Pattern Replacement

Steve Linton, St Andrews
Jim Propp, Massachusetts
Tom Roby, Connecticut

Adeline Pierrot, Paris
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Say that two n-permutations are replacement-equivalent if they differ by an adjacent transposition $a_i a_{i+1} \leftrightarrow a_{i+1} a_i$, where both inequalities $a_i < a_{i+2}$ and $a_{i+1} < a_{i+2}$ hold.

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2  1  3

1  2  3
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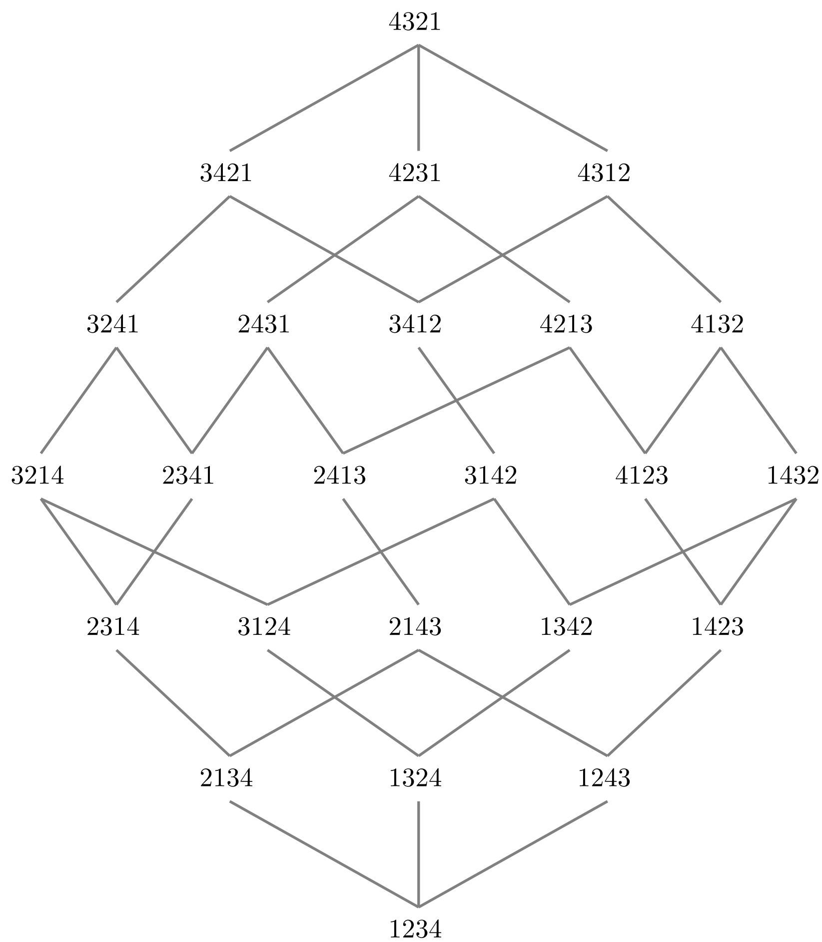
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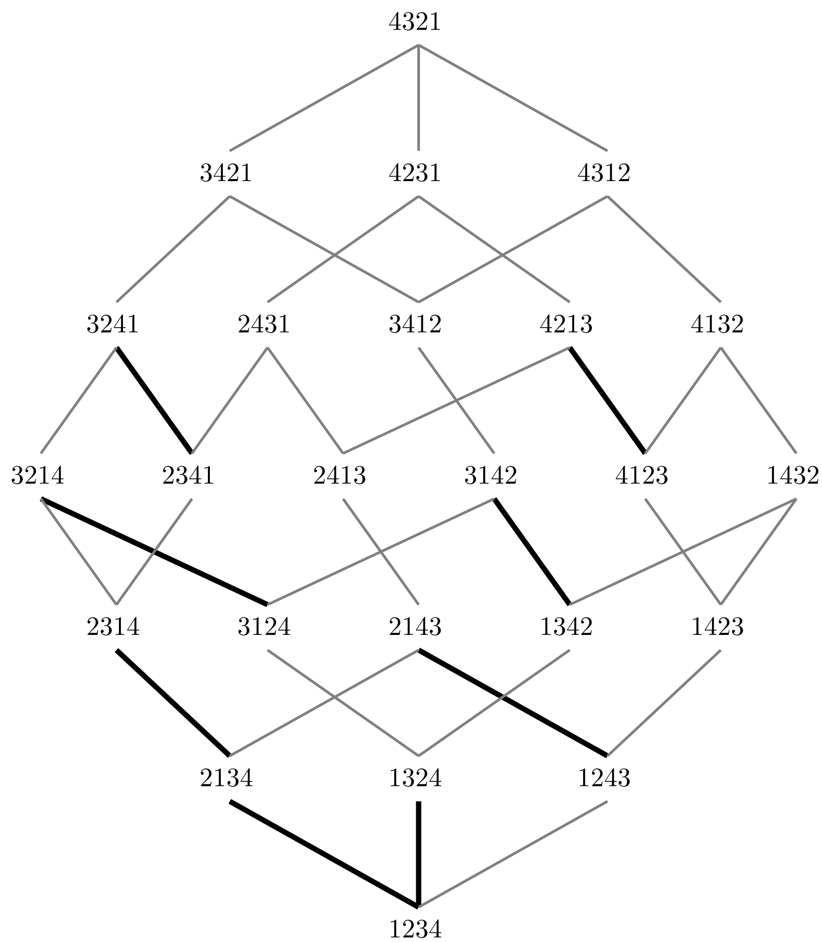
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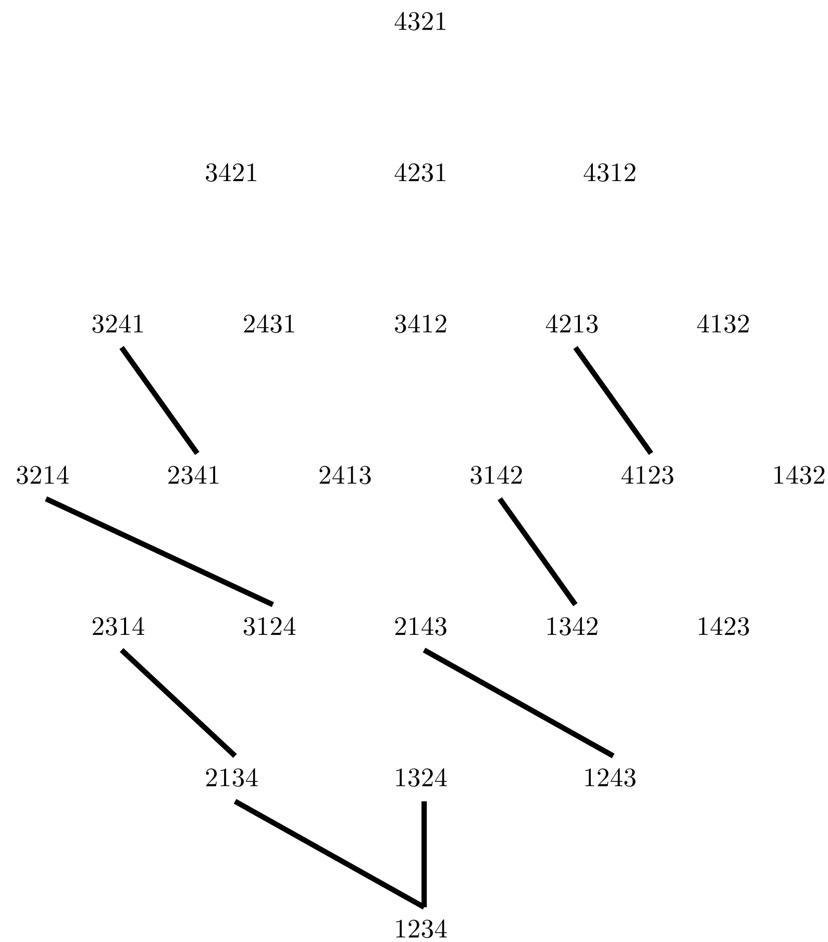
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Origins

From: James Propp <jpropp@cs.uml.edu>
Date: Sun, 3 May 2009 22:48:03 -0400
Subject: combinatorics question

:
:

How many permutations pi_1 pi_2 ... pi_n of 1,2,...,n are reachable from 1,2,...,n by means of some combination of the restricted swap operations s_1,s_2,...,s_{n-1} where s_i swaps pi_{i-1} and pi_i if both numbers are less than pi_{i+1} (and does nothing otherwise)?

For instance, 31245 is reachable because of the sequence of operations 12345->13245->13425->31425->31245. Note that all these swap operations are reversible: if s_i turns pi into pi', then s_i also turns pi' into pi. This enables us to see that 54321 is not reachable, because there's no way to do any restricted swaps to change it into something else.

I have a conjectural characterization of the reachable permutations that implies a very nice formula for the final answer, but I can't prove it.

Shall I tell you what I know about the problem, or would you like to play with it on your own for a while first?

Jim

Links

The **plactic monoid** over some totally ordered alphabet (often the positive integers) is the monoid with the following presentation:

The generators are the letters of the alphabet.

The relations are the elementary Knuth transformations $yzx = yxz$ whenever $x < y \leq z$ and $xzy = zxy$ whenever $x \leq y < z$.

Two words are called Knuth equivalent if they represent the same element of the plactic monoid, or in other words if one can be obtained from the other by a sequence of elementary Knuth transformations.

source: http://en.wikipedia.org/wiki/Plactic_monoid

The **Chinese monoid** is a monoid generated by a totally ordered alphabet with the relations $cba = cab = bca$ for every $a \leq b \leq c$. It was discovered by Duchamp & Krob (1994) during their classification of monoids with growth similar to that of the plactic monoid.

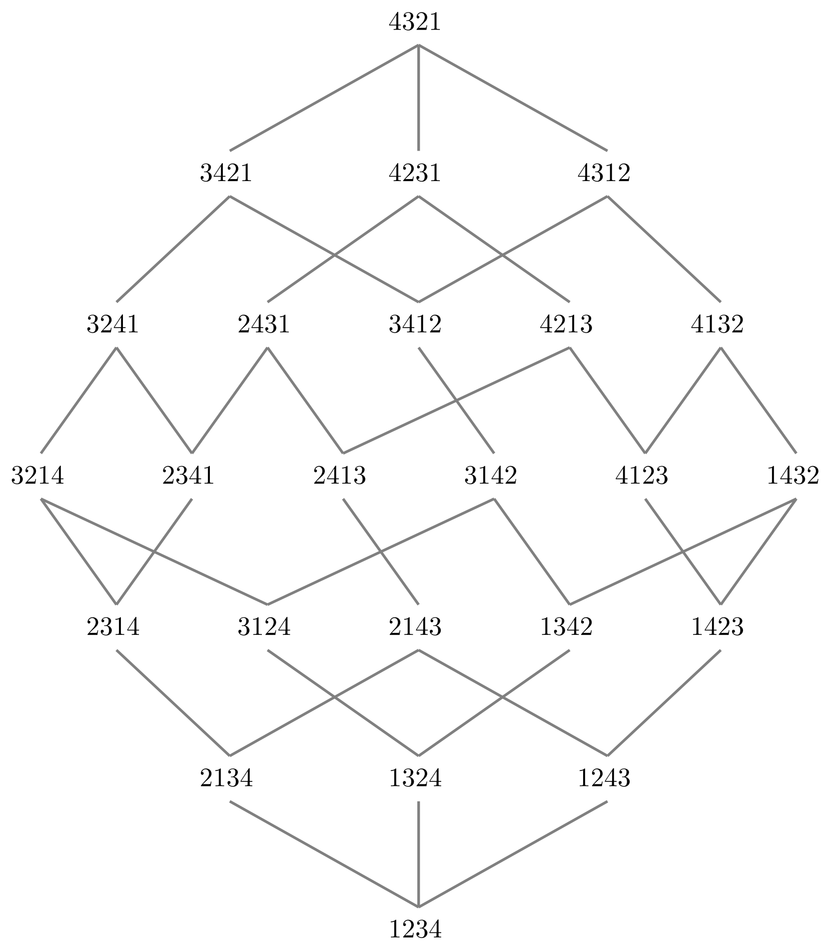
http://en.wikipedia.org/wiki/Chinese_monoid

Links

Problem 11452. Proposed by Donald Knuth, Stanford University, Stanford, CA.

Say that the permutations $a_1 a_2 \ldots a_k a_{k+1} \ldots a_n$ and $a_k \ldots a_2 a_1 a_{k+1} \ldots a_n$ are equivalent when $k = n$ or when a_{k+1} exceeds all of $a_1, \ldots, a_k$. Also say that two permutations are equivalent whenever they can be obtained fom each other by a sequence of such flips. For example, $321 \equiv 123 \equiv 213 \equiv 312$ and $132 \equiv 231$. Show that the number of equivalence classes is equal to the Euler secant-and-tangent number for all n. (The nth secant-and-tangent number counts the number of "up-down" permutations of length n, namely the permutations like 25341 that alternately rise and fall, beginning with a rise.)

source: American Mathematical Monthly, August-September 2009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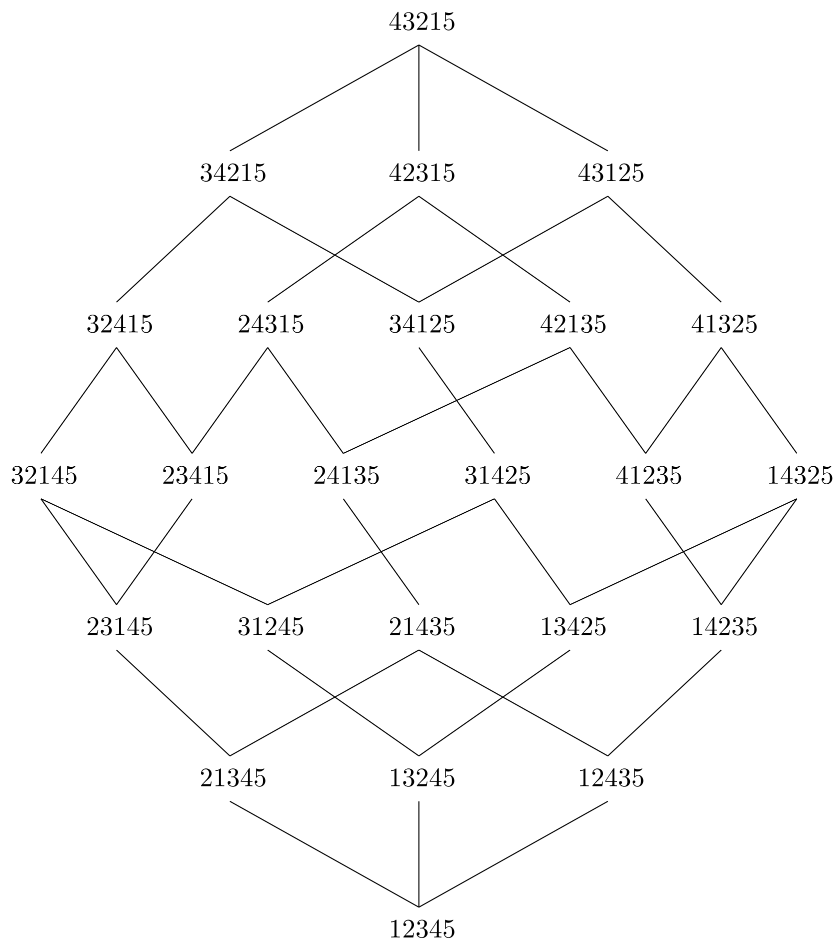
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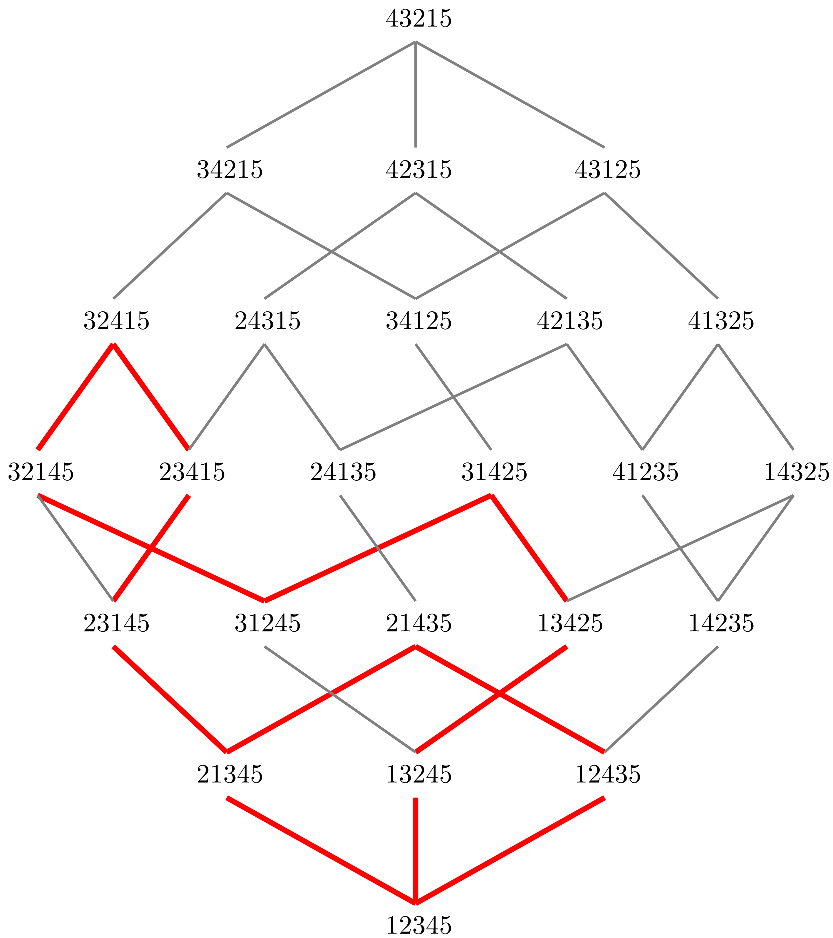
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Theorem. The number of n-permutations replacement-equivalent to the identity is $\lfloor n/2 \rfloor ! \lceil n/2 \rceil !$

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i.e., for $n = 2r$,

$$\#\{\pi : \pi \leftrightarrow 123 \ldots n\} = r!r!$$

and for $n = 2r + 1$,

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Proof that this is an upper bound.

The largest element must be in the rightmost position.

This implies that the second-largest element must be in one of the three rightmost positions.

This implies that the third-largest element ...

Now, placing the elements from largest to smallest, we have the following number of choices for each placement:

$1 \cdot 2 \cdot 3 \cdot \ldots \cdot \lceil n/2 \rceil \cdot \lfloor n/2 \rfloor \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

Proof of equality.

It remains to show that all permutations meeting these constraints are in fact reachable.

Imagine a target permutation meeting the constraints. That is, the first element (even case) or first two elements (odd case) are less than $\lceil n/2 \rceil + 1$, the next two elements are less than $\lceil n/2 \rceil + 2$, etc.

`Target:   kgfOdiPahNQcbTRjUmSVWXelYZ`

Step one. Advance all the "large" elements as far as they will go by rippling them forward:

```
.............NOPQRSTUVWXYZ
............N.OPQRSTUVWXYZ
............NO.PQRSTUVWXYZ
............NOP.QRSTUVWXYZ
............NOPQ.RSTUVWXYZ
             :
............NOPQRSTUVWXY.Z
             :
             :
...........NOPQRSTUVWX.Y.Z
             :
             :
..........NOPQRSTUVW.X.Y.Z
             :
             :
             :
.N.O.P.Q.R.S.T.U.V.W.X.Y.Z
```

```
Target:   kgfOdiPahNQcbTRjUmSVWXelYZ

          .N.O.P.Q.R.S.T.U.V.W.X.Y.Z
```

Now observe that the "small" elements can be permuted freely while leaving the "large" elements in place.

```
fRjS  ->  fjRS  ->  jfRS  ->  jRfS
```

Step two. Using this observation, move the correct element into the first position. (In the odd case, move the two correct elements into the first two positions.) Because the target permutation obeys the constraints, this element (or pair of elements) will be small compared with the fixed skeleton of large elements which is facilitating their movement.

```
kN.O.P.Q.R.S.T.U.V.W.X.Y.Z
```

Continue to place elements two at a time:

```
kN.O.P.Q.R.S.T.U.V.W.X.Y.Z
kgfO.P.Q.R.S.T.U.V.W.X.Y.Z
kgfOdP.Q.R.S.T.U.V.W.X.Y.Z
kgfOdiPQ.R.S.T.U.V.W.X.Y.Z
kgfOdiPahR.S.T.U.V.W.X.Y.Z
            :
kgfOdiPahNQcbTRjUmSVWXelYZ
```

□

Back to Jim

```
From: James Propp <jpropp@cs.uml.edu>
Date: Wed, 8 Jul 2009 17:07:07 -0400
Subject: two hundred and ten questions

:
:

I'd like to know the partition of n! determined by the transitive
closure of each of the following seven relations on S_n:

:
:

The two most interesting
numbers are probably the number of components and the size of
the component containing the permutation 1,2,3,...,n.

:
:

I should say that I want this information for _three_ distinct
interpretations of what "123 <--> 213" means:
(a) In the narrowest sense, it could mean that if pi(i+1) = pi(i)+1
    and pi(i+2) = pi(i)+2, then you can swap the values of pi(i) and
    pi(i+1).
(b) More broadly, it could mean that if pi(i) < pi(i+1) < pi(i+2),
    then you can swap the values of pi(i) and pi(i+1).
(c) More broadly still, it could mean that if pi(i) < pi(j) < pi(k)
    for i < j < k, then you can swap the values of pi(i) and pi(j).

Jim
```

Number of classes	§ 1 neither ("classical")	§ 2 only indices adjacent	§ 3 indices and values adjacent
$123 \leftrightarrow 132$ $123 \leftrightarrow 213$	[5, 14, 42, 132, 429] Catalan	[5, 16, 62, 284, 1507, 9104]	[5, 20, 102, 626, 4458, 36144]
$123 \leftrightarrow 321$	[5, 10, 3, 1, 1, 1] trivial	[5, 16, 60, 260, 1260, 67442]	[5, 20, 102, 626, 4458, 36144]
$123 \leftrightarrow 132 \leftrightarrow 213$	[4, 8, 16, 32, 64, 128] powers of 2	[4, 10, 26, 76, 232, 764] involutions	[4, 17, 89, 556, 4011, 32843]
$123 \leftrightarrow 132 \leftrightarrow 321$ $123 \leftrightarrow 213 \leftrightarrow 321$	[4, 2, 1, 1, 1, 1] trivial	[4, 8, 14, 27, 68, 159, 496]	[4, 16, 84, 536, 3912, 32256]
$123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 321$	[3, 2, 1, 1, 1, 1] trivial	[3, 4, 5, 8, 11, 20, 29, 57]	[3, 13, 71, 470, 3497]

Size of class with ι_n	§ 1 neither ("classical")	§ 2 only indices adjacent	§ 3 indices and values adjacent
$123 \leftrightarrow 132$ $123 \leftrightarrow 213$	[2, 6, 24, 120, 720] (n-1)!	[2, 4, 12, 36, 144, 576, 2880] product of two factorials	[2, 3, 5, 8, 13, 21, 34, 55] Fibonacci numbers
$123 \leftrightarrow 321$	[2, 4, 24, 720] trivial	[2, 3, 6, 10, 20, 35, 70, 126] central binomial coefficients	[2, 3, 4, 6, 9, 13, 19, 28] A000930
$123 \leftrightarrow 132 \leftrightarrow 213$	[3, 13, 71, 461] connected A003319	[3, 7, 35, 135, 945, 5193] terms are always odd	[3, 4, 8, 12, 21, 33, 55, 88] A052952
$123 \leftrightarrow 132 \leftrightarrow 321$ $123 \leftrightarrow 213 \leftrightarrow 321$	[3, 23, 120, 720] trivial	[3, 9, 54, 285, 2160, 15825] formulae for odd/even	[3, 5, 9, 17, 31, 57, 105, 193] tribonacci numbers A000213
$123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 321$	[3, 23, 120, 720] trivial	[4, 21, 116, 713, 5030]	[4, 6, 13, 23, 44, 80, 149, 273] tribonacci A000073 $-[n$ even$]$

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$123 \leftrightarrow 231$		product of two factorials	
$123 \leftrightarrow 132 \leftrightarrow 312$		$n!/2$	
$123 \leftrightarrow 132 \leftrightarrow 231$		$n!\binom{n-1}{\lfloor (n-1)/2 \rfloor}/2^{n-1}$	
$123 \leftrightarrow 231 \leftrightarrow 321$		formulae for odd/even	
$123 \leftrightarrow 231 \leftrightarrow 312$		[3, 8, 45, 313, 2310]	conserve parity of inversions
$123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 231$		$n! - (n-1)!$	
$123 \leftrightarrow 132 \leftrightarrow 231 \leftrightarrow 312$		$(n-1)!\lceil \frac{n}{2} \rceil$	
$123 \leftrightarrow 132 \leftrightarrow 231 \leftrightarrow 312$		$n! - (n-1)$	
$123 \leftrightarrow 132 \leftrightarrow 312 \leftrightarrow 321$		$n! - 2$	
$123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 321$		$n! - c(\lfloor \frac{n-1}{2} \rfloor) - c(\lfloor \frac{n}{2} \rfloor)$	
5 patterns, not incl. 321		$n! - 1$	
other sets of 5 patterns		$n!$	

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$123 \leftrightarrow 231 \leftrightarrow 312$		[3, 8, 45, 313, 2310]	conserve parity of inversions
$123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 231$		$n! - (n-1)!$	
$123 \leftrightarrow 132 \leftrightarrow 231 \leftrightarrow 312$		$(n-1)!\lceil \frac{n}{2} \rceil$	
$123 \leftrightarrow 132 \leftrightarrow 231 \leftrightarrow 312$		$n! - (n-1)$	
$123 \leftrightarrow 132 \leftrightarrow 312 \leftrightarrow 321$		$n! - 2$	
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5 patterns, not incl. 321		$n! - 1$	
other sets of 5 patterns		$n!$	

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123 ↔ 231 ↔ 321		formulae for odd/even	
123 ↔ 231 ↔ 312		[3, 8, 45, 313, 2310]	conserve parity of inversions
123 ↔ 132 ↔ 213 ↔ 231		$n! - (n-1)!$	
123 ↔ 132 ↔ 231 ↔ 312		$(n-1)!\lceil \frac{n}{2} \rceil$	
123 ↔ 132 ↔ 231 ↔ 312		$n! - (n-1)$	
123 ↔ 132 ↔ 312 ↔ 321		$n! - 2$	
123 ↔ 132 ↔ 213 ↔ 321		$n! - c(\lfloor \frac{n-1}{2} \rfloor) - c(\lfloor \frac{n}{2} \rfloor)$	
5 patterns, not incl. 321		$n! - 1$	
other sets of 5 patterns		$n!$	

Theorem. The number of permutations reachable from the identity under $123 \leftrightarrow 132 \leftrightarrow 321 =$
$\frac{3}{2}(k)(k+1)(2k-1)!$, ($n = 2k+1$ odd).
$\frac{3}{2}(k)(k-\frac{1}{3})(2k-2)! - (2k-3)!!$, ($n = 2k$ even).
(Here $(2k-3)!! = (2k-3)\ldots(3)(1)$, the product of odd numbers less than or equal to $(2k-3)$.)

Half of proof.

First, the element (1) cannot occupy a position of even index, because it can only participate in a swap as a "1", and then the swap either leaves it fixed or moves it by two positions.

Second, the element (2) cannot occupy a position of odd index to the left of (1), because if it winds up to the left of (1), it last moves there via a move $\rightarrow 321$, and this places (2) into a position of even index. Then it stays on the left of the (1), so it must play the role of "1" in any future swaps.

Additionally, in the even case, permutations of the following form are not reachable: Working from right to left, fill the positions in order $n-1, n, n-3, n-2, n-5, n-4 \ldots 3, 4, 1, 2$, according to the following rule. When filling positions of odd index, the smallest available element must be chosen; the subsequent selection of an element to place to its right is then unconstrained.

Theorem. The permutations reachable from the identity under $123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 231$ are those which do not begin with n, of which are there are $n! - (n-1)!$

Proof. First check that we can reach any target permutation which terminates in n. Beginning with the identity (which increases from left to right), we will build the target from left to right, one element at a time.

At each step, to place the kth element, we locate it, together with the (larger) element which follows it (which exists because the n is going to stay in position at the end). Advance this pair of elements to the correct position using $123 \to 231$, then use $231 \to 213$ once, and $132 \to 123$ repeatedly, to restore the following element to its original location.

Now to reach an arbitrary permutation, modify it to place the n in terminal position. This can be reached. Now advance the n using $123 \to 132$ and $213 \to 231$ as necessary.

Theorem. The permutations reachable from the identity under $123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 231$ are those which do not begin with n, of which are there are $n! - (n-1)!$

Example.

```
target:   egdhabcf



          abcdEFgh
          abcEFdgh
          abEFcdgh
          aEFbcdgh
          EFabcdgh
          EaFbcdgh
          EabFcdgh
          EabcFdgh
          EabcdFgh

          eabcdfGH
             :
          eGHabcdf
             :
          eGabcdfH

          egabcDFh
             :
          egDFabch
             :
          egDabcFh

          egdabcfH
          egdabcHf
          egdabHcf
          egdaHbcf
          egdHabcf
```

Theorem. The only permutations not reachable from the identity under $123 \to 132 \to 213 \to 321$ are isolated.

Proof. Suppose σ is not isolated and is has a minimal number of inversions for its equivalence class. First, σ contains no 132, 213 or 321, because any such string could be replaced by 123, reducing the number of inversions.

But σ is not isolated, and so contains a 123.

If σ is not the identity, it contains a 123 preceded by a descent, or a 123 followed by a descent.

In the first case, we have $abcx$ with $a < b < c$ and $b > x$, and we can make the transformations $abcx \to acbx \to axbc$, which has two fewer inversions. In the second case we have $xabc$ with $a < b < c$ and $x > b$ and $xabc \to xbac \to abxc$ again gives a permutation with two fewer inversions.

So σ must be the identity.

ahnkT

ouY