

Permutation Patterns in Algebraic Geometry

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We start with some definitions.

Permutations and patterns

A **permutation** in \mathfrak{S}_n is a bijection $\pi: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$.

We will use one-line notation for permutations, for example,

$\pi = 32415$ is the permutation in \mathfrak{S}_5 that sends

$$1 \mapsto 3$$

$$2 \mapsto 2$$

$$3 \mapsto 4$$

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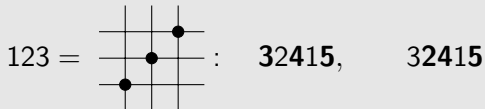
Patterns are also permutations but we are interested in how they occur in other permutations ...

Patterns inside permutations

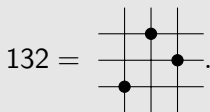
Given a pattern p we say that it **occurs** in a permutation π if π contains a subsequence that is order-equivalent to p . If p does not occur in π we say that π **avoids** the pattern p . Let $\mathfrak{S}_n(p)$ denote the set of permutations in \mathfrak{S}_n that avoid the pattern p .

Example

The permutation $\pi = 32415$ has two occurrences of the pattern



It avoids the pattern

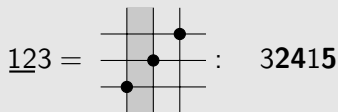


Vincular patterns

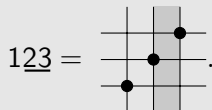
Babson and Steingrímsson (2000) defined **generalized patterns**, or **vincular patterns**, where conditions are placed on the locations of the occurrence.

Example

The permutation $\pi = 32415$ has one occurrence of the pattern



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Motivation for vincular patterns

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- Many interesting sequences of integers come up when we count the permutations avoiding a pattern p . For example if p is any classical pattern of length 3 then

$$|\mathfrak{S}_n(p)| = n\text{-th Catalan number} = \frac{1}{n+1} \binom{2n}{n}.$$

However Claesson showed in 2001 that

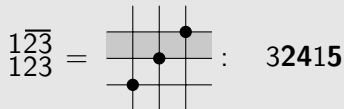
$$|\mathfrak{S}_n(\underline{123})| = n\text{-th Bell number}.$$

Bivincular patterns

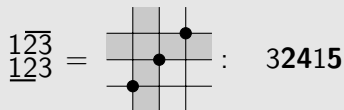
Bousquet-Mélou, Claesson, Dukes, and Kitaev (2010) defined **bivincular patterns** as vincular patterns with extra restrictions on the values in an occurrence.

Example

The permutation $\pi = 32415$ has one occurrence of the pattern



This is not an occurrence of $\overline{123}$. But it is an occurrence of



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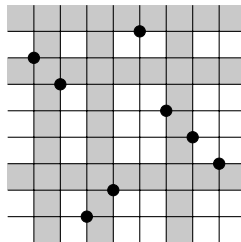
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- New Wilf-equivalence: For example the patterns

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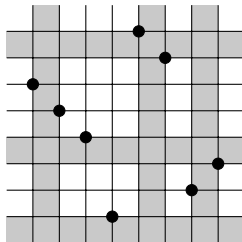


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(Complete) flags

We will only consider complete flags in \mathbb{C}^m so we will simply refer to them as **flags**. A flag is a sequence of vector-subspaces of \mathbb{C}^m

$$E_{\bullet} = (E_1 \subset E_2 \subset \cdots \subset E_m = \mathbb{C}^m),$$

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We want to consider special subsets of this flag manifold ...

Schubert cells in $Fl(\mathbb{C}^m)$

If we choose a basis f_1, f_2, \dots, f_m , for \mathbb{C}^m then we can fix a **reference flag**

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for $1 \leq p, q \leq m$. Let's look at an example.

... from Geometry

A Schubert cell in $Fl(\mathbb{C}^3)$

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Schubert varieties in $Fl(\mathbb{C}^m)$

Given a Schubert cell X_π° we define the **Schubert variety** as the closure

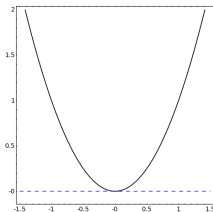
$$X_\pi = \overline{X_\pi^\circ},$$

in the Zariski topology.

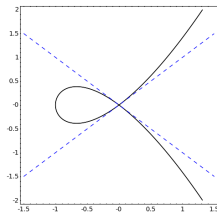
We will now show how pattern avoidance can be used to describe geometric properties of Schubert varieties.

Smooth, factorial and Gorenstein varieties

Pictorial definition of smoothness: the tangent space at every point has the right dimension.



(a) $y - x^2 = 0$.



(b) $y^2 - x^2 - x^3 = 0$.

Figure: Compare the single tangent direction in subfigure 1(a) with the two tangent directions in subfigure 1(b).

Smooth, factorial and Gorenstein varieties

Algebraic definitions: a variety:

X is	if
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Description in terms of patterns

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$$\text{avoiding } \begin{array}{c} 12\overline{345} \\ 3\underline{15}24 \end{array} = \begin{array}{|c|c|c|c|} \hline & & \bullet & \\ \hline \bullet & & & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{c} 12\overline{345} \\ 24\underline{15}3 \end{array} = \begin{array}{|c|c|c|c|} \hline & & & \bullet \\ \hline \bullet & & & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline & & \bullet & \\ \hline \end{array}.$$

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- The second condition of factoriality, avoiding 1324 , is weakened to the avoidance of two infinite families of bivincular patterns, which we now describe.

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Given such a permutation π , with a descent at d , we construct its **associated partition** $\lambda(\pi)$ as the partition inside a bounding box with dimensions $d \times (n - d)$, whose lower border is the lattice path that starts at the lower left corner of the box and whose i -th step is vertical if i is weakly to the left of the position d , and horizontal otherwise.

Example

The permutation

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has a unique descent at $d = 5$.

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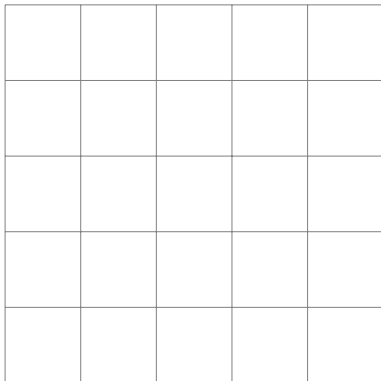


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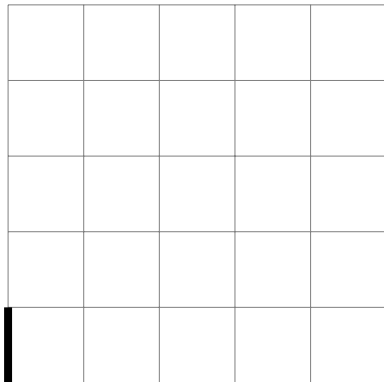


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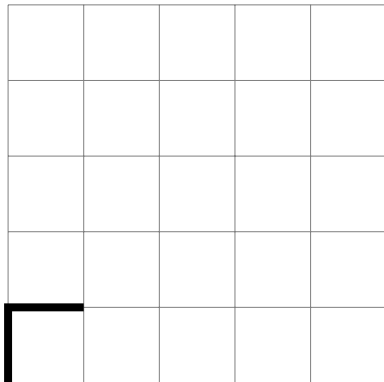


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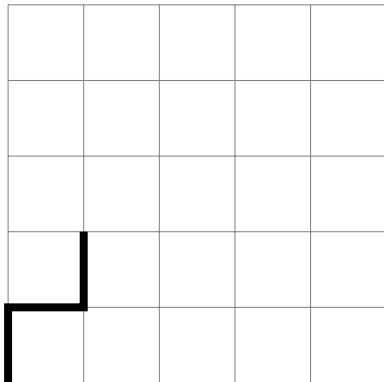


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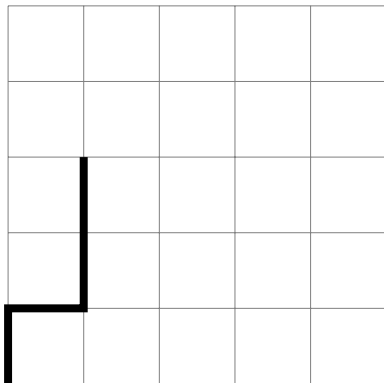


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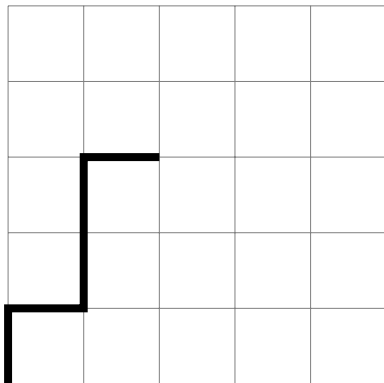


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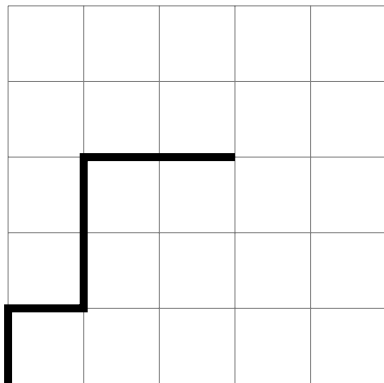


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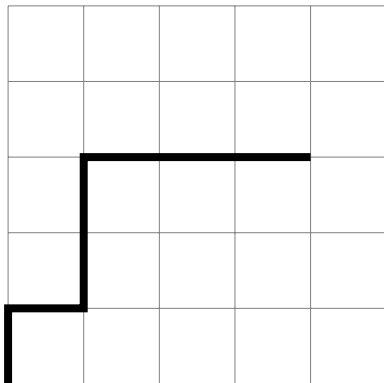


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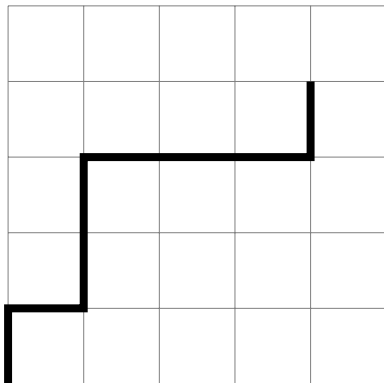


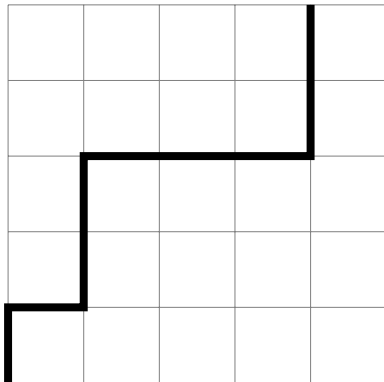
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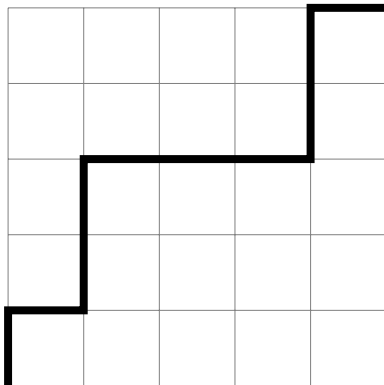


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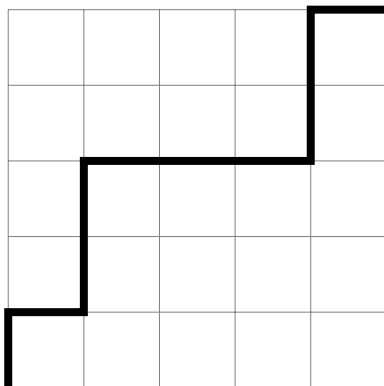


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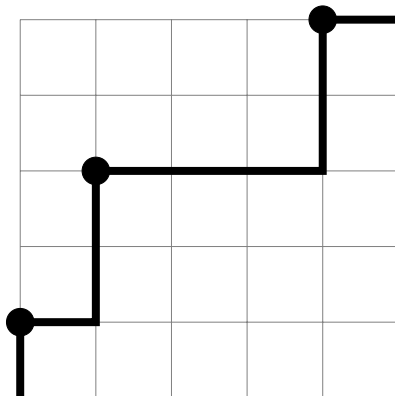


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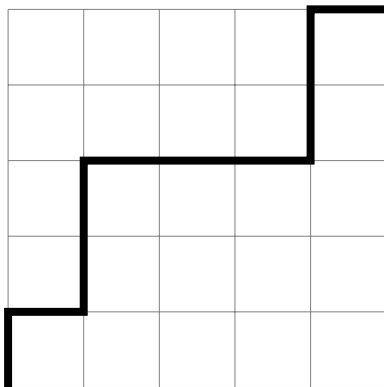


Figure: Outer corners of $\pi = 13489 \downarrow 2567|10$.

Outer corners of the partition

If we want to translate this condition into pattern avoidance then it is actually better to consider the **outer corners** of the partition.

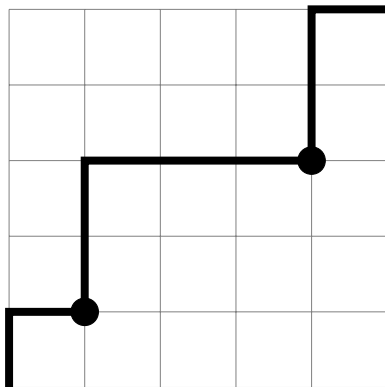


Figure: Outer corners of $\pi = 13489 \downarrow 2567|10$.

Depth and width of outer corners

We see that all the inner corners lie on the same diagonal if and only if each outer corner has the same **depth** and **width**.

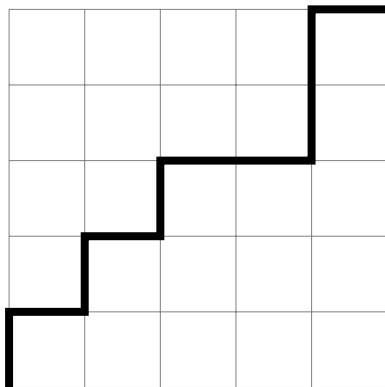
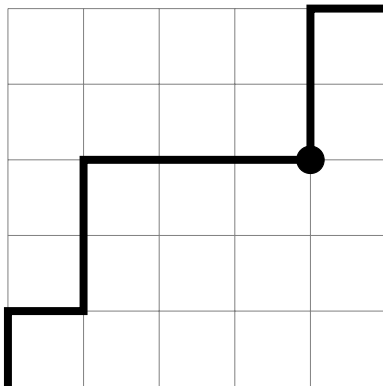


Figure: $\pi = 13589 \downarrow 2467|10$.

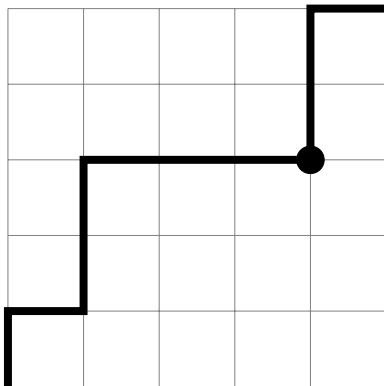
Detecting too wide outer corners

Let's go back to the permutation $\pi = 13489 \downarrow 2567|10$, and consider the outer corner that is too wide



Detecting too wide outer corners

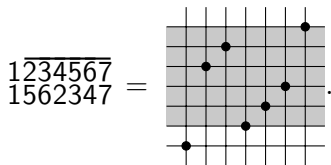
Let's go back to the permutation $\pi = 13489 \downarrow 2567|10$, and consider the outer corner that is too wide



This outer corner comes from the subsequence $13489 \downarrow 2567|10$.

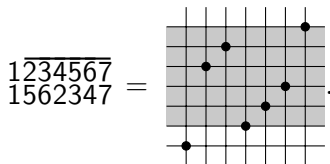
Detecting too wide outer corners cont.

The shape of this outer corner can be detected with the bivincular pattern



Detecting too wide outer corners cont.

The shape of this outer corner can be detected with the bivincular pattern



In general, we can detect too wide outer corners with the patterns

$$\overline{12345} / \overline{14235}, \overline{1234567} / \overline{1562347}, \overline{123456789} / \overline{167823459}, \dots, \overline{12 \dots \dots \dots k} / \overline{1 \ell+1 \dots 2 \dots \ell k}, \dots$$

Detecting too wide outer corners cont.

The shape of this outer corner can be detected with the bivincular pattern

$$\overline{1234567} / \overline{1562347} = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \bullet \\ \hline & & & \bullet & & & \\ \hline & & \bullet & & & & \\ \hline & & & & & \bullet & \\ \hline & & & & \bullet & & \\ \hline & \bullet & & & & & \\ \hline \end{array}.$$

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and too deep outer corners with the patterns

$$\overline{12345} / \overline{13425}, \overline{1234567} / \overline{1456237}, \overline{123456789} / \overline{156782349}, \dots, \overline{12 \dots \dots \dots k} / \overline{1 \ell+1 \dots 2 \dots \ell k}, \dots$$

Summary

The Schubert variety

X_π is	if
smooth	π avoids 2143 and 1324
factorial	π avoids $\underline{2}143$ and 1324
Gorenstein	π avoids $\frac{12345}{3\underline{1}524}$ and $\frac{12345}{24\underline{1}53}, \dots$

Summary

The Schubert variety

X_π is	if
smooth	π avoids 2143 and 1324
factorial	π avoids $2\underline{1}43$ and 1324
Gorenstein	π avoids $\frac{12345}{3\underline{1}524}$ and $\frac{12345}{24\underline{1}53}, \dots$

... and the two infinite corner families — remember that this is modulo a technical detail I have omitted.

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- The description is in terms of patterns only and one doesn't need to construct the associated partition.
- It is very easy to see on the pattern level that smooth implies factorial implies Gorenstein.

We end with some open problems.

Other smoothness properties

- A variety is a **local complete intersection** if it can be described by the expected number of equations. This condition is in between factoriality and Gorensteinness and I'm working with Woo on giving a pattern description.

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- Where do the **mesh patterns** patterns fit into this story?

The end! Questions?