-n q^{n+1} + \sum_{k=1}^{n} q^k \right) = q \log q \left(-n q^n + \frac{q^n - 1}{q-1} \right).$$

The result follows upon solving for S. $\square$

We shall also need some elementary results on the distribution of primes in $\mathbf{F}_q[t]$, which we collect here. These will be easy consequences of the following

Theorem 2 (Prime Number Theorem for $\mathbf{F}_q[t]$)**.** *Let $\mathbf{F}_q$ be a finite field. Let $\nu_q(n)$ denote the number of (monic) prime polynomials of degree n in $\mathbf{F}_q[t]$. For $n \geq 1$, $\sum_{d|n} d\nu_q(d) = q^n$. Thus*

$$\nu_q(n) = \frac{1}{n} \sum_{d|n} q^d \mu(n/d) = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n} \right).$$

Proof. We consider the prime factorization of $t^{q^n} - t$ in $\mathbf{F}_q[t]$. If $\pi(t)$ is a monic prime of degree $d|n$, say $n = dk$, then $t^{q^d} \equiv t \bmod \pi(t)$ by the analog of Fermat's little theorem; repeatedly raising both sides to the q^d then yields $t^{q^{kd}} \equiv t^{q^{(k-1)d}} \equiv \cdots \equiv t \bmod \pi(t)$, so that $\pi(t)|t^{q^n} - t$.

Conversely, if $\pi(t)|t^{q^n} - x$, choose (the coset of) $g(t)$ to be a generator of the multiplicative group $\mathbf{F}_q[t]/(\pi(t))^*$. Then $g(t)^{q^n} = g(t^{q^n}) \equiv g(t) \mod \pi(t)$, as $t^{q^n} \equiv t \bmod \pi(t)$. As g is a generator, this implies $q^d - 1|q^n - 1$, which forces $d|n$.

Let

$$t^{q^n} - t = \pi_1(t)\pi_2(t)\cdots\pi_k(t)$$

be the unique factorization of $t^{q^n} - t$ into monic irreducibles. By what was said above, every monic irreducible of degree dividing n appears as one of the π_i, and every π_i is a monic irreducible of degree dividing n. Also, no π_i occurs more than once, for otherwise by the product rule π_i divides the formal derivative of $t^{q^n} - t$, which is -1. It follows that

$$t^{q^n} - t = \prod_{\pi(t):\deg \pi | n} \pi(t).$$

Comparing degrees, $q^n = \sum_{d|n} d\nu_q(d)$.

The formula for $\nu_q(n)$ follows by Möbius inversion. Since

$$\sum_{d|n} q^d \mu(n/d) = q^n + O\left(q^{n/2} + nq^{n/3}\right) = q^n + O\left(q^{n/2}\right),$$

the final estimate follows. □

Define the analog of the von-Mangoldt function by

$$\Lambda(f) = \begin{cases} \log|\pi| & \text{if } f = \pi^k, \\ 0 & \text{otherwise} \end{cases}$$

and the analog of the Möbius function by

$$\mu(f) = \begin{cases} 0 & \text{if } \pi^2 \mid f \text{ for any } \pi, \\ (-1)^k & \text{if } f \text{ is a product of } k \text{ distinct primes.} \end{cases}$$

Then just as over $\mathbf{Z}$, we have

Lemma 2. *For every f, $\sum_{d|f} \Lambda(d) = \log|f|$. Also, $\sum_{d|f} \mu(d) \log|d| = -\Lambda(f)$.*

Proof. To prove the first claim, note that

$$\sum_{d|f} \Lambda(d) = \sum_{\pi^k|f} \log|\pi| = \sum_{\pi^k \| f} k \log|\pi| = \log|f|.$$

The second claim then follows by a suitable generalization of Möbius inversion. We can give a direct proof as follows:

Write $f = \pi_1^{a_1}\cdots\pi_k^{a_k}$. To evaluate the left hand side it suffices to restrict the sum to squarefree divisors d. Expanding out $\log|d|$ formally, we see

$$\sum_{\substack{d|f \\ d \text{ squarefree}}} \mu(d)\log|d| = \sum_{\pi|f} \log|\pi| \sum_{j=1}^{k} (-1)^j \binom{k-1}{j-1}.$$

If $k = 0$ so that $f = 1$, the right hand sum is empty. If $k = 1$, then $f = \pi_1^{a_1}$, and the right hand side evaluates to $-\log|\pi_1|$. If $k \geq 2$, the inner sum is equal to $-(1-1)^{k-1} = 0$. Thus in any case the result follows. □

Lemma 3. *For $n \geq 0$,*

$$\Psi(n) := \sum_{\deg f \leq n} \Lambda(f) = \log q \frac{q^{n+1} - q}{q - 1} = O(q^n)$$

Remark. This is another form of the prime number theorem for $\mathbf{F}_q[t]$. It is a remarkable fact worth noting that we have a simple exact formula for this sum. This observation is made in [Sny].

Proof.

$$\begin{aligned} \sum_{\deg f \leq n} \Lambda(f) &= \log q \sum_{\substack{k \geq 1, \pi: \\ k \deg \pi \leq n}} \deg \pi \\ &= \log q \sum_{1 \leq r \leq n} \sum_{k \deg \pi = r} \deg \pi \\ &= \log q \sum_{1 \leq r \leq n} \sum_{\deg \pi | r} \deg \pi \\ &= \log q \sum_{1 \leq r \leq n} q^r, \end{aligned}$$

where the last equality follows Theorem 2. The corollary emerges upon evaluating the geometric series.

From this the big-O estimate is clear. □

Lemma 4. *For $n \geq 0$,*

$$\begin{aligned} \sum_{\deg f \leq n} \frac{\Lambda(f)}{|f|} &= \sum_{\deg \pi \leq n} \frac{\log|\pi|}{|\pi|} + \sum_{2 \deg \pi \leq n} \frac{\log|\pi|}{|\pi|^2} + \cdots \\ &= \sum_{\deg \pi \leq n} \frac{\log|\pi|}{|\pi|} + O(1). \end{aligned}$$

Also,

$$\sum_{\deg \pi \leq n} \frac{\log|\pi|}{|\pi|} = \log(q^n) + O(1).$$

Proof. The first line follows by rearranging the sum, just as when proving the analogous statement over $\mathbf{Z}$. It remains to estimate

$$\sum_{\deg f \leq n} \frac{\Lambda(f)}{|f|} - \sum_{\deg \pi \leq n} \frac{\log|\pi|}{|\pi|} = \sum_{2 \deg \pi \leq n} \frac{\log|\pi|}{|\pi|^2} + \sum_{3 \deg \pi \leq n} \frac{\log|\pi|}{|\pi|^3} \cdots$$

This is bounded by

$$\begin{aligned}\sum_{k\geq 2}\sum_{2\deg\pi\leq n}\frac{\log|\pi|}{|\pi|^k} &= \sum_{\deg\pi\leq n/2}\frac{\log|\pi|}{|\pi|\,(|\pi|-1)}\\ &\leq 2\sum_{k\leq n/2}q^{-2k}\sum_{\pi:\deg\pi=k}k\log q\\ &\leq 2\log q\sum_{k\leq n/2}kq^{-k}\ \ll\ 1.\end{aligned}$$

By the proof of Lemma 3, $\Psi(k)-\Psi(k-1)=q^k\log q$ when $k\geq 1$. Hence

$$\begin{aligned}\sum_{\deg f\leq n}\frac{\Lambda(f)}{|f|} &= \sum_{k=1}^{n}\left(\Psi(k)-\Psi(k-1)\right)q^{-k}\\ &= n\log q\ =\ \log\left(q^n\right).\end{aligned}$$

Referring to the first part of the lemma now finishes the proof. □

3. Characters and L-series

We briefly review properties of characters of finite abelian groups. For details and proofs we refer the reader to either of [Apo98] or [IR91].

By a character of a finite abelian group G we mean a homomorphism $\chi\colon G\to\mathbf{C}^*$. The characters form a group of order $|G|$ (in fact isomorphic to G) under pointwise multiplication. Since $1=\chi(1)=\chi(g^{|G|})=\chi(g)^{|G|}$, each character takes values that are necessarily roots of unity, so that $\chi^{-1}=\overline{\chi}$.

For us, $G=\mathbf{F}_q[t]/(m)^*$, the group of units mod m. We can think of χ as a function on $\mathbf{F}_q[t]$ by identifying a polynomial with its residue class mod m and defining $\chi(p)=0$ for those p not relatively prime to m. The functions we obtain in this manner are called *Dirichlet characters* mod m. The character arising from the trivial homomorphism is denoted χ_0 and called the *principal character*.

The characters of a finite abelian group satisfy certain *orthogonality relations*. For us these take the following form:

Lemma 5 (Orthogonality relations)**.** *Let χ,ψ be characters mod m, and $u,v\in\mathbf{F}_q[t]$ with $(v,m)=1$. Then*

(1) $\frac{1}{\varphi(m)}\sum_{u \bmod m}\chi(u)\overline{\psi(u)}=\delta(\chi,\psi)$

(2) $\frac{1}{\varphi(m)}\sum_{\chi}\chi(u)\overline{\chi(v)}=\delta(u,v)$

Here $\delta(u,v)=1$ if $u\equiv v \bmod m$, 0 otherwise, and $\delta(\chi,\psi)=1$ if $\chi=\psi$, 0 otherwise.

Proof. The proof is the same as in the integer case; see, e.g., Theorem 6.16 in [Apo98] or Theorem 16.3.2 in [IR91]. □

As an example, taking $\psi=\chi_0$, we may deduce from the first relation above that $\sum_{u \bmod m}\chi(u)=0$ for any nonprincipal character χ.

We can now define the L-series we will be concerned with. For χ a nonprincipal character, set

$$L(\chi):=\sum_{k=0}^{\infty}\sum_{\deg f=k}\frac{\chi(f)}{|f|}=\sum_{k=0}^{\infty}\frac{c_k}{q^k},$$

where $c_k = \sum_{\deg f = k} \chi(f)$.

Ostensibly this is an infinite sum. However, we will now show that when $k \geq \deg m$, $c_k = 0$, so that in the definition of $L(\chi)$ we need only sum up to $k = \deg m - 1$.

To see this, let $r \bmod m$ be an arbitrary residue class, where we may assume $r = 0$ or $\deg r < \deg m$. When $k \geq m$, the polynomials f of degree k with $f \equiv r \bmod m$ are exactly the polynomials $f = mQ + r$, where $Q \in \mathbf{F}_q[t]$ is monic of degree $\deg f - \deg m$.

Thus there are $q^{\deg f - \deg m}$ such polynomials for each residue class mod m. Since this number is independent of r, it follows from our application of the orthogonality relations above that $c_k = \sum_{\deg f = k} \chi(f) = 0$.

We can now establish the non-vanishing of $L(\chi)$ for real, nonprincipal χ. Whereas this is one of the most difficult steps in the proof of Dirichlet's theorem over $\mathbf{Z}$, here a fairly easy argument works.

Theorem 3. *If $\chi \neq \chi_0$ is a real, nonprincipal Dirichlet character* $\bmod m$*, then* $L(\chi) \neq 0$.

Proof (Landau, [Kor19]*).* Define $F(f) := \sum_{d \mid f} \chi(d)$. If $f = \pi_1^{a_1} \cdots \pi_k^{a_k}$ is the factorization of f into monic irreducibles,

$$F(f) = \prod_{1 \leq j \leq k} (1 + |\pi_j| + \ldots + |\pi_j|^{a_j}) = \prod_{1 \leq j \leq k} F(\pi_j^{a_j}).$$

Now if $\chi(\pi) = 0$ then $F(\pi^l) = 1$; if $\chi(\pi) = 1$ then $F(\pi^l) = l + 1$. Finally, if $\chi(\pi) = -1$, then

$$F(\pi^l) = \begin{cases} 0 & \text{if } l \text{ is odd} \\ 1 & \text{if } l \text{ is even} \end{cases}$$

Hence $F(\pi^l) \geq 0$ always with $F(\pi^l) \geq 1$ if l is even. Consequently $F(f) \geq 0$ always, and $F(f) \geq 1$ if F is a square.

For natural numbers z, define $S(z) := \sum_{\deg f \leq z} F(f)$. Then for $z \geq \deg m - 1$,

$$\begin{aligned} S(z) &= \sum_{\deg f \leq z} \chi(d) \sum_{\deg d \leq z - \deg f} 1 \\ &= \sum_{k=0}^{z} c_k \sum_{l=0}^{z-k} q^l \\ &= \sum_{k=0}^{\deg m - 1} c_k \frac{q^{z-k+1} - 1}{q - 1} \\ &= \frac{q^{z+1}}{q-1} L(\chi) + c, \end{aligned}$$

where c is constant. Thus if $L(\chi) = 0$, $S(z) = c$ for $z \geq \deg m - 1$. But

$$S(z) \geq \sum_{\substack{f : f = \blacksquare \\ \deg f \leq z}} 1 \geq \sum_{\deg f \leq z/2} 1 = \sum_{0 \leq k \leq z/2} q^k \to \infty.$$

This contradiction completes the proof. $\square$

4. The Proof Proper

We are now ready to move to the heart of the argument: We will study the functions $A_\chi(n) := \sum_{\deg f \le n} \chi(f)\Lambda(f)/|f|$.

Theorem 4. *For $n \ge 0$, $A_{\chi_0}(n) = \log(q^n) + O(1)$.*

Proof. We have

$$A_{\chi_0}(n) = \sum_{\substack{\deg f \le n \\ (f,m)=1}} \frac{\Lambda(f)}{|f|} = \sum_{\substack{\deg \pi \le n \\ \pi \nmid m}} \frac{\log|\pi|}{|\pi|} + R(n),$$

where

$$\begin{aligned} R(n) &= \sum_{\substack{2\deg \pi \le n \\ \pi \nmid m}} \frac{\log|\pi|}{|\pi|^2} + \sum_{\substack{3\deg \pi \le n \\ \pi \nmid m}} \frac{\log|\pi|}{|\pi|^2} + \cdots \\ &\ll \sum_{\deg f \le n} \frac{\Lambda(f)}{|f|} - \sum_{\deg \pi \le n} \frac{\log|\pi|}{|\pi|} \ll 1 \end{aligned}$$

by Lemma 4. Hence

$$A_{\chi_0}(n) = \sum_{\substack{\deg \pi \le n \\ \pi \nmid m}} \frac{\log|\pi|}{|\pi|} + O(1) = \sum_{\deg \pi \le n} \frac{\log|\pi|}{|\pi|} + O(1) = \log(q^n) + O(1),$$

again by Lemma 4. □

The rest of this section is devoted to showing that for $\chi \ne \chi_0$, $A_\chi(n) = O(1)$. Theorem 1 will follow from this result, Theorem 4, and the orthogonality relations.

We first prove the following estimate on the error incurred if we replace an L-series by one of its partial sums:

Lemma 6. *Let χ be a nonprincipal character. Then for $n \ge 0$,*

$$L(\chi) - \sum_{k=0}^{n} \frac{c_k}{q^k} = O(q^{-n}).$$

Proof. As shown in the last section, the function on the left vanishes for $n \ge \deg m - 1$. Hence an acceptable value of the implied constant is

$$q^{\deg m - 2} \max_{0 \le l \le \deg m - 2} \left| L(\chi) - \sum_{k=0}^{l} \frac{c_k}{q^k} \right|.$$

This completes the proof. □

Lemma 7. *Let χ be a nonprincipal character. For $n \ge 0$,*

$$L(\chi) \sum_{\deg f \le n} \frac{\chi(f)\Lambda(f)}{|f|} = O(1).$$

Proof. By Lemma 2,

$$\sum_{\deg f \le n} \frac{\chi(f)\log|f|}{|f|} = \sum_{\deg f \le n} \frac{\chi(f)}{|f|} \sum_{d|f} \Lambda(d) = \sum_{\deg d \le n} \frac{\chi(d)\Lambda(d)}{|d|} \sum_{\deg f \le n - \deg d} \frac{\chi(f)}{|f|},$$

and by Lemma 6, this is equal to $L(\chi)\sum_{\deg d\le n}\chi(d)\Lambda(d)/|d| + *$, where

$$R(n) = O\left(\sum\nolimits_{\deg d\le n}\frac{\Lambda(d)}{|d|}q^{\deg d-n}\right) = O\left(q^{-n}\sum\nolimits_{\deg d\le n}\Lambda(d)\right) = O(1)$$

by Lemma 3. Since for $n \ge \deg m - 1$,

$$\sum_{\deg f\le n}\frac{\chi(f)\log|f|}{|f|} = \sum_{0\le k\le n}\frac{k\log q}{q^k}c_k = \sum_{k=0}^{\deg m-1}\frac{k\log q}{q^k}c_k = c,$$

a constant, the result follows. □

From this we deduce immediately

Corollary 2. *If $L(\chi)\neq 0$ for the nonprincipal character χ, then $A_\chi(n) = O(1)$.*

We now turn to determining what the behavior of $A_\chi(n)$ would be like were $L(\chi) = 0$. To this end we prove

Lemma 8. *Let χ be a nonprincipal character. Then for $n\ge 0$,*

$$\log(q^n) + A_\chi(n) = L(\chi)\sum_{\deg f\le n}\frac{\chi(f)\mu(f)}{|f|}\log\frac{q^n}{|f|} + O(1).$$

In particular, if $L(\chi) = 0$, $A_\chi(n) = -\log(q^n) + O(1)$.

Proof. We evaluate $\sum_{\deg f\le n}\frac{\chi(f)}{|f|}\sum_{d|f}\mu(d)\log\frac{q^n}{|d|}$ in two different ways. Since

$$\begin{aligned}\sum_{d|f}\mu(d)\log\frac{q^n}{|d|} &= \sum_{d|f}\mu(d)\log(q^n) - \sum_{d|f}\mu(d)\log|d| \\ &= \log(q^n)\prod_{\pi|f}(1+\mu(\pi)) + \Lambda(f) \\ &= \begin{cases}\log(q^n) & \text{if } f = 1,\\ \Lambda(f) & \text{otherwise.}\end{cases}\end{aligned}$$

we have on the one hand

$$\sum_{\deg f\le n}\frac{\chi(f)}{|f|}\sum_{d|f}\mu(d)\log\frac{q^n}{|d|} = \log(q^n) + \sum_{\deg f\le n}\frac{\chi(f)\Lambda(f)}{|f|}.$$

On the other hand, reversing the order of summation, the expression becomes

$$\sum_{\deg d\le n}\frac{\chi(d)\mu(d)}{|d|}\log\frac{q^n}{|d|}\sum_{\deg h\le n-\deg d}\frac{\chi(h)}{|h|} = L(\chi)\sum_{\deg d\le n}\frac{\chi(d)\mu(d)}{|d|}\log\frac{q^n}{|d|} + R(n),$$

where

$$R(n) = O\left(q^{-n}\sum_{\deg d\le n}\log\frac{q^n}{|d|}\right) = O\left(q^{-n}\left(\log(q^n)\sum_{k=0}^{n}q^k - \sum_{\deg d\le n}\log|d|\right)\right)$$

After expanding the geometric series and subtracting the expression obtained in Lemma 1, we see

$$\begin{aligned}\log(q^n)\sum_{k=0}^{n}q^k - \sum_{\deg d\le n}\log|d| &= \log(q^n)\left(\frac{-1}{q-1}\right) + \log q\frac{q}{q-1}\frac{q^n-1}{q-1} \\ &\ll \log(q^n) + q^n \ll q^n,\end{aligned}$$

so that $R(n) = O(1)$. Comparing the two expressions yields the claim of the theorem. ☐

Putting together this result with that of Corollary 2, we see we have shown

Lemma 9. *Let χ be a nonprincipal character. Then for $n \geq 0$,*

$$A_\chi(n) = O(1) + \log(n) \begin{cases} 0 & \text{if } L(\chi) \neq 0, \\ -1 & \text{otherwise.} \end{cases}$$

Corollary 3 (Nonvanishing of $L(\chi)$ for non-real χ)**.** *If χ is a character that assumes at least one non-real value, $L(\chi) \neq 0$.*

Proof. By Lemmas 5, 9 and Theorem 4,

$$\varphi(m) \sum_{\substack{f \equiv 1 \bmod m \\ \deg f \leq n}} \frac{\Lambda(f)}{|f|} = \sum_\chi A_\chi(n) = (1 - V)\log(q^n) + O(1).$$

Since the left hand side is nonnegative for all n, we must have $V \leq 1$.

But if $L(\chi) = 0$ for a nonreal character χ, then $0 = \overline{L(\chi)} = L(\overline{\chi})$ as well. Since χ assumes a non-real value, $\chi \neq \overline{\chi}$, and hence $V \geq 2$, contradicting the above. ☐

Since by Corollary 3 and Theorem 3, $L(\chi) \neq 0$ for every nonprincipal χ, Corollary 2 implies the promised

Corollary 4. *If χ is any nonprincipal character, $A_\chi(n) = O(1)$.*

5. The Denouement

Proof of Theorem 1. By Lemma 5, Theorem 4, and Corollary 4,

$$\varphi(m) \sum_{\substack{\deg f \leq n \\ f \equiv a \bmod m}} \frac{\Lambda(f)}{|f|} = \sum_\chi \overline{\chi}(a) A_\chi(n) = \log(q^n) + O(1).$$

Hence

$$\begin{aligned} \frac{1}{\varphi(m)} \log(q^n) + O(1) &= \sum_{\substack{\deg f \leq n \\ f \equiv a \bmod m}} \frac{\Lambda(f)}{|f|} \\ &= \sum_{\substack{\deg \pi \leq n \\ \pi \equiv a \bmod m}} \frac{\log|\pi|}{|\pi|} + O\left(\sum_{\deg f \leq n} \frac{\Lambda(f)}{|f|} - \sum_{\deg \pi \leq n} \frac{\log|\pi|}{|\pi|} \right) \\ &= \sum_{\substack{\deg \pi \leq n \\ \pi \equiv a \bmod m}} \frac{\log|\pi|}{|\pi|} + O(1), \end{aligned}$$

by Lemma 4. Rearranging gives the the theorem. ☐

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