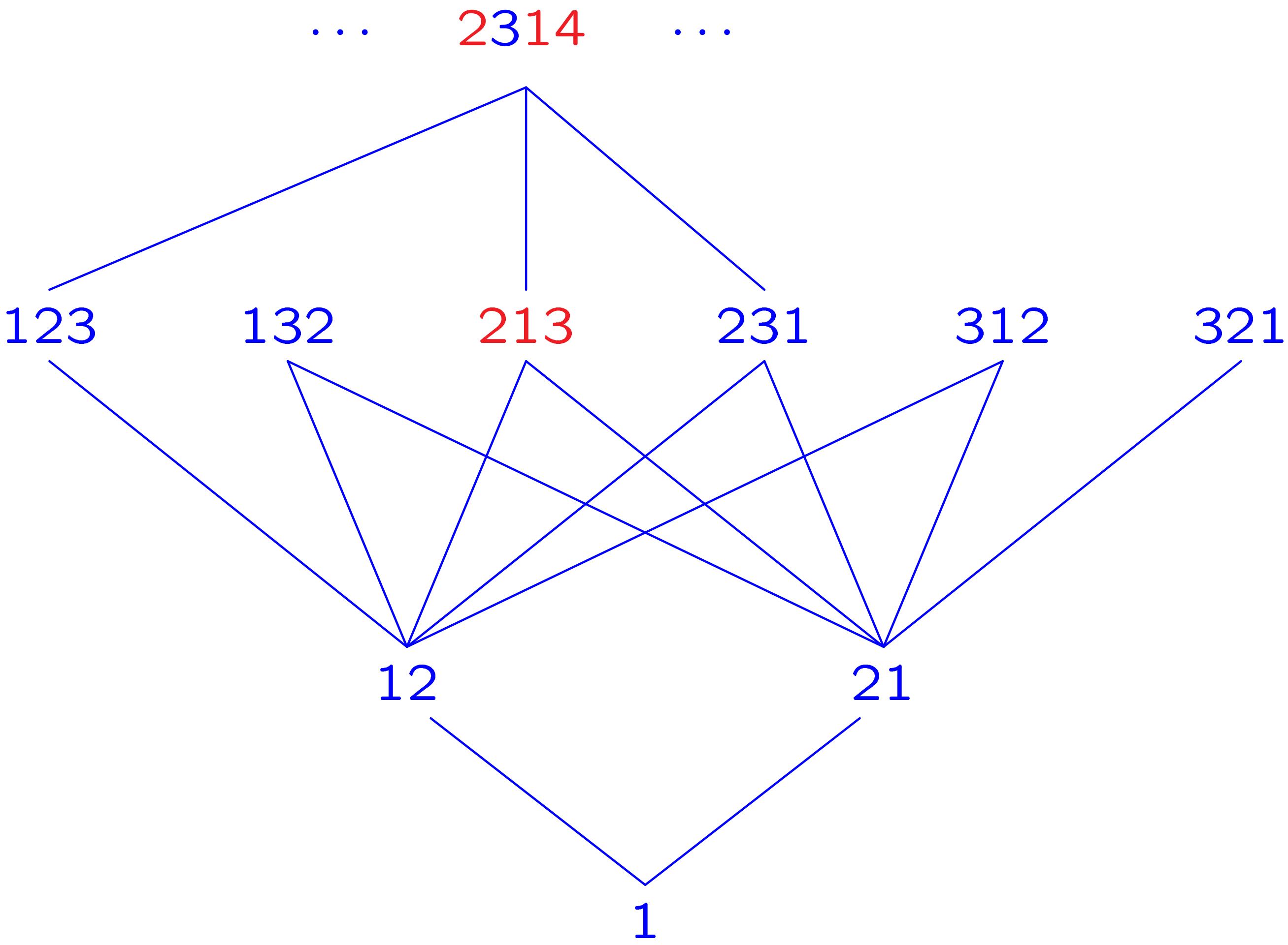
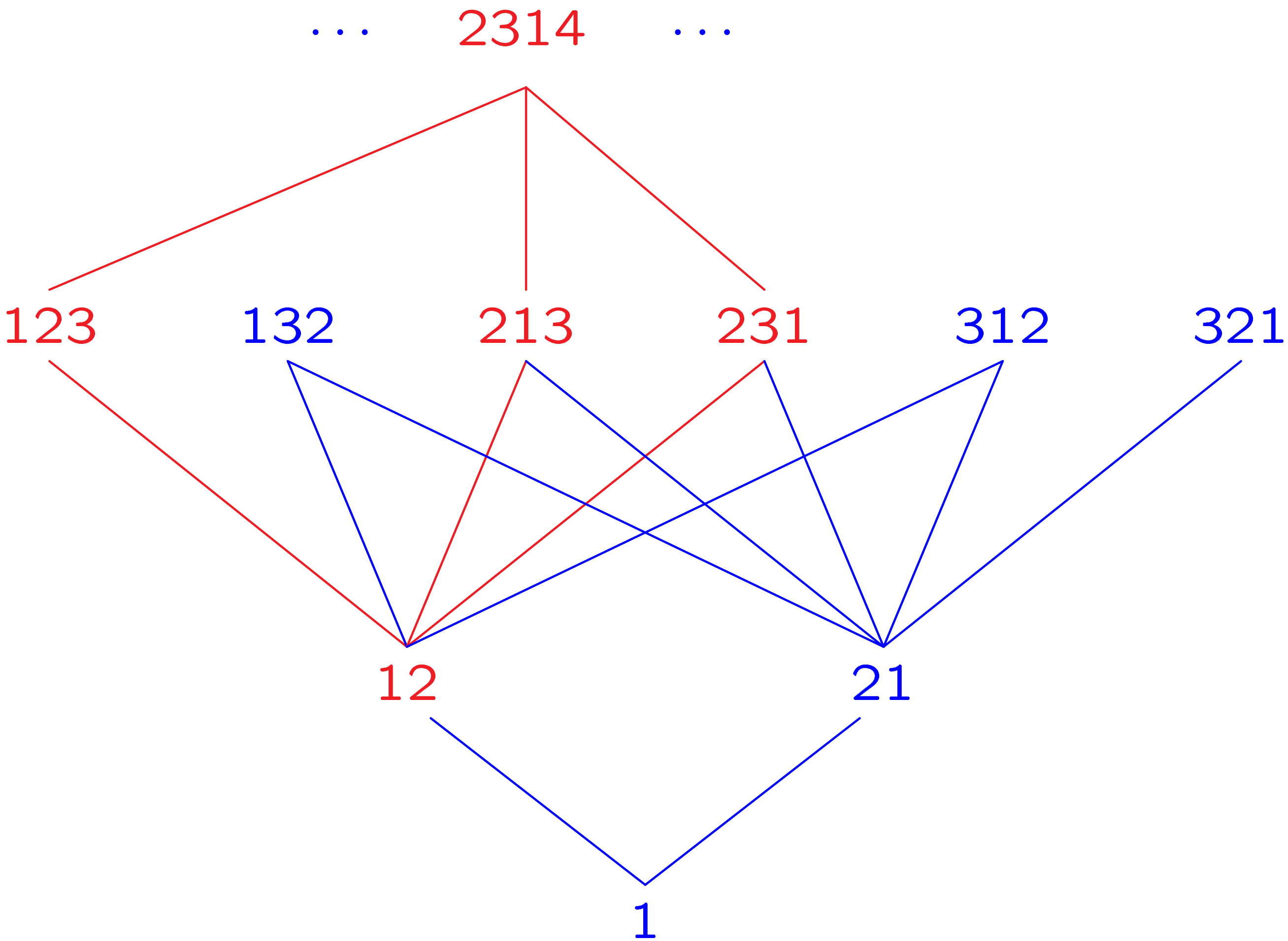
Permutation patterns and the Möbius function

Alex Burstein, Vít Jelínek, Eva Jelínková and
Einar Steingrímsson

(Take two)

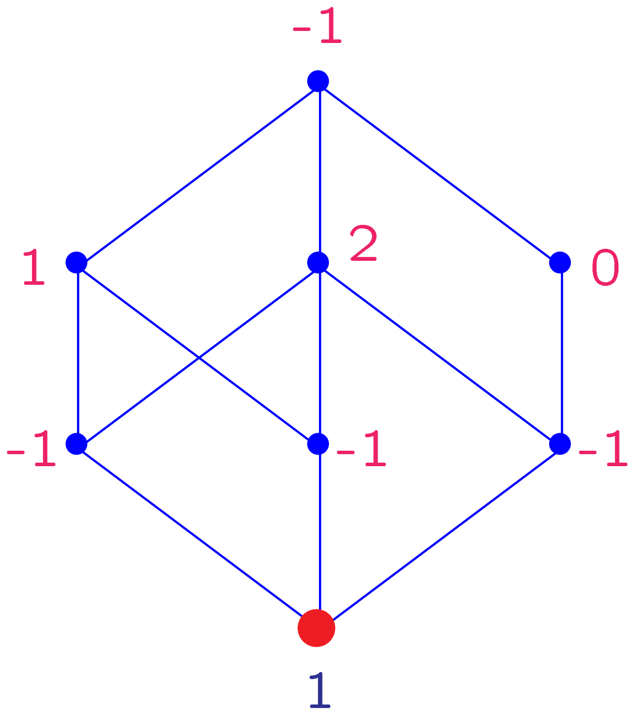


The poset of permutations w.r.t. pattern containment



The interval [12, 2314]

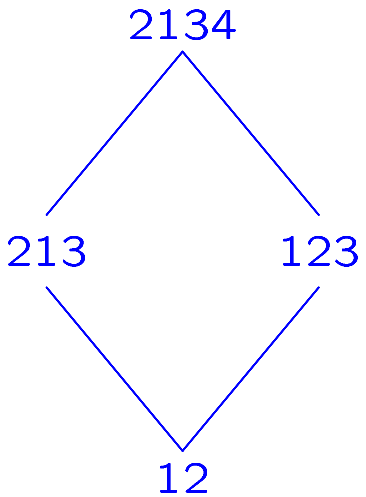
Computing the Möbius function $\mu(\bullet, y)$



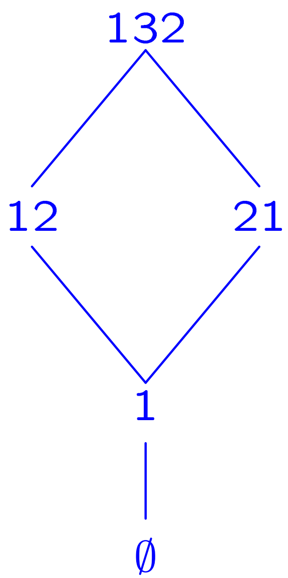
The Möbius function is defined by $\mu(x, x) = 1$ and

$$\sum_{x \leqslant t \leqslant y} \mu(x, t) = 0 \quad \text{if } x < y$$

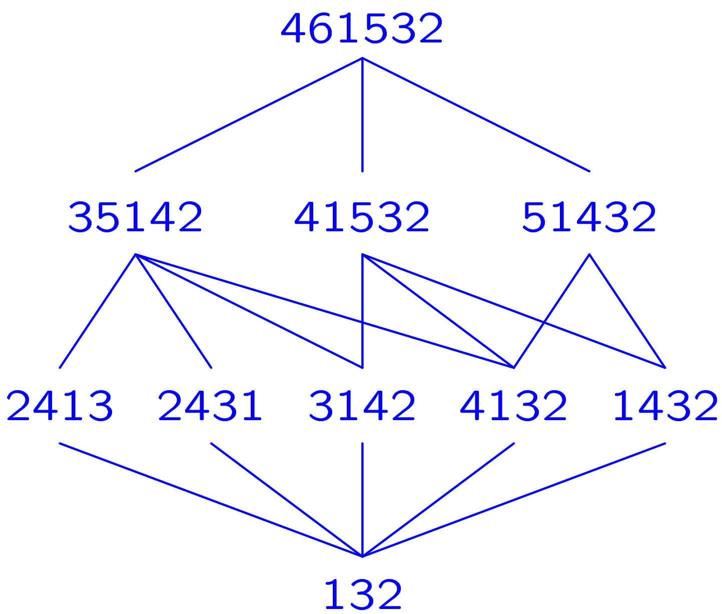
Some examples



$\mu(12, 2134) = 1$

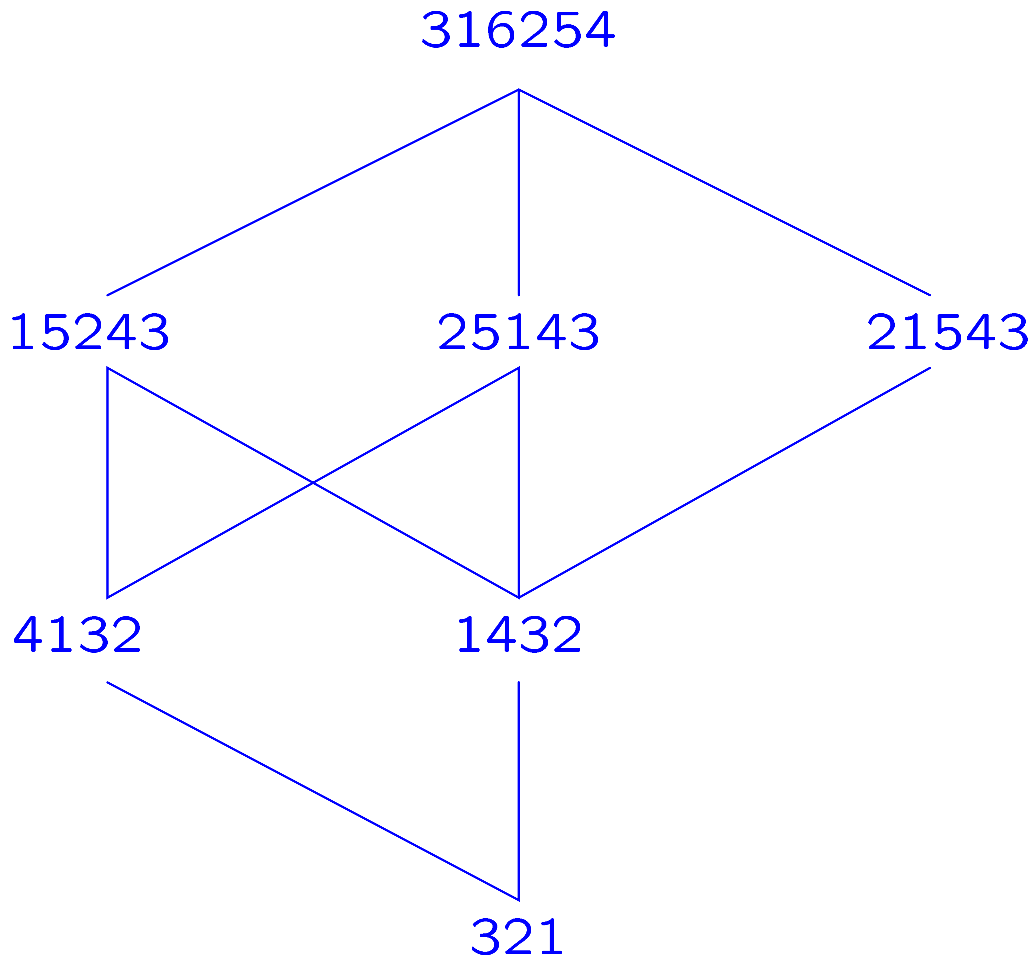


$\mu(\emptyset, 132) = 0$



$\mu(132, 461532) = -2$

P :



Theorem (Philip Hall): The Möbius function of P is given by $\sum_i (-1)^i C_i$, where C_i is the number of chains of length i in P that contain both the minimal and maximal elements.

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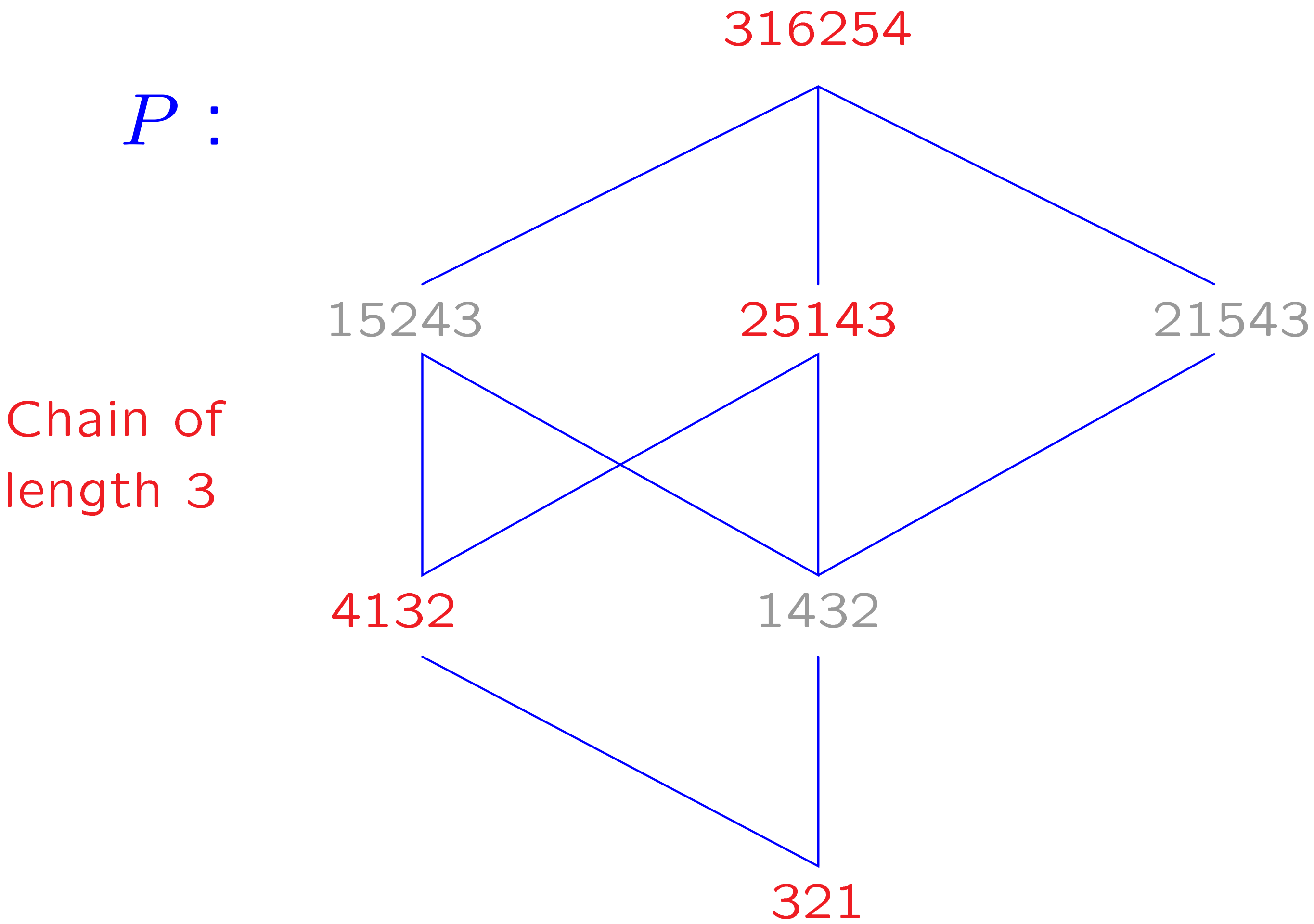

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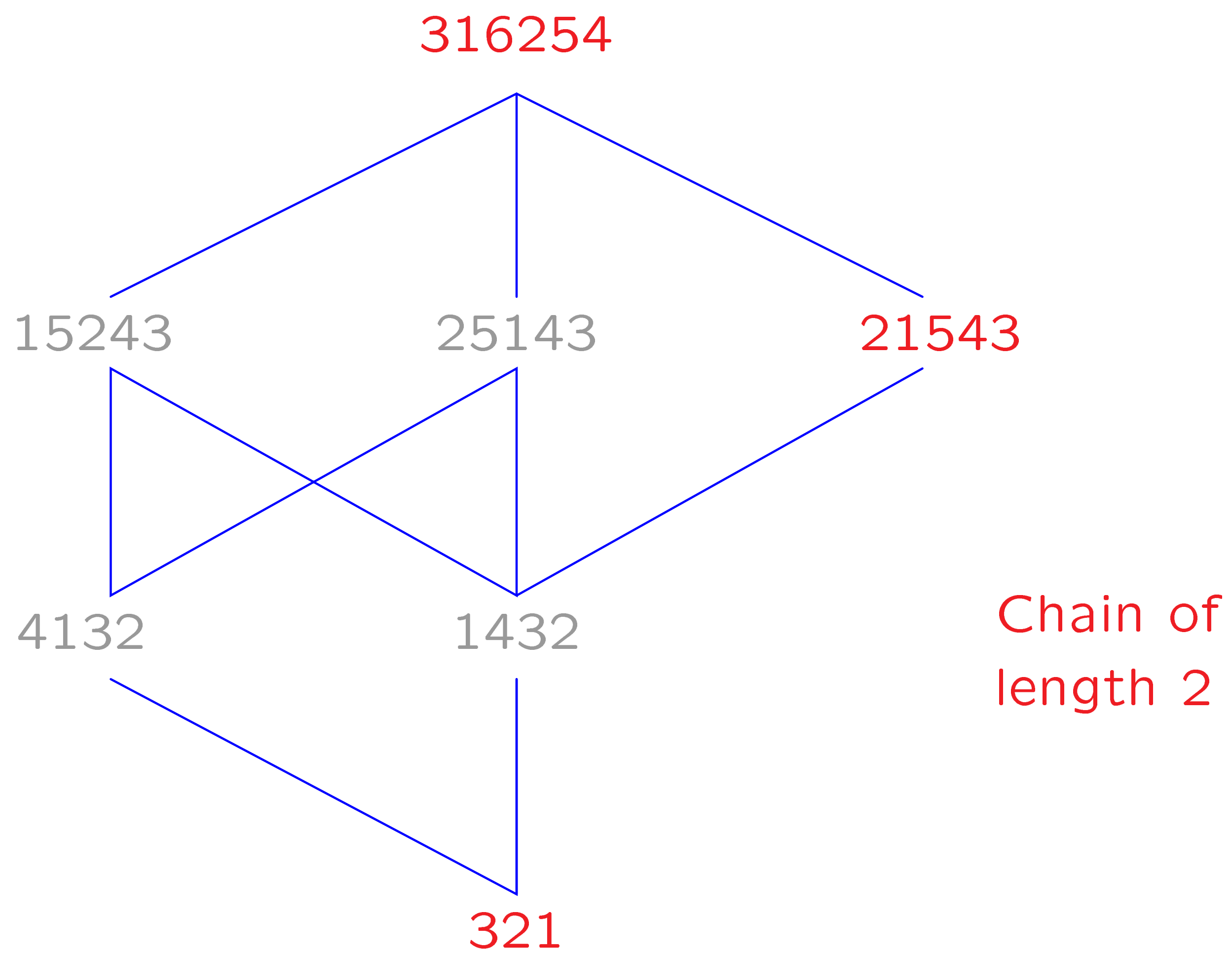

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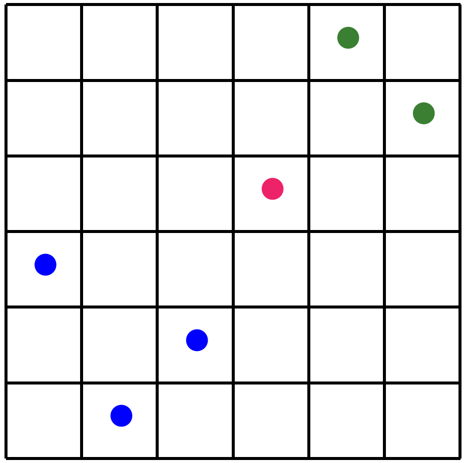
A permutation is *decomposable* if it is the *direct sum* of two or more (nonempty) permutations:



3 1 2 4 6 5

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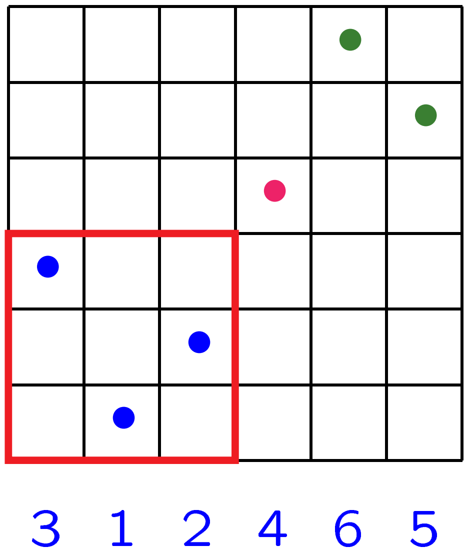

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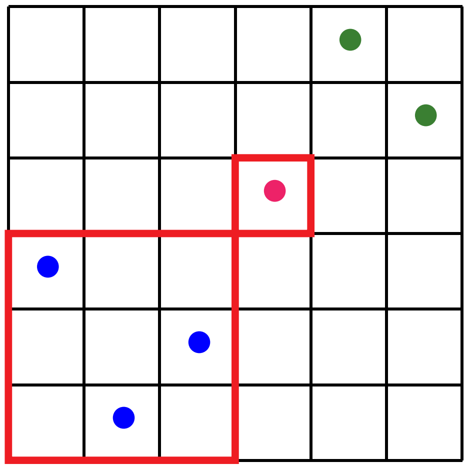

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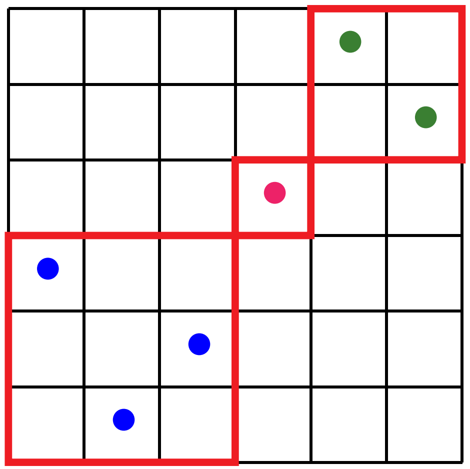

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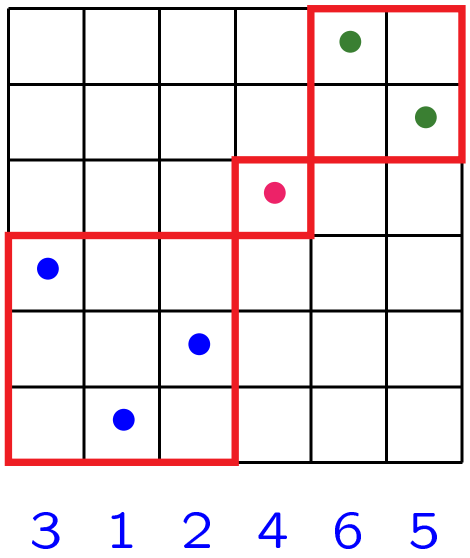

$312465 = 312$

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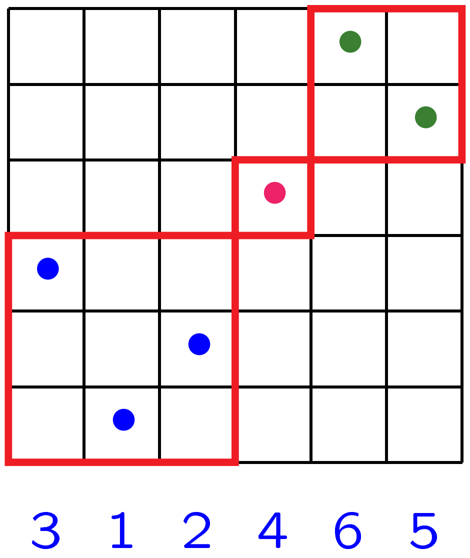

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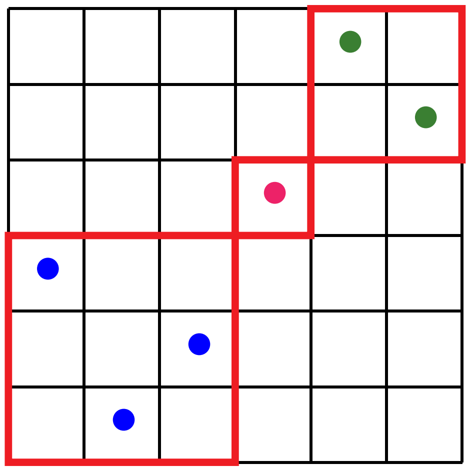

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$$312465 = 312 \oplus 1 \oplus 21$$

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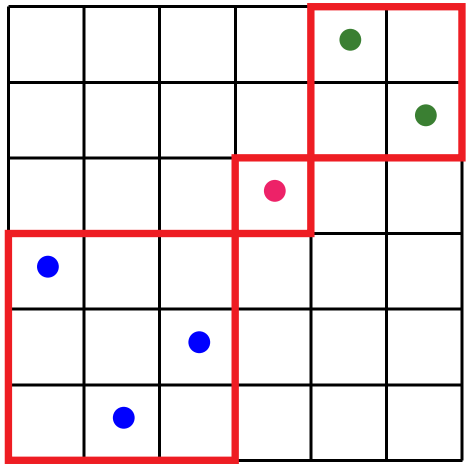

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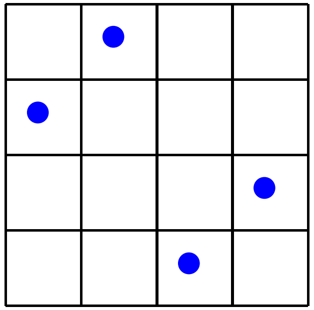

3 4 1 2

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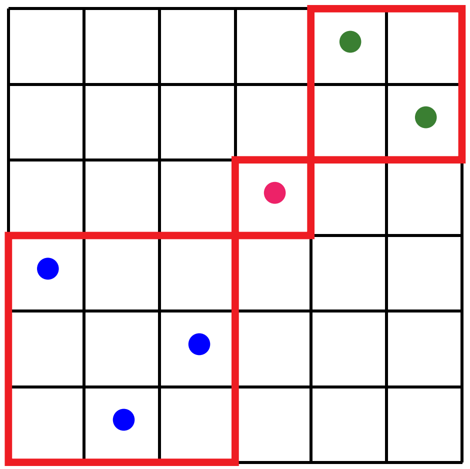

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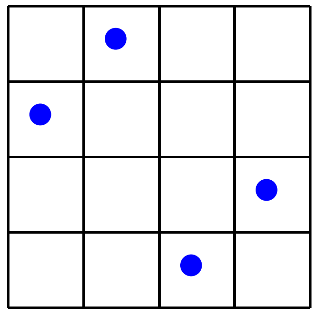

3 4 1 2

Indecomposable

We write $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_n$ *only* if each π_i is indecomposable

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First Recurrence: Let $\ell \geqslant 0$ and $k \geqslant 1$ be maximal so that $\sigma_1 = \sigma_2 = \cdots = \sigma_\ell = 1$ and $\pi_1 = \pi_2 = \cdots = \pi_k = 1$. Then

$$\mu(\sigma, \pi) = \begin{cases} 0 & \text{if } \ell \leqslant k-2 \\ -\mu(\sigma_{\geqslant k}, \pi_{>k}) & \text{if } \ell = k-1 \\ \mu(\sigma_{>k}, \pi_{>k}) - \mu(\sigma_{\geqslant k}, \pi_{>k}) & \text{if } \ell \geqslant k \end{cases}$$

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Example:

$\mu(132, 1237564) = 0$

$\mu(132, 126453) = -\mu(21, 4231) = -2$

$\mu(132, 13524) = \mu(21, 2413) - \mu(132, 2413) = 3 - (-1) = 4$

Main Theorem: Suppose $\pi_1 \neq 1$. Let $k \geqslant 1$ be maximal so that $\pi_1 = \pi_2 = \cdots = \pi_k$. Then

$$\mu(\sigma, \pi) = \sum_{i=1}^{m} \sum_{j=1}^{k} \mu(\sigma_{\leqslant i}, \pi_1)\mu(\sigma_{>i}, \pi_{>j})$$

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Corollary: If $\sigma = a \oplus b$ and $\pi = c \oplus d$, where $c, d \neq 1, c \neq d$, then

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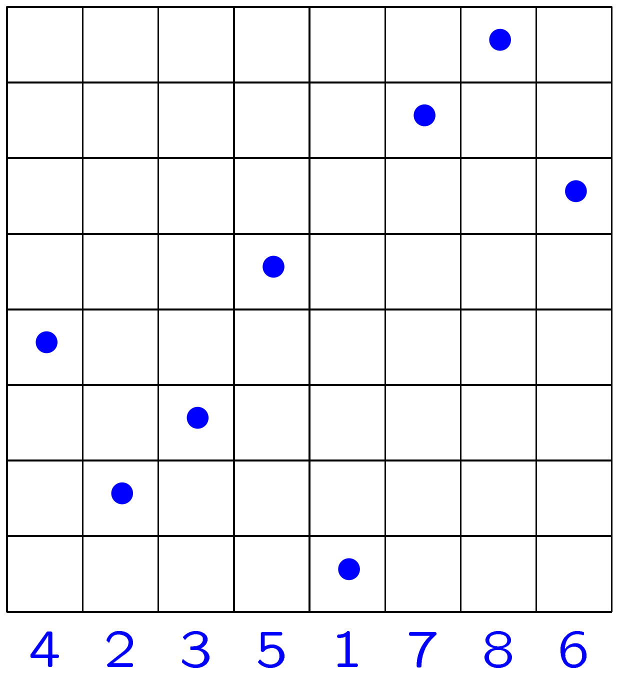
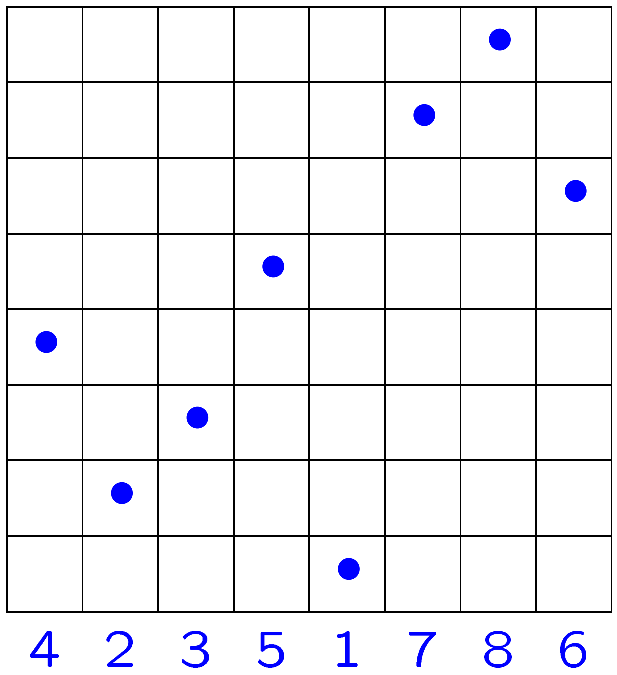
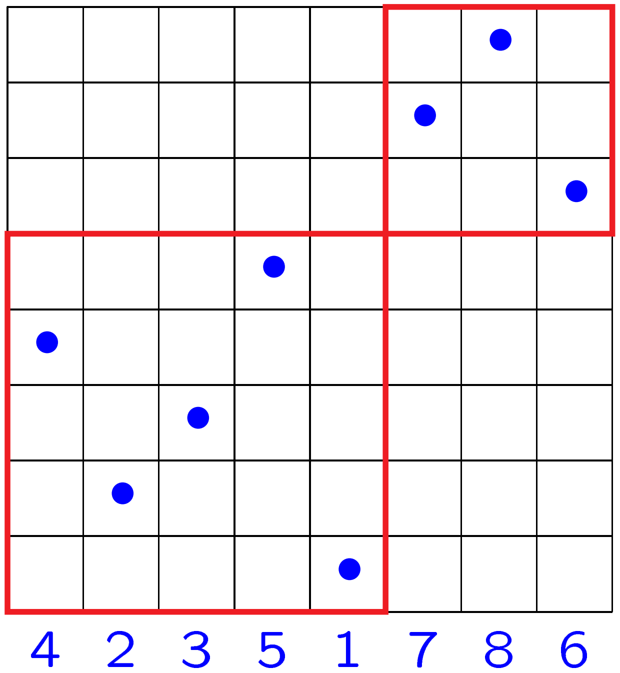
Corollary: If σ is indecomposable (so $m = 1$), then

- $\mu(\sigma, \pi) = \mu(\sigma, \pi_1)$ if $\pi = \pi_1 \oplus \pi_1 \oplus \cdots \oplus \pi_1$
- $\mu(\sigma, \pi) = -\mu(\sigma, \pi_1)$ if $\pi = \pi_1 \oplus \pi_1 \oplus \cdots \oplus \pi_1 \oplus 1$ $(\pi_1 \neq 1)$
- $\mu(\sigma, \pi) = 0$ otherwise


4 2 3 5 1 7 8 6



A permutation is *separable* if it can be generated from 1 by direct sums and *skew sums*.



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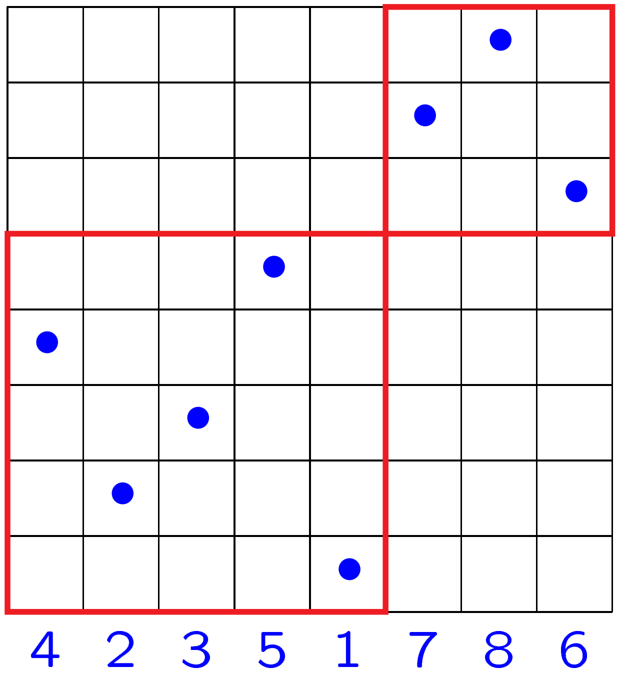

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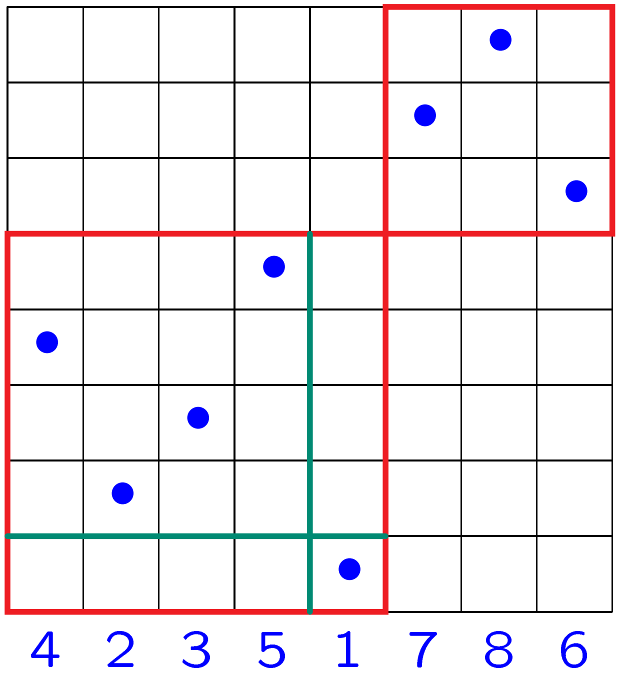
$$42351786 = 42351 \oplus 231$$



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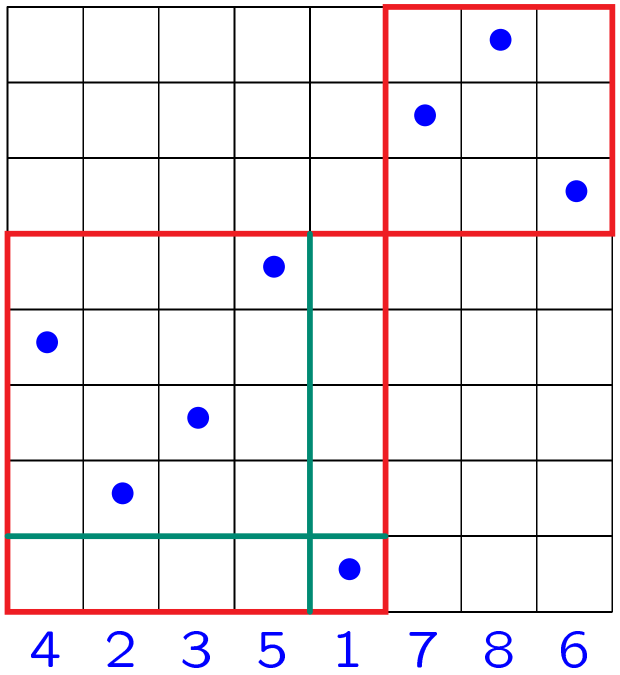

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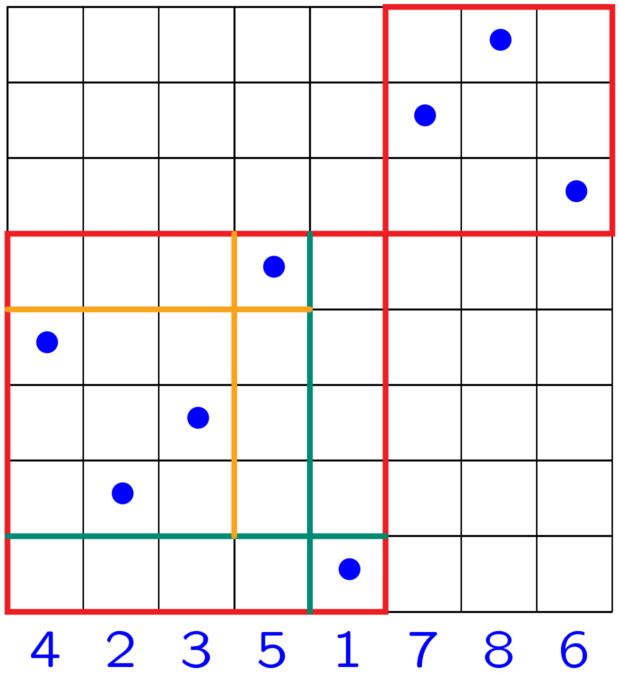

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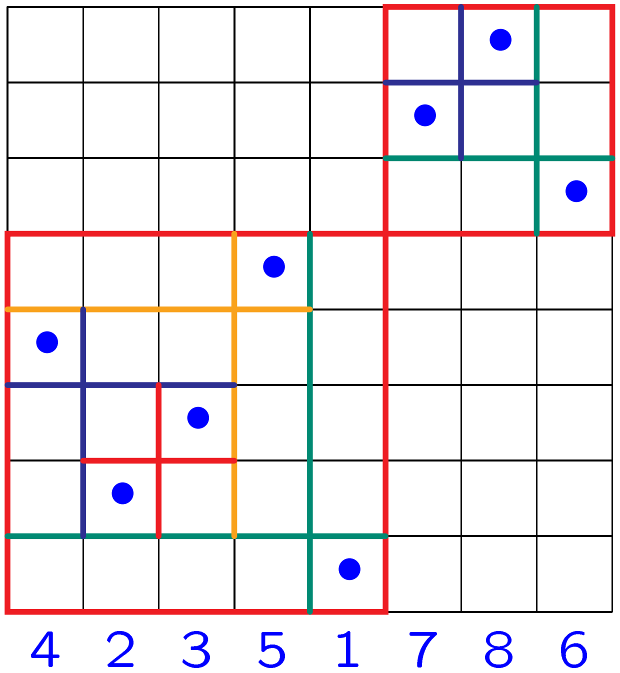

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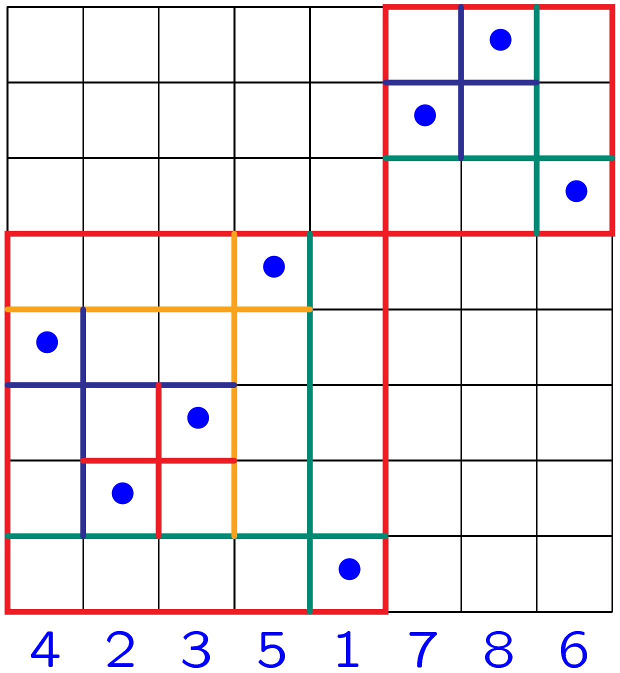

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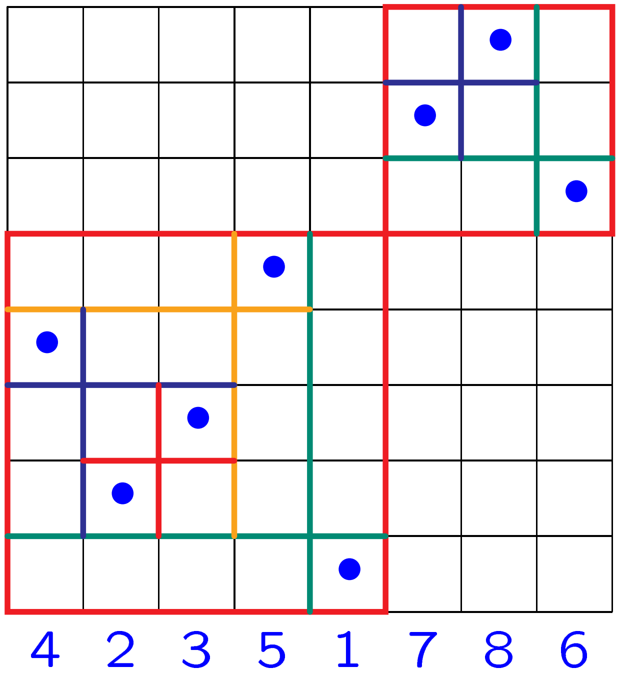

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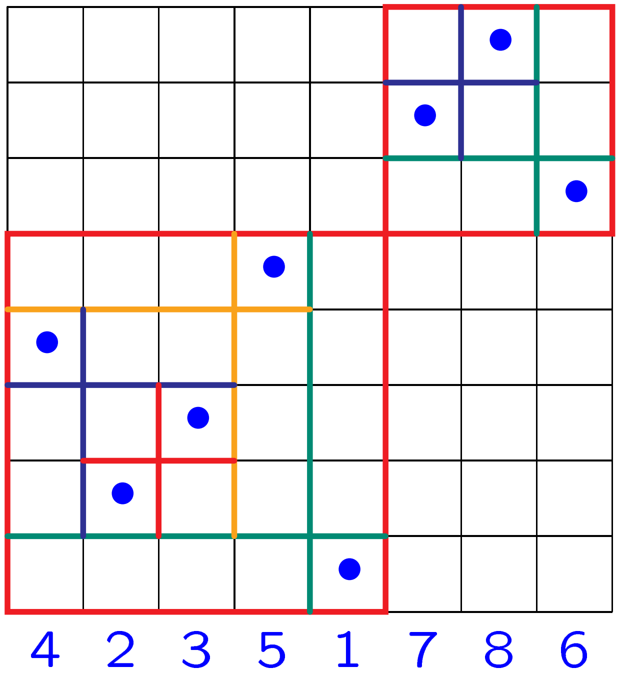

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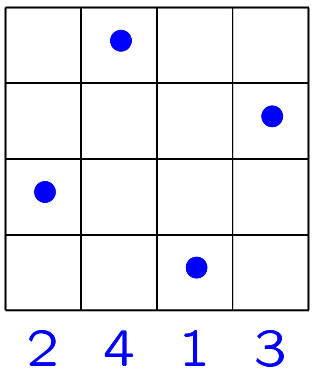

Not separable

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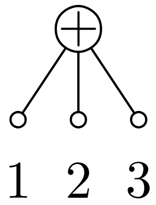
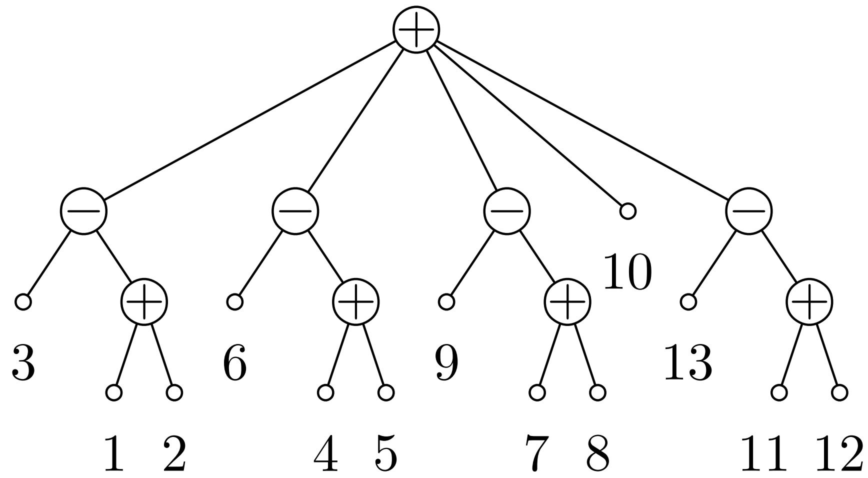


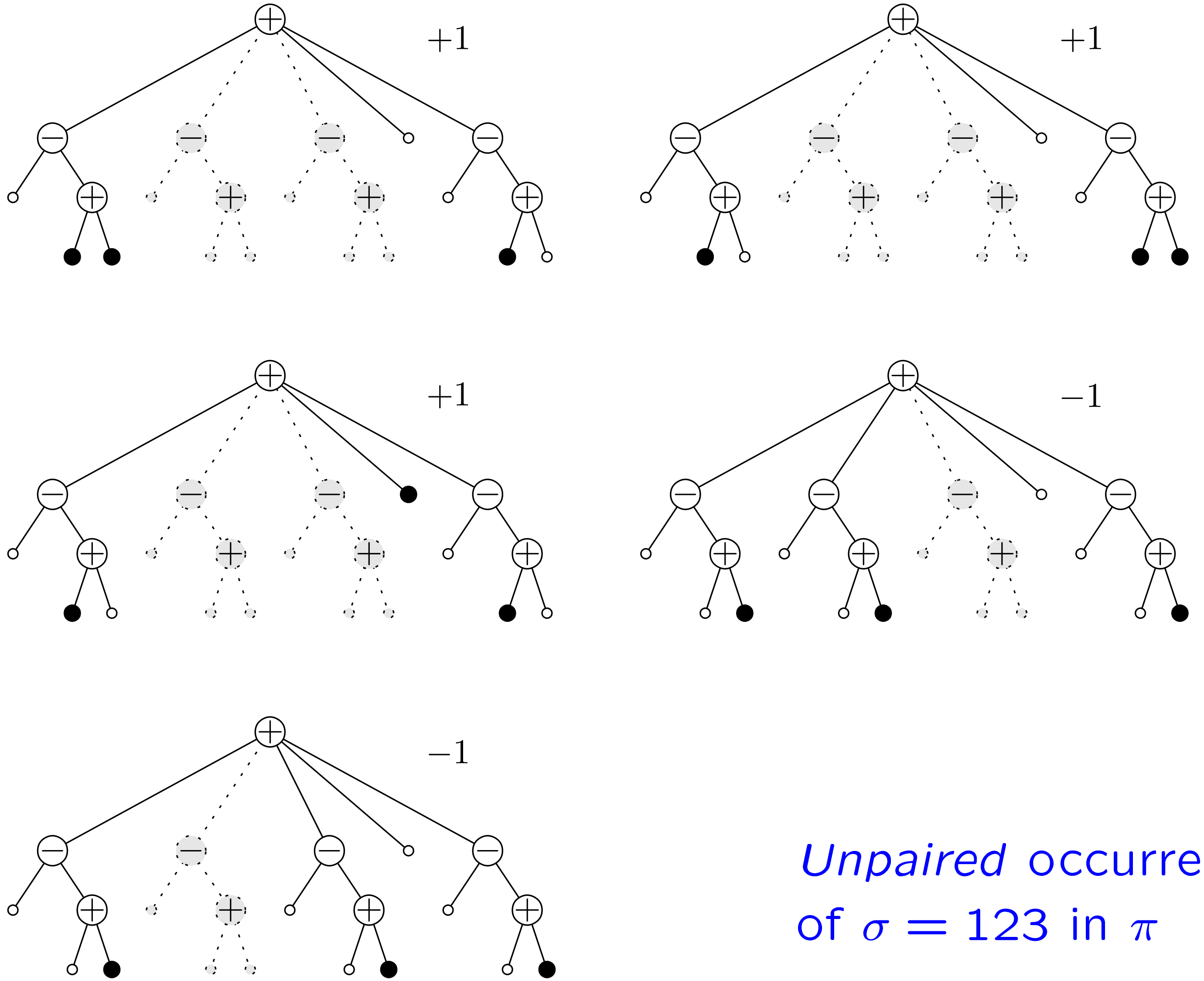
$\sigma = 123$



$\pi = 3, 1, 2, 6, 4, 5, 9, 7, 8, 10, 13, 11, 12$

The *separating trees* of σ and π

(σ and π separable)



Unpaired occurrences of $\sigma = 123$ in π

Theorem: If σ and π are separable permutations, then

$$\mu(\sigma, \pi) = \sum_{\mathsf{X}} (-1)^{\mathsf{parity(X)}}$$

where the sum is over *unpaired* occurrences of σ in π.

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$$|\mu(\sigma, \pi)| \leqslant \sigma(\pi)$$

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where $\sigma(\pi)$ is the number of occurrences of σ in π.

In particular: If π avoids 132 then $|\mu(\sigma, \pi)| \leqslant \sigma(\pi)$

Inflations

Let $\pi[\pi_1, \ldots, \pi_n]$ be the permutation obtained by replacing i in π by π_i after incrementing the letters of π_i so that they are larger than those corresponding to π_{i-1} in the inflated permutation and smaller than those corresponding to π_{i+1}.

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Example: $2413[312, 1, 21, 12] = 47831265$

Theorem: If π is *simple* of length n and none of $\pi_1, \ldots, \pi_n$ contains π then

$$\mu(\pi, \pi[\pi_1, \ldots, \pi_n]) = \prod_{i=1}^{n} \mu(1, \pi_i)$$

A class $\mathcal{C}$ of permutations is *sum-closed* if

$$\sigma, \pi \in \mathcal{C} \;\Rightarrow\; \sigma \oplus \pi \in \mathcal{C}$$

A class $\mathcal{C}$ of permutations is *sum-closed* if   skew-closed

$$\sigma, \pi \in \mathcal{C} \ \Rightarrow \ \sigma \oplus \pi \in \mathcal{C} \qquad \sigma \ominus \pi \in \mathcal{C}$$

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Theorem: Suppose σ is neither decomposable nor skew-decomposable. Let $\mathcal{C}$ be any set of permutations. Then

$$\max\{|\mu(\sigma,\pi)| : \pi \in \mathsf{cl}(\mathcal{C})\} = \max\{|\mu(\sigma,\pi)| : \pi \in \mathcal{C}\}.$$

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The computation of $\mu(\sigma,\pi)$ for $\pi \in \mathrm{cl}(\mathcal{C})$ can be efficiently reduced to the computation of the values $\mu(\sigma,\tau)$ for $\tau \in \mathcal{C}$.

More results

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- In particular, $\mu(\sigma,\pi)$ is unbounded
- If π is separable then $\mu(1,\pi) \in \{0,1,-1\}$

More results

- $\mu(135\ldots 2k\text{-}1\ 2k\ldots 42, 135\ldots 2n-1\ 2n\ldots 42) = \binom{n+k-1}{n-k}$
- In particular, $\mu(\sigma,\pi)$ is unbounded
- If π is separable then $\mu(1,\pi) \in \{0,1,-1\}$
- If σ is indecomposable and $\pi = \pi_1 \oplus 1 \oplus \pi_2$ then $\mu(\sigma,\pi) = 0$

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- Which intervals $[\sigma, \pi]$ are shellable?
- Conjecture: $max_{\pi \in S_n} |\mu(1, \pi)|$ is unbounded as a function of n

To do:

Find better general theorems that solve some of the open problems, and unify some of those and the above results

.