Shuffles of Permutations

Camillia Smith Barnes

Department of Mathematical Sciences
Sweet Briar College
cbarnes@sbc.edu

Permutation Patterns ✻ Dartmouth College ✻ 11 August 2010

Outline

1. Background
   - Definitions
   - Problem: How Many Distinct Shuffles?
2. Distinct Shuffles of Permutations
   - Motivation & Special Cases
   - Towards Enumeration Theorem
3. Goals
   - Partial Ordering on Symmetric Group for Shuffles
   - Minimal & Maximal Permutations for Shuffles
   - Further Generalizations

Outline

Words, Letters, Length, & Supports

- a *word* is a string of symbols
  e.g. 12342
- a *letter* is a symbol used in a word
  e.g. 3
- the *length* of a word is $\#$ of letters
  e.g. length of 12342 is 5
- the *support* of a word is the set of letters in word
  e.g. $\text{supp}(12342) = \{1, 2, 3, 4\}$

Words, Letters, Length, & Supports

- a *word* is a string of symbols
  e.g. 12342
- a *letter* is a symbol used in a word
  e.g. 3
- the *length* of a word is # of letters
  e.g. length of 12342 is 5
- the *support* of a word is the set of letters in word
  e.g. $\text{supp}(12342) = \{1, 2, 3, 4\}$

Words, Letters, Length, & Supports

- a *word* is a string of symbols
  e.g. 12342
- a *letter* is a symbol used in a word
  e.g. 3
- the *length* of a word is $\#$ of letters
  e.g. length of 12342 is 5
- the *support* of a word is the set of letters in word
  e.g. $\text{supp}(12342) = \{1, 2, 3, 4\}$

Words, Letters, Length, & Supports

- a *word* is a string of symbols
  e.g. 12342
- a *letter* is a symbol used in a word
  e.g. 3
- the *length* of a word is $\#$ of letters
  e.g. length of 12342 is 5
- the *support* of a word is the set of letters in word
  e.g. $\text{supp}(12342) = \{1, 2, 3, 4\}$

Intuitive Definition of Shuffle

- a *shuffle* of the words 123 and 456 is obtained by interspersing the letters of these words
- the letters of each original word must stay in order
- for example, 142563

Intuitive Definition of Shuffle

- a *shuffle* of the words 123 and 456 is obtained by interspersing the letters of these words
- the letters of each original word must stay in order
- for example, 142563

Intuitive Definition of Shuffle

- a *shuffle* of the words 123 and 456 is obtained by interspersing the letters of these words
- the letters of each original word must stay in order
- for example, 142563

Formal Definition of Shuffle

Given two words $u = a_1 a_2 \ldots a_m$ and $v = b_1 b_2 \ldots b_n$,

- concatenate u and v:

$$c_1 c_2 \ldots c_{m+n} = a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n$$

- then permute entries to obtain a *shuffle* of u with v:

$$c_{\rho(1)} c_{\rho(2)} \cdots c_{\rho(m+n)}$$

- where ρ is a permutation on $m + n$ elements satisfying order-preserving conditions
  - $\rho^{-1}(1) < \rho^{-1}(2) < \cdots < \rho^{-1}(m)$
  - $\rho^{-1}(m+1) < \rho^{-1}(m+2) < \cdots < \rho^{-1}(m+n)$

Formal Definition of Shuffle

Given two words $u = a_1 a_2 \ldots a_m$ and $v = b_1 b_2 \ldots b_n$,

- concatenate u and v:

$$c_1 c_2 \ldots c_{m+n} = a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n$$

- then permute entries to obtain a *shuffle* of u with v:

$$c_{\rho(1)} c_{\rho(2)} \cdots c_{\rho(m+n)}$$

- where ρ is a permutation on $m + n$ elements satisfying order-preserving conditions
  - $\rho^{-1}(1) < \rho^{-1}(2) < \cdots < \rho^{-1}(m)$
  - $\rho^{-1}(m+1) < \rho^{-1}(m+2) < \cdots < \rho^{-1}(m+n)$

Formal Definition of Shuffle

Given two words $u = a_1 a_2 \ldots a_m$ and $v = b_1 b_2 \ldots b_n$,

- concatenate u and v:

$$c_1 c_2 \ldots c_{m+n} = a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n$$

- then permute entries to obtain a *shuffle* of u with v:

$$c_{\rho(1)} c_{\rho(2)} \cdots c_{\rho(m+n)}$$

- where ρ is a permutation on $m + n$ elements satisfying order-preserving conditions
  - $\rho^{-1}(1) < \rho^{-1}(2) < \cdots < \rho^{-1}(m)$
  - $\rho^{-1}(m+1) < \rho^{-1}(m+2) < \cdots < \rho^{-1}(m+n)$

Formal Definition of Shuffle

Given two words $u = a_1 a_2 \ldots a_m$ and $v = b_1 b_2 \ldots b_n$,

- concatenate u and v:

$$c_1 c_2 \ldots c_{m+n} = a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n$$

- then permute entries to obtain a *shuffle* of u with v:

$$c_{\rho(1)} c_{\rho(2)} \cdots c_{\rho(m+n)}$$

- where ρ is a permutation on $m + n$ elements satisfying order-preserving conditions
  - $\rho^{-1}(1) < \rho^{-1}(2) < \cdots < \rho^{-1}(m)$
  - $\rho^{-1}(m+1) < \rho^{-1}(m+2) < \cdots < \rho^{-1}(m+n)$

Formal Definition of Shuffle

Given two words $u = a_1 a_2 \ldots a_m$ and $v = b_1 b_2 \ldots b_n$,

- concatenate u and v:

$$c_1 c_2 \ldots c_{m+n} = a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n$$

- then permute entries to obtain a *shuffle* of u with v:

$$c_{\rho(1)} c_{\rho(2)} \cdots c_{\rho(m+n)}$$

- where ρ is a permutation on $m + n$ elements satisfying order-preserving conditions
  - $\rho^{-1}(1) < \rho^{-1}(2) < \cdots < \rho^{-1}(m)$
  - $\rho^{-1}(m+1) < \rho^{-1}(m+2) < \cdots < \rho^{-1}(m+n)$

Example: Shuffles of 123 with 456

All shuffles of the words 123 and 456:

123456	142536	412356	415623
124356	142563	412536	451236
124536	145236	412563	451263
124563	145263	415236	451623
142356	145623	415263	456123

- Notation: $\mathfrak{sh}(123, 456) = \{\text{shuffles of 123 with 456}\}$

Example: Shuffles of 123 with 456

All shuffles of the words 123 and 456:

123456	142536	412356	415623
124356	142563	412536	451236
124536	145236	412563	451263
124563	145263	415236	451623
142356	145623	415263	456123

- Notation: $\mathfrak{sh}(123, 456) = \{\text{shuffles of 123 with 456}\}$

Example: Shuffles of 123 with 456

All shuffles of the words 123 and 456:

123456	142536	412356	415623
124356	142563	412536	451236
124536	145236	412563	451263
124563	145263	415236	451623
142356	145623	415263	456123

- Notation: $\mathfrak{sh}(123, 456) = \{\text{shuffles of } 123 \text{ with } 456\}$

Outline

1. Background
   - Definitions
   - Problem: How Many Distinct Shuffles?
2. Distinct Shuffles of Permutations
   - Motivation & Special Cases
   - Towards Enumeration Theorem
3. Goals
   - Partial Ordering on Symmetric Group for Shuffles
   - Minimal & Maximal Permutations for Shuffles
   - Further Generalizations

Example: Counting Shuffles of 123 with 456

- None of entries in word 123 appear in word 456, so

$$\#\text{ of distinct shuffles} = \#\mathfrak{sh}(123, 456) = \binom{6}{3} = 20$$

- In fact, if words u and v have disjoint supports, then

$$\#\mathfrak{sh}(u, v) = \binom{\text{length of } u + \text{ length of } v}{\text{length of } u}$$

Example: Counting Shuffles of 123 with 456

- None of entries in word 123 appear in word 456, so

$$\#\text{ of distinct shuffles} = \#\mathfrak{sh}(123, 456) = \binom{6}{3} = 20$$

- In fact, if words u and v have disjoint supports, then

$$\#\mathfrak{sh}(u, v) = \binom{\text{length of } u + \text{ length of } v}{\text{length of } u}$$

Example with Non-Disjoint Supports: Permuations

All *distinct* shuffles of permutation words 123 and 312:

12**33**12	132312	131232	312132	3**1122**3
123132	132132	131**22**3	3**1212**3	
123123	132123	312312	3**11**232	

- Entries that could come from either word in **black**
- Shuffles with black entries have multiplicity > 1
- eg 3**1122**3 could be
  311223, 311223, 311223, or 311223

Example with Non-Disjoint Supports: Permuations

All *distinct* shuffles of permutation words 123 and 312:

12**33**12	132312	131232	312132	**3**1**122**3
123132	132132	131**22**3	**3**1**212**3	
123123	132123	312312	**311**232	

- Entries that could come from either word in **black**
- Shuffles with black entries have multiplicity > 1
- eg 3**1122**3 could be
  311223, 311223, 311223, or 311223

Example with Non-Disjoint Supports: Permuations

All *distinct* shuffles of permutation words 123 and 312:

12**33**12	132312	131232	312132	**3**1**122**3
123132	132132	131**22**3	**3**1**212**3	
123123	132123	312312	**3**1**1**232	

- Entries that could come from either word in **black**
- Shuffles with black entries have multiplicity > 1
- eg 3**1122**3 could be 311223, 311223, 311223, or 311223

Example with Non-Disjoint Supports: Permuations

All *distinct* shuffles of permutation words 123 and 312:

12**33**12	132312	131232	312132	**3**1**1**2**2**3
123132	132132	131**22**3	**3**12**12**3	
123123	132123	312312	**311**232	

- Entries that could come from either word in **black**
- Shuffles with black entries have multiplicity > 1
- eg 3**1122**3 could be
  311223, 311223, 311223, or 311223

Outline

Shuffles of Identity Permutation with Itself

All *distinct* shuffles of 123 with itself:

123123
121323
121233
112323
112233

- $\#\mathfrak{sh}(123, 123) = 5$

Shuffles of Identity Permutation with Itself

All *distinct* shuffles of 123 with itself:

123123
121323
121233
112323
112233

- $\#\mathfrak{sh}(123, 123) = 5$

Why count shuffles of permutations?

Fun Fact:

$$\#\mathfrak{sh}(\mathrm{id}_n, \mathrm{id}_n) = \frac{1}{n+1}\binom{2n}{n} = C_n \qquad (n^{th} \text{ Catalan } \#)$$

Proof: Construct bijection:
$\mathfrak{sh}(\mathrm{id}_n, \mathrm{id}_n) \longrightarrow \{\text{ballot sequences of length } 2n\}$

Why count shuffles of permutations?

Fun Fact:

$$\#\mathfrak{sh}(\mathrm{id}_n, \mathrm{id}_n) = \frac{1}{n+1}\binom{2n}{n} = C_n \qquad (n^{th} \text{ Catalan } \#)$$

Proof: Construct bijection:
$\mathfrak{sh}(\mathrm{id}_n, \mathrm{id}_n) \longrightarrow \{\text{ballot sequences of length } 2n\}$

Why count shuffles of permutations?

Fun Fact:

$$\#\mathfrak{sh}(\mathrm{id}_n, \mathrm{id}_n) = \frac{1}{n+1}\binom{2n}{n} = C_n \qquad (n^{th} \text{ Catalan } \#)$$

Proof: Construct bijection:

$\mathfrak{sh}(\mathrm{id}_n, \mathrm{id}_n) \longrightarrow \{$ballot sequences of length $2n\}$

Bijection Between Shuffles & Ballot Sequences

- ballot sequence has
  - n $+1$'s
  - n -1's
  - all partial sums nonnegative
- Substitute $+1$ for 1^{st} occurrence of 1 through n, substitute -1 for 2^{nd} occurrence
- eg $121323 \longmapsto ++-+--$
- $\#\{\text{ballot sequences of length } 2n\} = C_n$

Bijection Between Shuffles & Ballot Sequences

- ballot sequence has
  - n $+1$'s
  - n -1's
  - all partial sums nonnegative
- Substitute $+1$ for 1^{st} occurrence of 1 through n, substitute -1 for 2^{nd} occurrence
- eg $121323 \longmapsto ++-+--$
- $\#\{\text{ballot sequences of length } 2n\} = C_n$

Bijection Between Shuffles & Ballot Sequences

- ballot sequence has
  - n $+1$'s
  - n -1's
  - all partial sums nonnegative
- Substitute $+1$ for 1^{st} occurrence of 1 through n, substitute -1 for 2^{nd} occurrence
- eg $121323 \longmapsto ++-+--$
- $\#\{\text{ballot sequences of length } 2n\} = C_n$

Bijection Between Shuffles & Ballot Sequences

- ballot sequence has
  - n $+1$'s
  - n -1's
  - all partial sums nonnegative
- Substitute $+1$ for 1^{st} occurrence of 1 through n, substitute -1 for 2^{nd} occurrence
- eg $121323 \longmapsto ++-+--$
- $\#\{\text{ballot sequences of length } 2n\} = C_n$

Bijection Between Shuffles & Ballot Sequences

- ballot sequence has
  - n $+1$'s
  - n -1's
  - all partial sums nonnegative
- Substitute $+1$ for 1^{st} occurrence of 1 through n, substitute -1 for 2^{nd} occurrence
- eg $121323 \longmapsto ++-+--$
- $\#\{\text{ballot sequences of length } 2n\} = C_n$

Bijection Between Shuffles & Ballot Sequences

- ballot sequence has
  - n $+1$'s
  - n -1's
  - all partial sums nonnegative
- Substitute $+1$ for 1^{st} occurrence of 1 through n, substitute -1 for 2^{nd} occurrence
- eg 121323 $\longmapsto$ $++-+--$
- $\#\{\text{ballot sequences of length } 2n\} = C_n$

Bijection Between Shuffles & Ballot Sequences

- ballot sequence has
  - n $+1$'s
  - n -1's
  - all partial sums nonnegative
- Substitute $+1$ for 1^{st} occurrence of 1 through n, substitute -1 for 2^{nd} occurrence
- eg 121323 $\longmapsto$ $++-+--$
- $\#\{$ballot sequences of length $2n\} = C_n$

Example: Shuffles of Identity with Reverse

The *distinct* shuffles of the words 123 and 321 are:

12**33**21	132321	13**22**13	312321	321213
123231	132123	31**22**31	312123	32**11**23
123213	13**22**31	31**22**13	321231	

- Shuffles with **bolded** entries occur exactly twice
- Take *total* number of shuffles (counted with multiplicities) & *subtract* number of duplicates

Example: Shuffles of Identity with Reverse

The *distinct* shuffles of the words 123 and 321 are:

12**33**21	132321	13**22**13	312321	321213
123231	132123	31**22**31	312123	32**11**23
123213	13**22**31	31**22**13	321231	

- Shuffles with **bolded** entries occur exactly twice
- Take *total* number of shuffles (counted with multiplicities) & *subtract* number of duplicates

Example: Shuffles of Identity with Reverse

The *distinct* shuffles of the words 123 and 321 are:

12**33**21	132321	13**22**13	312321	321213
123231	132123	31**22**31	312123	32**11**23
123213	13**22**31	31**22**13	321231	

- Shuffles with **bolded** entries occur exactly twice
- Take *total* number of shuffles (counted with multiplicities) & *subtract* number of duplicates

Example: Shuffles of Identity with Reverse

The *distinct* shuffles of the words 123 and 321 are:

12**33**21	132321	13**22**13	312321	321213
123231	132123	31**22**31	312123	32**11**23
123213	13**22**31	31**22**13	321231	

- Shuffles with **bolded** entries occur exactly twice
- Take *total* number of shuffles (counted with multiplicities) & *subtract* number of duplicates

Counting Shuffles of Identity with Reverse

Shuffles of the words 123 and 321 of multiplicity 2 are:

123321	312231
132231	312213
132213	321123

- Count duplicates by constructing bijection: $\{w \in \mathfrak{sh}(\mathrm{id}_n, \mathrm{rev}_n) \mid \mu(w) = 2\} \longrightarrow \mathfrak{sh}(+^{n-1}, -^{n-1})$
- Notation: use rev_n instead of ω_0 to avoid double subscript as n varies

Counting Shuffles of Identity with Reverse

Shuffles of the words 123 and 321 of multiplicity 2 are:

123321	312231
132231	312213
132213	321123

- Count duplicates by constructing bijection:
  $\{w \in \mathfrak{sh}(\mathrm{id}_n, \mathrm{rev}_n) \mid \mu(w) = 2\} \longrightarrow \mathfrak{sh}(+^{n-1}, -^{n-1})$
- Notation: use rev_n instead of ω_0
  to avoid double subscript as n varies

Counting Shuffles of Identity with Reverse

Shuffles of the words 123 and 321 of multiplicity 2 are:

123321	312231
132231	312213
132213	321123

- Count duplicates by constructing bijection:
  $\{w \in \mathfrak{sh}(\mathrm{id}_n, \mathrm{rev}_n) \mid \mu(w) = 2\} \longrightarrow \mathfrak{sh}(+^{n-1}, -^{n-1})$
- Notation: use rev_n instead of ω_0
  to avoid double subscript as n varies

Bijection:

$$\{w \in \mathfrak{sh}(\mathrm{id}_n, \mathrm{rev}_n) \mid \mu(w) = 2\} \longrightarrow \mathfrak{sh}(+^{n-1}, -^{n-1})$$

- Excise double entries
- Replace entries from id_n with $+$
- Replace entries from rev_n with $-$
- Examples:

123321	$\longrightarrow$	1221	$\longrightarrow$	$++--$
132231	$\longrightarrow$	1331	$\longrightarrow$	$+-+-$
132213	$\longrightarrow$	1313	$\longrightarrow$	$+--+$

- Can compute $\#\mathfrak{sh}(123, 321) = \binom{6}{3} - \binom{4}{2} = 14$
- In general, $\#\mathfrak{sh}(\mathrm{id}_m, \mathrm{rev}_n) = \binom{m+n}{m} - \binom{m+n-2}{m-1}$

Bijection: $\{w \in \mathfrak{sh}(\mathrm{id}_n, \mathrm{rev}_n) \mid \mu(w) = 2\} \longrightarrow \mathfrak{sh}(+^{n-1}, -^{n-1})$

- Excise double entries
- Replace entries from id_n with $+$
- Replace entries from rev_n with $-$
- Examples:

123321	$\longrightarrow$	1221	$\longrightarrow$	$++--$
132231	$\longrightarrow$	1331	$\longrightarrow$	$+-+-$
132213	$\longrightarrow$	1313	$\longrightarrow$	$+--+$

- Can compute $\#\mathfrak{sh}(123, 321) = \binom{6}{3} - \binom{4}{2} = 14$
- In general, $\#\mathfrak{sh}(\mathrm{id}_m, \mathrm{rev}_n) = \binom{m+n}{m} - \binom{m+n-2}{m-1}$

Bijection: $\{w \in \mathfrak{sh}(\mathrm{id}_n, \mathrm{rev}_n) \mid \mu(w) = 2\} \longrightarrow \mathfrak{sh}(+^{n-1}, -^{n-1})$

- Excise double entries
- Replace entries from id_n with $+$
- Replace entries from rev_n with $-$
- Examples:

123321	$\longrightarrow$	1221	$\longrightarrow$	$++--$
132231	$\longrightarrow$	1331	$\longrightarrow$	$+-+-$
132213	$\longrightarrow$	1313	$\longrightarrow$	$+--+$

- Can compute $\#\mathfrak{sh}(123, 321) = \binom{6}{3} - \binom{4}{2} = 14$
- In general, $\#\mathfrak{sh}(\mathrm{id}_m, \mathrm{rev}_n) = \binom{m+n}{m} - \binom{m+n-2}{m-1}$

Bijection: $\{w \in \mathfrak{sh}(\mathrm{id}_n, \mathrm{rev}_n) \mid \mu(w) = 2\} \longrightarrow \mathfrak{sh}(+^{n-1}, -^{n-1})$

- Excise double entries
- Replace entries from id_n with $+$
- Replace entries from rev_n with $-$
- Examples:

123321	$\longrightarrow$	1221	$\longrightarrow$	$++--$
132231	$\longrightarrow$	1331	$\longrightarrow$	$+-+-$
132213	$\longrightarrow$	1313	$\longrightarrow$	$+--+$

- Can compute $\#\mathfrak{sh}(123, 321) = \binom{6}{3} - \binom{4}{2} = 14$
- In general, $\#\mathfrak{sh}(\mathrm{id}_m, \mathrm{rev}_n) = \binom{m+n}{m} - \binom{m+n-2}{m-1}$

Bijection:
$\{w \in \mathfrak{sh}(\mathrm{id}_n, \mathrm{rev}_n) \mid \mu(w) = 2\} \longrightarrow \mathfrak{sh}(+^{n-1}, -^{n-1})$

- Excise double entries
- Replace entries from id_n with $+$
- Replace entries from rev_n with $-$
- Examples:

123321	$\longrightarrow$	1221	$\longrightarrow$	$++--$
132231	$\longrightarrow$	1331	$\longrightarrow$	$+-+-$
132213	$\longrightarrow$	1313	$\longrightarrow$	$+--+$

- Can compute $\#\mathfrak{sh}(123, 321) = \binom{6}{3} - \binom{4}{2} = 14$
- In general, $\#\mathfrak{sh}(\mathrm{id}_m, \mathrm{rev}_n) = \binom{m+n}{m} - \binom{m+n-2}{m-1}$

Bijection:

$\{w \in \mathfrak{sh}(\mathrm{id}_n, \mathrm{rev}_n) \mid \mu(w) = 2\} \longrightarrow \mathfrak{sh}(+^{n-1}, -^{n-1})$

- Excise double entries
- Replace entries from id_n with $+$
- Replace entries from rev_n with $-$
- Examples:

123321	$\longrightarrow$	1221	$\longrightarrow$	$++--$
132231	$\longrightarrow$	1331	$\longrightarrow$	$+-+-$
132213	$\longrightarrow$	1313	$\longrightarrow$	$+--+$

- Can compute $\#\mathfrak{sh}(123, 321) = \binom{6}{3} - \binom{4}{2} = 14$
- In general, $\#\mathfrak{sh}(\mathrm{id}_m, \mathrm{rev}_n) = \binom{m+n}{m} - \binom{m+n-2}{m-1}$

Outline

Multiplicities of Shuffles

- *WLOG:* Enough to enumerate shuffles of each permutation with identity permutation
- Claim: Multiplicity of shuffle of two permutations always a *power of 2*
- Reason: If a shuffle has multiplicity > 1, there must be one or more blocks of letters that could come from either permutation
- e.g. for $121233 \in \mathfrak{sh}(123, 123)$:
  - each 12 block could come from either copy of 123
  - each 3 could come from either copy of 123
- Call such a contiguous block of consecutive letters a *consecutive identity subword* (or *idword*)

Multiplicities of Shuffles

- *WLOG:* Enough to enumerate shuffles of each permutation with identity permutation
- Claim: Multiplicity of shuffle of two permutations always a *power of 2*
- Reason: If a shuffle has multiplicity > 1, there must be one or more blocks of letters that could come from either permutation
- e.g. for $121233 \in \mathfrak{sh}(123, 123)$:
  - each 12 block could come from either copy of 123
  - each 3 could come from either copy of 123
- Call such a contiguous block of consecutive letters a *consecutive identity subword* (or *idword*)

Multiplicities of Shuffles

- *WLOG:* Enough to enumerate shuffles of each permutation with identity permutation
- Claim: Multiplicity of shuffle of two permutations always a *power of 2*
- Reason: If a shuffle has multiplicity > 1, there must be one or more blocks of letters that could come from either permutation
- e.g. for $121233 \in \mathfrak{sh}(123, 123)$:
    - each 12 block could come from either copy of 123
    - each 3 could come from either copy of 123
- Call such a contiguous block of consecutive letters a *consecutive identity subword* (or *idword*)

Multiplicities of Shuffles

- *WLOG:* Enough to enumerate shuffles of each permutation with identity permutation
- Claim: Multiplicity of shuffle of two permutations always a *power of 2*
- Reason: If a shuffle has multiplicity > 1, there must be one or more blocks of letters that could come from either permutation
- e.g. for $121233 \in \mathfrak{sh}(123, 123)$:
  - each 12 block could come from either copy of 123
  - each 3 could come from either copy of 123
- Call such a contiguous block of consecutive letters a *consecutive identity subword* (or *idword*)

Multiplicities of Shuffles

- *WLOG:* Enough to enumerate shuffles of each permutation with identity permutation
- Claim: Multiplicity of shuffle of two permutations always a *power of 2*
- Reason: If a shuffle has multiplicity > 1, there must be one or more blocks of letters that could come from either permutation
- e.g. for $121233 \in \mathfrak{sh}(123, 123)$:
  - each 12 block could come from either copy of 123
  - each 3 could come from either copy of 123
- Call such a contiguous block of consecutive letters a *consecutive identity subword* (or *idword*)

Multiplicities of Shuffles

- *WLOG:* Enough to enumerate shuffles of each permutation with identity permutation
- Claim: Multiplicity of shuffle of two permutations always a *power of 2*
- Reason: If a shuffle has multiplicity > 1, there must be one or more blocks of letters that could come from either permutation
- e.g. for $121233 \in \mathfrak{sh}(123, 123)$:
  - each 12 block could come from either copy of 123
  - each 3 could come from either copy of 123
- Call such a contiguous block of consecutive letters a *consecutive identity subword* (or *idword*)

Multiplicities of Shuffles

- *WLOG:* Enough to enumerate shuffles of each permutation with identity permutation
- Claim: Multiplicity of shuffle of two permutations always a *power of 2*
- Reason: If a shuffle has multiplicity > 1, there must be one or more blocks of letters that could come from either permutation
- e.g. for $121233 \in \mathfrak{sh}(123, 123)$:
  - each 12 block could come from either copy of 123
  - each 3 could come from either copy of 123
- Call such a contiguous block of consecutive letters a *consecutive identity subword* (or *idword*)

Examples: Idwords

- Example: $312123 \in \mathfrak{sh}(123, 312)$ has idword 12 "locally shuffled with itself" (or *local-shuffled*)
  - multiplicity $\mu(312123) = 2$
- Example: $311223 \in \mathfrak{sh}(123, 312)$ has idwords 1 and 2 local-shuffled
  - multiplicity $\mu(311223) = 4$
  - could say 311223 has idword 12 local-shuffled
  - but need to decompose longer idwords into smallest possible pieces to get maximal number simultaneously local-shuffled

Examples: Idwords

- Example: $312123 \in \mathfrak{sh}(123, 312)$ has idword 12 "locally shuffled with itself" (or *local-shuffled*)
  - multiplicity $\mu(312123) = 2$
- Example: $311223 \in \mathfrak{sh}(123, 312)$ has idwords 1 and 2 local-shuffled
  - multiplicity $\mu(311223) = 4$
  - could say 311223 has idword 12 local-shuffled
  - but need to decompose longer idwords into smallest possible pieces to get maximal number simultaneously local-shuffled

Examples: Idwords

- Example: $312123 \in \mathfrak{sh}(123, 312)$ has idword 12 "locally shuffled with itself" (or *local-shuffled*)
  - multiplicity $\mu(312123) = 2$
- Example: $311223 \in \mathfrak{sh}(123, 312)$ has idwords 1 and 2 local-shuffled
  - multiplicity $\mu(311223) = 4$
  - could say 311223 has idword 12 local-shuffled
  - but need to decompose longer idwords into smallest possible pieces to get maximal number simultaneously local-shuffled

Examples: Idwords

- Example: $312123 \in \mathfrak{sh}(123, 312)$ has idword 12 "locally shuffled with itself" (or *local-shuffled*)
  - multiplicity $\mu(312123) = 2$
- Example: $311223 \in \mathfrak{sh}(123, 312)$ has idwords 1 and 2 local-shuffled
  - multiplicity $\mu(311223) = 4$
  - could say 311223 has idword 12 local-shuffled
  - but need to decompose longer idwords into smallest possible pieces to get maximal number simultaneously local-shuffled

Examples: Idwords

- Example: $312123 \in \mathfrak{sh}(123, 312)$ has idword 12 “locally shuffled with itself” (or *local-shuffled*)
  - multiplicity $\mu(312123) = 2$
- Example: $311223 \in \mathfrak{sh}(123, 312)$ has idwords 1 and 2 local-shuffled
  - multiplicity $\mu(311223) = 4$
  - could say 311223 has idword 12 local-shuffled
  - but need to decompose longer idwords into smallest possible pieces to get maximal number simultaneously local-shuffled

Examples: Idwords

- Example: $312123 \in \mathfrak{sh}(123, 312)$ has idword 12 "locally shuffled with itself" (or *local-shuffled*)
  - multiplicity $\mu(312123) = 2$
- Example: $311223 \in \mathfrak{sh}(123, 312)$ has idwords 1 and 2 local-shuffled
  - multiplicity $\mu(311223) = 4$
  - could say 311223 has idword 12 local-shuffled
  - but need to decompose longer idwords into smallest possible pieces to get maximal number simultaneously local-shuffled

Sketched Proof of Claim: Multiplicities are Powers of 2

- For a given shuffle w, can construct an *automorphism group* of permutations acting on letters of w so that w still "looks" the same
- automorphism group for shuffle $w \in \mathfrak{sh}(u, v)$ is subgroup of $\mathfrak{S}_{m+n}$ where $u \in \mathfrak{S}_m$, $v \in \mathfrak{S}_n$

Sketched Proof of Claim: Multiplicities are Powers of 2

- For a given shuffle w, can construct an *automorphism group* of permutations acting on letters of w so that w still "looks" the same
- automorphism group for shuffle $w \in \mathfrak{sh}(u, v)$ is subgroup of $\mathfrak{S}_{m+n}$ where $u \in \mathfrak{S}_m$, $v \in \mathfrak{S}_n$

Examples: Shuffle Automorphism Groups

- Example: $312123 \in \mathfrak{sh}(123, 312)$ has automorphism group $\langle(24)(35)\rangle$
  - $\langle(24)(35)\rangle$ order 2 $\implies$ $\mu(312123) = 2$ in $\mathfrak{sh}(123, 312)$
- Example: $311223 \in \mathfrak{sh}(123, 312)$ has automorphism group $\langle(23), (45)\rangle$
  - $\langle(24), (35)\rangle$ order 4 $\implies$ $\mu(311223) = 4$ in $\mathfrak{sh}(123, 312)$

Examples: Shuffle Automorphism Groups

- Example: $312123 \in \mathfrak{sh}(123, 312)$
  has automorphism group $\langle(24)(35)\rangle$
  - $\langle(24)(35)\rangle$ order 2 $\implies \mu(312123) = 2$ in $\mathfrak{sh}(123, 312)$
- Example: $311223 \in \mathfrak{sh}(123, 312)$
  has automorphism group $\langle(23), (45)\rangle$
  - $\langle(24), (35)\rangle$ order 4 $\implies \mu(311223) = 4$ in $\mathfrak{sh}(123, 312)$

Examples: Shuffle Automorphism Groups

- Example: $312123 \in \mathfrak{sh}(123, 312)$
  has automorphism group $\langle(24)(35)\rangle$
  - $\langle(24)(35)\rangle$ order 2 $\implies \mu(312123) = 2$ in $\mathfrak{sh}(123, 312)$
- Example: $311223 \in \mathfrak{sh}(123, 312)$
  has automorphism group $\langle(23), (45)\rangle$
  - $\langle(24), (35)\rangle$ order 4 $\implies \mu(311223) = 4$ in $\mathfrak{sh}(123, 312)$

Examples: Shuffle Automorphism Groups

- Example: $312123 \in \mathfrak{sh}(123, 312)$ has automorphism group $\langle (24)(35) \rangle$
  - $\langle (24)(35) \rangle$ order 2 $\implies \mu(312123) = 2$ in $\mathfrak{sh}(123, 312)$
- Example: $311223 \in \mathfrak{sh}(123, 312)$ has automorphism group $\langle (23), (45) \rangle$
  - $\langle (24), (35) \rangle$ order 4 $\implies \mu(311223) = 4$ in $\mathfrak{sh}(123, 312)$

Sketched Proof of Claim, Continued

- permutation shuffle automorphism groups always generated by disjoint permutations of order 2
- each automorphism group has order 2^k for some integer $k \geq 0$
- order of automorphism group is multiplicity of shuffle

Sketched Proof of Claim, Continued

- permutation shuffle automorphism groups
  always generated by disjoint permutations of order 2
- each automorphism group has order 2^k
  for some integer $k \geq 0$
- order of automorphism group is multiplicity of shuffle

Sketched Proof of Claim, Continued

- permutation shuffle automorphism groups
  always generated by disjoint permutations of order 2
- each automorphism group has order 2^k
  for some integer $k \geq 0$
- order of automorphism group is multiplicity of shuffle

Overview of Strategy

- Take total number of shuffles with multiplicity, subtract duplicates
- Essentially an application of **Principle of Inclusion-Exclusion**
- i.e. take an alternating sum of set cardinalities that subtracts the correct number of repeated shuffles from the binomial coefficient

Overview of Strategy

- Take total number of shuffles with multiplicity, subtract duplicates
- Essentially an application of **Principle of Inclusion-Exclusion**
- i.e. take an alternating sum of set cardinalities that subtracts the correct number of repeated shuffles from the binomial coefficient

Overview of Strategy

- Take total number of shuffles with multiplicity, subtract duplicates
- Essentially an application of **Principle of Inclusion-Exclusion**
- i.e. take an alternating sum of set cardinalities that subtracts the correct number of repeated shuffles from the binomial coefficient

Using Inclusion-Exclusion

- Let $T_j^\sigma = \#$shuffles of σ with id_m, *counted with multiplicity*, that have j or more local-shuffled idwords
- Example: How many times is $w = 11623434565277 \in \mathfrak{sh}(\text{id}_7, 1634527)$ counted in $T_2^{1634527}$?
  - w has 3 local-shuffled idwords: 11623434565277
  - choose 2 out of 3 of local-shuffled idwords; $\binom{3}{2} = 3$ choices
  - remaining local-shuffled idword can be interpreted $2^{3-2} = 2$ ways
  - w is counted $\binom{3}{2}2^{3-2} = 6$ times in $T_2^{1634527}$

Using Inclusion-Exclusion

- Let $T_j^\sigma = \#$shuffles of σ with id_m, *counted with multiplicity*, that have j or more local-shuffled idwords
- Example: How many times is $w = 11623434565277 \in \mathfrak{sh}(\mathrm{id}_7, 1634527)$ counted in $T_2^{1634527}$?
  - w has 3 local-shuffled idwords: 11623434565277
  - choose 2 out of 3 of local-shuffled idwords; $\binom{3}{2} = 3$ choices
  - remaining local-shuffled idword can be interpreted $2^{3-2} = 2$ ways
  - w is counted $\binom{3}{2}2^{3-2} = 6$ times in $T_2^{1634527}$

Using Inclusion-Exclusion

- Let $T_j^\sigma = \#$shuffles of σ with id_m, *counted with multiplicity*, that have j or more local-shuffled idwords
- Example: How many times is $w = 11623434565277 \in \mathfrak{sh}(\mathrm{id}_7, 1634527)$ counted in $T_2^{1634527}$?
  - w has 3 local-shuffled idwords: 11623434565277
  - choose 2 out of 3 of local-shuffled idwords; $\binom{3}{2} = 3$ choices
  - remaining local-shuffled idword can be interpreted $2^{3-2} = 2$ ways
  - w is counted $\binom{3}{2}2^{3-2} = 6$ times in $T_2^{1634527}$

Using Inclusion-Exclusion

- Let $T_j^\sigma = \#$shuffles of σ with id_m, *counted with multiplicity*, that have j or more local-shuffled idwords
- Example: How many times is $w = 11623434565277 \in \mathfrak{sh}(\mathrm{id}_7, 1634527)$ counted in $T_2^{1634527}$?
  - w has 3 local-shuffled idwords: 11623434565277
  - choose 2 out of 3 of local-shuffled idwords; $\binom{3}{2} = 3$ choices
  - remaining local-shuffled idword can be interpreted $2^{3-2} = 2$ ways
  - w is counted $\binom{3}{2}2^{3-2} = 6$ times in $T_2^{1634527}$

Using Inclusion-Exclusion

- Let $T_j^\sigma = \#$shuffles of σ with id_m, *counted with multiplicity*, that have j or more local-shuffled idwords
- Example: How many times is $w = 11623434565277 \in \mathfrak{sh}(\text{id}_7, 1634527)$ counted in $T_2^{1634527}$?
  - w has 3 local-shuffled idwords: 11623434565277
  - choose 2 out of 3 of local-shuffled idwords; $\binom{3}{2} = 3$ choices
  - remaining local-shuffled idword can be interpreted $2^{3-2} = 2$ ways
  - w is counted $\binom{3}{2}2^{3-2} = 6$ times in $T_2^{1634527}$

Using Inclusion-Exclusion

- Let $T_j^\sigma = \#$shuffles of σ with id_m, *counted with multiplicity*, that have j or more local-shuffled idwords
- Example: How many times is $w = 11623434565277 \in \mathfrak{sh}(\mathrm{id}_7, 1634527)$ counted in $T_2^{1634527}$?
  - w has 3 local-shuffled idwords: 11623434565277
  - choose 2 out of 3 of local-shuffled idwords; $\binom{3}{2} = 3$ choices
  - remaining local-shuffled idword can be interpreted $2^{3-2} = 2$ ways
  - w is counted $\binom{3}{2}2^{3-2} = 6$ times in $T_2^{1634527}$

Using Inclusion-Exclusion, Continued

- In general, can show that

$$T_j^\sigma = \sum_{k=j}^{m} \binom{k}{j} 2^{k-j} N_k^\sigma$$

where $N_k^\sigma = \#\{w \in \mathfrak{sh}(\mathrm{id}_m, \sigma) \mid \mu(w) = 2^k\}$

- Hence, thanks to binomial theorem & changing order of summation,

$$\#\mathfrak{sh}(\mathrm{id}_m, \sigma) = \sum_{j=0}^{m} (-1)^j T_j^\sigma$$

Using Inclusion-Exclusion, Continued

- In general, can show that

$$T_j^\sigma = \sum_{k=j}^{m} \binom{k}{j} 2^{k-j} N_k^\sigma$$

where $N_k^\sigma = \#\{w \in \mathfrak{sh}(\mathrm{id}_m, \sigma) \mid \mu(w) = 2^k\}$

- Hence, thanks to binomial theorem & changing order of summation,

$$\#\mathfrak{sh}(\mathrm{id}_m, \sigma) = \sum_{j=0}^{m} (-1)^j T_j^\sigma$$

Example: Shuffles of 123 with 312

- For example,

$$\#\mathfrak{sh}(\mathrm{id}_3, 312) = T_0^{312} - T_1^{312} + T_2^{312} - T_3^{312}$$

- We know $T_0^{312} = \binom{6}{3} = 20$

Example: Shuffles of 123 with 312

- For example,

$$\#\mathfrak{sh}(\mathrm{id}_3, 312) = T_0^{312} - T_1^{312} + T_2^{312} - T_3^{312}$$

- We know $T_0^{312} = \binom{6}{3} = 20$

Counting Shuffles of 123 with 312

- Notation: $*$ denotes concatenation, [] denotes empty word
- Fix double 1's: $\mathfrak{sh}([], 3) * 11 * \mathfrak{sh}(23, 2)$, get $\binom{1}{0} \cdot C_0 \cdot \binom{3}{2} = 3$
- Fix double 2's: $\mathfrak{sh}(1, 31) * 22 * \mathfrak{sh}(3, [])$, get $\binom{3}{1} \cdot C_0 \cdot \binom{1}{1} = 3$
- Fix double 3's: $\mathfrak{sh}(12, []) * 33 * \mathfrak{sh}([], 12)$, get $\binom{2}{2} \cdot C_0 \cdot \binom{2}{0} = 1$
- Fix unique indecomposable shuffle of 12 with itself: $\mathfrak{sh}([], 3) * 1212 * \mathfrak{sh}(3, [])$, get $\binom{1}{0} \cdot C_1 \cdot \binom{1}{1} = 1$
- $T_1^{312} = 3 + 3 + 1 + 1 = 8$

Counting Shuffles of 123 with 312

- Notation: $*$ denotes concatenation, [] denotes empty word
- Fix double 1's: $\mathfrak{sh}([], 3) * 11 * \mathfrak{sh}(23, 2)$, get $\binom{1}{0} \cdot C_0 \cdot \binom{3}{2} = 3$
- Fix double 2's: $\mathfrak{sh}(1, 31) * 22 * \mathfrak{sh}(3, [])$, get $\binom{3}{1} \cdot C_0 \cdot \binom{1}{1} = 3$
- Fix double 3's: $\mathfrak{sh}(12, []) * 33 * \mathfrak{sh}([], 12)$, get $\binom{2}{2} \cdot C_0 \cdot \binom{2}{0} = 1$
- Fix unique indecomposable shuffle of 12 with itself: $\mathfrak{sh}([], 3) * 1212 * \mathfrak{sh}(3, [])$, get $\binom{1}{0} \cdot C_1 \cdot \binom{1}{1} = 1$
- $T_1^{312} = 3 + 3 + 1 + 1 = 8$

Counting Shuffles of 123 with 312

- Notation: $*$ denotes concatenation, [] denotes empty word
- Fix double 1's: $\mathfrak{sh}([],3) * 11 * \mathfrak{sh}(23,2)$, get $\binom{1}{0} \cdot C_0 \cdot \binom{3}{2} = 3$
- Fix double 2's: $\mathfrak{sh}(1,31) * 22 * \mathfrak{sh}(3,[])$, get $\binom{3}{1} \cdot C_0 \cdot \binom{1}{1} = 3$
- Fix double 3's: $\mathfrak{sh}(12,[]) * 33 * \mathfrak{sh}([],12)$, get $\binom{2}{2} \cdot C_0 \cdot \binom{2}{0} = 1$
- Fix unique indecomposable shuffle of 12 with itself: $\mathfrak{sh}([],3) * 1212 * \mathfrak{sh}(3,[])$, get $\binom{1}{0} \cdot C_1 \cdot \binom{1}{1} = 1$
- $T_1^{312} = 3 + 3 + 1 + 1 = 8$

Counting Shuffles of 123 with 312

- Notation: $*$ denotes concatenation, [] denotes empty word
- Fix double 1's: $\mathfrak{sh}([],3) * 11 * \mathfrak{sh}(23,2)$, get $\binom{1}{0} \cdot C_0 \cdot \binom{3}{2} = 3$
- Fix double 2's: $\mathfrak{sh}(1,31) * 22 * \mathfrak{sh}(3,[])$, get $\binom{3}{1} \cdot C_0 \cdot \binom{1}{1} = 3$
- Fix double 3's: $\mathfrak{sh}(12,[]) * 33 * \mathfrak{sh}([],12)$, get $\binom{2}{2} \cdot C_0 \cdot \binom{2}{0} = 1$
- Fix unique indecomposable shuffle of 12 with itself: $\mathfrak{sh}([],3) * 1212 * \mathfrak{sh}(3,[])$, get $\binom{1}{0} \cdot C_1 \cdot \binom{1}{1} = 1$
- $T_1^{312} = 3 + 3 + 1 + 1 = 8$

Counting Shuffles of 123 with 312

- Notation: $*$ denotes concatenation, [] denotes empty word
- Fix double 1's: $\mathfrak{sh}([], 3) * 11 * \mathfrak{sh}(23, 2)$, get $\binom{1}{0} \cdot C_0 \cdot \binom{3}{2} = 3$
- Fix double 2's: $\mathfrak{sh}(1, 31) * 22 * \mathfrak{sh}(3, [])$, get $\binom{3}{1} \cdot C_0 \cdot \binom{1}{1} = 3$
- Fix double 3's: $\mathfrak{sh}(12, []) * 33 * \mathfrak{sh}([], 12)$, get $\binom{2}{2} \cdot C_0 \cdot \binom{2}{0} = 1$
- Fix unique indecomposable shuffle of 12 with itself: $\mathfrak{sh}([], 3) * 1212 * \mathfrak{sh}(3, [])$, get $\binom{1}{0} \cdot C_1 \cdot \binom{1}{1} = 1$
- $T_1^{312} = 3 + 3 + 1 + 1 = 8$

Counting Shuffles of 123 with 312

- Notation: $*$ denotes concatenation, [] denotes empty word
- Fix double 1's: $\mathfrak{sh}([],3) * 11 * \mathfrak{sh}(23,2)$, get $\binom{1}{0} \cdot C_0 \cdot \binom{3}{2} = 3$
- Fix double 2's: $\mathfrak{sh}(1,31) * 22 * \mathfrak{sh}(3,[])$, get $\binom{3}{1} \cdot C_0 \cdot \binom{1}{1} = 3$
- Fix double 3's: $\mathfrak{sh}(12,[]) * 33 * \mathfrak{sh}([],12)$, get $\binom{2}{2} \cdot C_0 \cdot \binom{2}{0} = 1$
- Fix unique indecomposable shuffle of 12 with itself: $\mathfrak{sh}([],3) * 1212 * \mathfrak{sh}(3,[])$, get $\binom{1}{0} \cdot C_1 \cdot \binom{1}{1} = 1$
- $T_1^{312} = 3 + 3 + 1 + 1 = 8$

Counting Shuffles of 123 with 312, continued

- Fix double 1's and double 2's: $\mathfrak{sh}([], 3) * 11 * \mathfrak{sh}([], []) * 22 * \mathfrak{sh}(3, [])$, get $\binom{1}{0} \cdot C_0 \cdot \binom{0}{0} \cdot C_0 \cdot \binom{1}{1} = 1$
  - $T_2^{312} = 1$
- $\nexists$ 3-set of idwords in 312 that can be simultaneously local-shuffled
  - $T_3^{312} = 0$
- Hence

$$\begin{aligned}\#\mathfrak{sh}(\mathrm{id}_3, 312) &= T_0^{312} - T_1^{312} + T_2^{312} - T_3^{312} \\ &= 20 - 8 + 1 - 0 \\ &= 13\end{aligned}$$

Counting Shuffles of 123 with 312, continued

- Fix double 1's and double 2's: $\mathfrak{sh}([],3) * 11 * \mathfrak{sh}([],[]) * 22 * \mathfrak{sh}(3,[])$, get $\binom{1}{0} \cdot C_0 \cdot \binom{0}{0} \cdot C_0 \cdot \binom{1}{1} = 1$
  - $T_2^{312} = 1$
- $\nexists$ 3-set of idwords in 312 that can be simultaneously local-shuffled
  - $T_3^{312} = 0$
- Hence

$$\begin{aligned}\#\mathfrak{sh}(\mathrm{id}_3, 312) &= T_0^{312} - T_1^{312} + T_2^{312} - T_3^{312} \\ &= 20 - 8 + 1 - 0 \\ &= 13\end{aligned}$$

Counting Shuffles of 123 with 312, continued

- Fix double 1's and double 2's: $\mathfrak{sh}([], 3) * 11 * \mathfrak{sh}([], []) * 22 * \mathfrak{sh}(3, [])$, get $\binom{1}{0} \cdot C_0 \cdot \binom{0}{0} \cdot C_0 \cdot \binom{1}{1} = 1$
  - $T_2^{312} = 1$
- $\nexists$ 3-set of idwords in 312 that can be simultaneously local-shuffled
  - $T_3^{312} = 0$
- Hence

$$\begin{aligned}\#\mathfrak{sh}(\mathrm{id}_3, 312) &= T_0^{312} - T_1^{312} + T_2^{312} - T_3^{312} \\ &= 20 - 8 + 1 - 0 \\ &= 13\end{aligned}$$

Counting Shuffles of 123 with 312, continued

- Fix double 1's and double 2's: $\mathfrak{sh}([], 3) * 11 * \mathfrak{sh}([], []) * 22 * \mathfrak{sh}(3, [])$, get $\binom{1}{0} \cdot C_0 \cdot \binom{0}{0} \cdot C_0 \cdot \binom{1}{1} = 1$
  - $T_2^{312} = 1$
- $\nexists$ 3-set of idwords in 312 that can be simultaneously local-shuffled
  - $T_3^{312} = 0$
- Hence

$$\begin{aligned}\#\mathfrak{sh}(\mathrm{id}_3, 312) &= T_0^{312} - T_1^{312} + T_2^{312} - T_3^{312} \\ &= 20 - 8 + 1 - 0 \\ &= 13\end{aligned}$$

Counting Shuffles of 123 with 312, continued

- Fix double 1's and double 2's: $\mathfrak{sh}([],3) * 11 * \mathfrak{sh}([],[]) * 22 * \mathfrak{sh}(3,[])$, get $\binom{1}{0} \cdot C_0 \cdot \binom{0}{0} \cdot C_0 \cdot \binom{1}{1} = 1$
  - $T_2^{312} = 1$
- $\nexists$ 3-set of idwords in 312 that can be simultaneously local-shuffled
  - $T_3^{312} = 0$
- Hence

$$\begin{aligned}\#\mathfrak{sh}(\mathrm{id}_3, 312) &= T_0^{312} - T_1^{312} + T_2^{312} - T_3^{312} \\ &= 20 - 8 + 1 - 0 \\ &= 13\end{aligned}$$

Overview of Enumeration Theorem

- Have theorem enumerating distinct shuffles of any two permutations
- Theorem provides complex but "computationally good" formula enumerating distinct shuffles of *any* two permutations
- Involves alternating sum of products of determinants
- Entries of determinants depend directly on inversions of (non-identity) permutation

Overview of Enumeration Theorem

- Have theorem enumerating distinct shuffles of any two permutations
- Theorem provides complex but “computationally good” formula enumerating distinct shuffles of *any* two permutations
- Involves alternating sum of products of determinants
- Entries of determinants depend directly on inversions of (non-identity) permutation

Overview of Enumeration Theorem

- Have theorem enumerating distinct shuffles of any two permutations
- Theorem provides complex but “computationally good” formula enumerating distinct shuffles of *any* two permutations
- Involves alternating sum of products of determinants
- Entries of determinants depend directly on inversions of (non-identity) permutation

Overview of Enumeration Theorem

- Have theorem enumerating distinct shuffles of any two permutations
- Theorem provides complex but "computationally good" formula enumerating distinct shuffles of *any* two permutations
- Involves alternating sum of products of determinants
- Entries of determinants depend directly on inversions of (non-identity) permutation

Examples of Maple Computations

> $CountShuffles([1, 2, 3], [3, 2, 1])$

$$14 \tag{18}$$

> $CountShuffles([1, 2, 3], [3, 1, 2])$

$$13 \tag{19}$$

> $CountShuffles([1, 3, 2], [2, 5, 3, 1, 4])$

$$43 \tag{20}$$

> $CountShuffles([1, 2, 3, 4, 5, 6], [4, 5, 6, 1, 2, 3])$

$$792 \tag{21}$$

> $CountShuffles([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], [7, 8, 9, 10, 11, 12, 13, 1, 2, 3, 4, 5, 6])$

$$10104590 \tag{22}$$

Outline

A "Quasi" Bruhat Ordering?

- Notation: for $\sigma \in \mathfrak{S}_n$, set $s(\sigma) = \#\mathfrak{sh}(\mathrm{id}_n, \sigma)$
- Would like to find partial ordering on $\mathfrak{S}_n$ such that $s(\sigma)$ is monotone increasing
- Bruhat ordering fails for $n = 4, 5, 6$, likely to fail for $n > 6$
- In most cases, $s(\sigma)$ increases as length of σ increases
- Exceptions such as
  - $s(3412) = 54$ whereas $s(4312) = 52$
  - $s(4312) = 52$ whereas $s(4321) = 50$
  - $s(2431) = 46$ whereas $s(4231) = 44$

A “Quasi” Bruhat Ordering?

- Notation: for $\sigma \in \mathfrak{S}_n$, set $s(\sigma) = \#\mathfrak{sh}(\mathrm{id}_n, \sigma)$
- Would like to find partial ordering on $\mathfrak{S}_n$ such that $s(\sigma)$ is monotone increasing
- Bruhat ordering fails for $n = 4, 5, 6$, likely to fail for $n > 6$
- In most cases, $s(\sigma)$ increases as length of σ increases
- Exceptions such as
  - $s(3412) = 54$ whereas $s(4312) = 52$
  - $s(4312) = 52$ whereas $s(4321) = 50$
  - $s(2431) = 46$ whereas $s(4231) = 44$

A "Quasi" Bruhat Ordering?

- Notation: for $\sigma \in \mathfrak{S}_n$, set $s(\sigma) = \#\mathfrak{sh}(\mathrm{id}_n, \sigma)$
- Would like to find partial ordering on $\mathfrak{S}_n$ such that $s(\sigma)$ is monotone increasing
- Bruhat ordering fails for $n = 4, 5, 6$, likely to fail for $n > 6$
- In most cases, $s(\sigma)$ increases as length of σ increases
- Exceptions such as
  - $s(3412) = 54$ whereas $s(4312) = 52$
  - $s(4312) = 52$ whereas $s(4321) = 50$
  - $s(2431) = 46$ whereas $s(4231) = 44$

A "Quasi" Bruhat Ordering?

- Notation: for $\sigma \in \mathfrak{S}_n$, set $s(\sigma) = \#\mathfrak{sh}(\mathrm{id}_n, \sigma)$
- Would like to find partial ordering on $\mathfrak{S}_n$ such that $s(\sigma)$ is monotone increasing
- Bruhat ordering fails for $n = 4, 5, 6$, likely to fail for $n > 6$
- In most cases, $s(\sigma)$ increases as length of σ increases
- Exceptions such as
  - $s(3412) = 54$ whereas $s(4312) = 52$
  - $s(4312) = 52$ whereas $s(4321) = 50$
  - $s(2431) = 46$ whereas $s(4231) = 44$

A "Quasi" Bruhat Ordering?

- Notation: for $\sigma \in \mathfrak{S}_n$, set $s(\sigma) = \#\mathfrak{sh}(\mathrm{id}_n, \sigma)$
- Would like to find partial ordering on $\mathfrak{S}_n$ such that $s(\sigma)$ is monotone increasing
- Bruhat ordering fails for $n = 4, 5, 6$, likely to fail for $n > 6$
- In most cases, $s(\sigma)$ increases as length of σ increases
- Exceptions such as
  - $s(3412) = 54$ whereas $s(4312) = 52$
  - $s(4312) = 52$ whereas $s(4321) = 50$
  - $s(2431) = 46$ whereas $s(4231) = 44$

A “Quasi” Bruhat Ordering?

- Notation: for $\sigma \in \mathfrak{S}_n$, set $s(\sigma) = \#\mathfrak{sh}(\mathrm{id}_n, \sigma)$
- Would like to find partial ordering on $\mathfrak{S}_n$ such that $s(\sigma)$ is monotone increasing
- Bruhat ordering fails for $n = 4, 5, 6$, likely to fail for $n > 6$
- In most cases, $s(\sigma)$ increases as length of σ increases
- Exceptions such as
  - $s(3412) = 54$ whereas $s(4312) = 52$
  - $s(4312) = 52$ whereas $s(4321) = 50$
  - $s(2431) = 46$ whereas $s(4231) = 44$

A “Quasi” Bruhat Ordering?

- Notation: for $\sigma \in \mathfrak{S}_n$, set $s(\sigma) = \#\mathfrak{sh}(\mathrm{id}_n, \sigma)$
- Would like to find partial ordering on $\mathfrak{S}_n$ such that $s(\sigma)$ is monotone increasing
- Bruhat ordering fails for $n = 4, 5, 6$, likely to fail for $n > 6$
- In most cases, $s(\sigma)$ increases as length of σ increases
- Exceptions such as
  - $s(3412) = 54$ whereas $s(4312) = 52$
  - $s(4312) = 52$ whereas $s(4321) = 50$
  - $s(2431) = 46$ whereas $s(4231) = 44$

A “Quasi” Bruhat Ordering?

- Notation: for $\sigma \in \mathfrak{S}_n$, set $s(\sigma) = \#\mathfrak{sh}(\mathrm{id}_n, \sigma)$
- Would like to find partial ordering on $\mathfrak{S}_n$ such that $s(\sigma)$ is monotone increasing
- Bruhat ordering fails for $n = 4, 5, 6$, likely to fail for $n > 6$
- In most cases, $s(\sigma)$ increases as length of σ increases
- Exceptions such as
  - $s(3412) = 54$ whereas $s(4312) = 52$
  - $s(4312) = 52$ whereas $s(4321) = 50$
  - $s(2431) = 46$ whereas $s(4231) = 44$

Outline

Find Minimal Permutation(s) for Shuffles

- For which permutation(s) $\sigma \in \mathfrak{S}_n$ is $s(\sigma)$ minimal for n?
- For $1 \leq n \leq 6$, $\min_{\sigma \in \mathfrak{S}_n} s(\sigma) = C_n$ (achieved by $\sigma = \mathrm{id}_n$)
- *Conjecture:* For all n, $\min_{\sigma \in \mathfrak{S}_n} s(\sigma) = C_n$, and minimum is achieved by $\sigma = \mathrm{id}_n$
- Can show that id_n gives a “local” minimum: each adjacent transposition has twice as many shuffles

Find Minimal Permutation(s) for Shuffles

- For which permutation(s) $\sigma \in \mathfrak{S}_n$ is $s(\sigma)$ minimal for n?
- For $1 \leq n \leq 6$, $\min_{\sigma \in \mathfrak{S}_n} s(\sigma) = C_n$ (achieved by $\sigma = \mathrm{id}_n$)
- *Conjecture:* For all n, $\min_{\sigma \in \mathfrak{S}_n} s(\sigma) = C_n$, and minimum is achieved by $\sigma = \mathrm{id}_n$
- Can show that id_n gives a “local” minimum: each adjacent transposition has twice as many shuffles

Find Minimal Permutation(s) for Shuffles

- For which permutation(s) $\sigma \in \mathfrak{S}_n$ is $s(\sigma)$ minimal for n?
- For $1 \leq n \leq 6$, $\min_{\sigma \in \mathfrak{S}_n} s(\sigma) = C_n$ (achieved by $\sigma = \mathrm{id}_n$)
- *Conjecture:* For all n, $\min_{\sigma \in \mathfrak{S}_n} s(\sigma) = C_n$, and minimum is achieved by $\sigma = \mathrm{id}_n$
- Can show that id_n gives a “local” minimum: each adjacent transposition has twice as many shuffles

Find Minimal Permutation(s) for Shuffles

- For which permutation(s) $\sigma \in \mathfrak{S}_n$ is $s(\sigma)$ minimal for n?
- For $1 \leq n \leq 6$, $\min_{\sigma \in \mathfrak{S}_n} s(\sigma) = C_n$ (achieved by $\sigma = \mathrm{id}_n$)
- *Conjecture:* For all n, $\min_{\sigma \in \mathfrak{S}_n} s(\sigma) = C_n$, and minimum is achieved by $\sigma = \mathrm{id}_n$
- Can show that id_n gives a “local” minimum: each adjacent transposition has twice as many shuffles

Find Maximal Permutation(s) for Shuffles

- For which permutation(s) $\sigma \in \mathfrak{S}_n$ is $s(\sigma)$ maximal for n?
- For $n = 4, 5, 6$, $s(\sigma)$ achieves maximum when $\sigma = 3412$, 34512, 456123, respectively
  - Note that $3412 = 12 \ominus 12$
  - $34512 = 123 \ominus 12$
  - $456123 = 123 \ominus 123$
- Does pattern hold for $n > 6$?
- *Conjecture:* For $n \geq 4$, maximal $s(\sigma)$ for $\sigma \in \mathfrak{S}_n$ achieved by $\mathrm{id}_{\lceil \frac{n}{2} \rceil} \ominus \mathrm{id}_{\lfloor \frac{n}{2} \rfloor}$ (or equivalently, $\mathrm{id}_{\lfloor \frac{n}{2} \rfloor} \ominus \mathrm{id}_{\lceil \frac{n}{2} \rceil}$)

Find Maximal Permutation(s) for Shuffles

- For which permutation(s) $\sigma \in \mathfrak{S}_n$ is $s(\sigma)$ maximal for n?
- For $n = 4, 5, 6$, $s(\sigma)$ achieves maximum when $\sigma =$ 3412, 34512, 456123, respectively
  - Note that $3412 = 12 \ominus 12$
  - $34512 = 123 \ominus 12$
  - $456123 = 123 \ominus 123$
- Does pattern hold for $n > 6$?
- *Conjecture:* For $n \geq 4$, maximal $s(\sigma)$ for $\sigma \in \mathfrak{S}_n$ achieved by $\mathrm{id}_{\lceil \frac{n}{2} \rceil} \ominus \mathrm{id}_{\lfloor \frac{n}{2} \rfloor}$ (or equivalently, $\mathrm{id}_{\lfloor \frac{n}{2} \rfloor} \ominus \mathrm{id}_{\lceil \frac{n}{2} \rceil}$)

Find Maximal Permutation(s) for Shuffles

- For which permutation(s) $\sigma \in \mathfrak{S}_n$ is $s(\sigma)$ maximal for n?
- For $n = 4, 5, 6$, $s(\sigma)$ achieves maximum when $\sigma = 3412$, 34512, 456123, respectively
  - Note that $3412 = 12 \ominus 12$
  - $34512 = 123 \ominus 12$
  - $456123 = 123 \ominus 123$
- Does pattern hold for $n > 6$?
- *Conjecture:* For $n \geq 4$, maximal $s(\sigma)$ for $\sigma \in \mathfrak{S}_n$ achieved by $\mathrm{id}_{\lceil \frac{n}{2} \rceil} \ominus \mathrm{id}_{\lfloor \frac{n}{2} \rfloor}$ (or equivalently, $\mathrm{id}_{\lfloor \frac{n}{2} \rfloor} \ominus \mathrm{id}_{\lceil \frac{n}{2} \rceil}$)

Find Maximal Permutation(s) for Shuffles

- For which permutation(s) $\sigma \in \mathfrak{S}_n$ is $s(\sigma)$ maximal for n?
- For $n = 4, 5, 6$, $s(\sigma)$ achieves maximum when $\sigma = 3412$, 34512, 456123, respectively
  - Note that $3412 = 12 \ominus 12$
  - $34512 = 123 \ominus 12$
  - $456123 = 123 \ominus 123$
- Does pattern hold for $n > 6$?
- *Conjecture:* For $n \geq 4$, maximal $s(\sigma)$ for $\sigma \in \mathfrak{S}_n$ achieved by $\mathrm{id}_{\lceil \frac{n}{2} \rceil} \ominus \mathrm{id}_{\lfloor \frac{n}{2} \rfloor}$ (or equivalently, $\mathrm{id}_{\lfloor \frac{n}{2} \rfloor} \ominus \mathrm{id}_{\lceil \frac{n}{2} \rceil}$)

Find Maximal Permutation(s) for Shuffles

- For which permutation(s) $\sigma \in \mathfrak{S}_n$ is $s(\sigma)$ maximal for n?
- For $n = 4, 5, 6$, $s(\sigma)$ achieves maximum when $\sigma = 3412$, 34512, 456123, respectively
  - Note that $3412 = 12 \ominus 12$
  - $34512 = 123 \ominus 12$
  - $456123 = 123 \ominus 123$
- Does pattern hold for $n > 6$?
- *Conjecture:* For $n \geq 4$, maximal $s(\sigma)$ for $\sigma \in \mathfrak{S}_n$ achieved by $\mathrm{id}_{\lceil \frac{n}{2} \rceil} \ominus \mathrm{id}_{\lfloor \frac{n}{2} \rfloor}$ (or equivalently, $\mathrm{id}_{\lfloor \frac{n}{2} \rfloor} \ominus \mathrm{id}_{\lceil \frac{n}{2} \rceil}$)

Find Maximal Permutation(s) for Shuffles

- For which permutation(s) $\sigma \in \mathfrak{S}_n$ is $s(\sigma)$ maximal for n?
- For $n = 4, 5, 6$, $s(\sigma)$ achieves maximum when $\sigma = 3412$, 34512, 456123, respectively
    - Note that $3412 = 12 \ominus 12$
    - $34512 = 123 \ominus 12$
    - $456123 = 123 \ominus 123$
- Does pattern hold for $n > 6$?
- *Conjecture:* For $n \geq 4$, maximal $s(\sigma)$ for $\sigma \in \mathfrak{S}_n$ achieved by $\mathrm{id}_{\lceil \frac{n}{2} \rceil} \ominus \mathrm{id}_{\lfloor \frac{n}{2} \rfloor}$ (or equivalently, $\mathrm{id}_{\lfloor \frac{n}{2} \rfloor} \ominus \mathrm{id}_{\lceil \frac{n}{2} \rceil}$)

Find Maximal Permutation(s) for Shuffles

- For which permutation(s) $\sigma \in \mathfrak{S}_n$ is $s(\sigma)$ maximal for n?
- For $n = 4, 5, 6$, $s(\sigma)$ achieves maximum when $\sigma = 3412$, 34512, 456123, respectively
  - Note that $3412 = 12 \ominus 12$
  - $34512 = 123 \ominus 12$
  - $456123 = 123 \ominus 123$
- Does pattern hold for $n > 6$?
- *Conjecture:* For $n \geq 4$, maximal $s(\sigma)$ for $\sigma \in \mathfrak{S}_n$ achieved by $\mathrm{id}_{\lceil \frac{n}{2} \rceil} \ominus \mathrm{id}_{\lfloor \frac{n}{2} \rfloor}$ (or equivalently, $\mathrm{id}_{\lfloor \frac{n}{2} \rfloor} \ominus \mathrm{id}_{\lceil \frac{n}{2} \rceil}$)

Outline

Shuffles of Multiset Permutations & *k*-Shuffles

- Enumerate distinct shuffles of multiset permutations
  eg compute $\#\mathfrak{sh}(12322, 33214)$
- Enumerate distinct *k*-shuffles of permutations
  eg compute $\#\mathfrak{sh}(132, 231, 1324)$

Shuffles of Multiset Permutations & *k*-Shuffles

- Enumerate distinct shuffles of multiset permutations
  eg compute $\#\mathfrak{sh}(12322, 33214)$
- Enumerate distinct *k*-shuffles of permutations
  eg compute $\#\mathfrak{sh}(132, 231, 1324)$

Acknowledgments

TYHAONUK!

Special thanks to:

- my dissertation advisor, Richard Stanley
- the organizers of PP2010

Acknowledgments

TYHAONUK!

Special thanks to:

- my dissertation advisor, Richard Stanley
- the organizers of PP2010

Acknowledgments

TYHAONUK!

Special thanks to:

- my dissertation advisor, Richard Stanley
- the organizers of PP2010

Selected References

W.F. Doran IV,
Shuffling lattices,
J. Combinatorial Theory Ser. A 66 (2002), 1-26.

R. Ehrenborg,
On posets and Hopf algebras,
Adv. Math. 119 (1996), 1-25.

C. Greene,
Posets of shuffles,
J. Combinatorial Theory Ser. A 47 (1988), 191-206.

P. Hersh,
Two generalizations of posets of shuffles,
J. Combinatorial Theory Ser. A 97 (2002), 1-26.

D.E. Knuth,
The Art of Computer Programming 3, Addison-Wesley, 1973.

D.E. Knuth,
The Art of Computer Programming 2, 3rd ed., Addison-Wesley, 1998.

R. Simion & R.P. Stanley,
Flag-symmetry of the poset of shuffles and a local action of the symmetric group,
Discrete Math. 204 (1999), 369-396.

R.P. Stanley,
Enumerative Combinatorics 1, Cambridge Univ. Press, 1997.

R.P. Stanley,
Enumerative Combinatorics 2, Cambridge Univ. Press, 1999.

Statement of Enumeration Theorem

Theorem

Let $\sigma \in S_n$ and assume $m \leq n$. Then
$\#\mathfrak{sh}(\mathrm{id}_m, \sigma)$

$$= \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\mathbf{a}=\{0=a_0<a_1<\cdots<a_{2k}<a_{2k+1}=m+1\}} (-1)^{h(\mathbf{a})} \prod_{r=0}^{k} \det Z^{\sigma}_{a_{2r},a_{2r+1}} \prod_{s=1}^{k} \det Y^{\sigma}_{a_{2s-1},a_{2s}},$$

where

$$h(\mathbf{a}) = m - \sum_{t=1}^{k} (a_{2t} - a_{2t-1}),$$

(continued on next slide...)

Statement of Enumeration Theorem, Continued

Theorem (continued)

and we define the matrices

$$Z^{\sigma}_{c,d} = [z^{\sigma}_{i,j}]_{c \leq i \leq d-1,\ c+1 \leq j \leq d},$$

with

$$z^{\sigma}_{i,j} = \begin{cases} 0, & i > j \\ 1, & i = j \\ 0, & 0 < i < j < m+1 \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j) \\ \binom{j-i-1+\sigma^{-1}(j)-\sigma^{-1}(i)-1}{j-i-1}, & 0 < i < j < m+1 \text{ and } \sigma^{-1}(i) < \sigma^{-1}(j) \\ \binom{j-1+\sigma^{-1}(j)-1}{j-1}, & i = 0,\ j < m+1 \\ \binom{m-i+n-\sigma^{-1}(i)}{m-i}, & j = m+1,\ i > 0 \\ \binom{m+n}{m}, & i = 0,\ j = m+1, \end{cases}$$

(continued on next slide...)

Statement of Enumeration Theorem, Continued

Theorem (continued)

and the matrices

$$Y^{\sigma}_{e,f} = [y^{\sigma}_{i,j}]_{e \leq i,j \leq f-1},$$

with

$$y^{\sigma}_{i,j} = \begin{cases} 0, & i-j > 1 \text{ or } \sigma^{-1}(i+1) \neq \sigma^{-1}(i)+1 \\ -1, & i-j = 1 \text{ and } \sigma^{-1}(i+1) = \sigma^{-1}(i)+1 \\ C_{j-i}, & i \leq j \text{ and } \sigma^{-1}(i+1) = \sigma^{-1}(i)+1 \end{cases}$$

and where

$$C_{j-i} = \frac{1}{j-i+1}\binom{2(j-i)}{j-i}, \text{ the } (j-i)^{th} \text{ Catalan number.}$$

Using Inclusion-Exclusion: More Details

- Recall $T_j^\sigma = \sum_{k=j}^{m} \binom{k}{j} 2^{k-j} N_k^\sigma$ where $N_k^\sigma = \#\{w \in \mathfrak{sh}(\mathrm{id}_m, \sigma) \mid \mu(w) = 2^k\}$
- So

$$
\begin{aligned}
\#\mathfrak{sh}(\mathrm{id}_n, \sigma) &= \sum_{k=0}^{m} N_k^\sigma \\
&= \sum_{k=0}^{m} N_k^\sigma (2-1)^k \\
&= \sum_{k=0}^{m} N_k^\sigma \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} (-1)^j \\
&= \sum_{j=0}^{m} (-1)^j \sum_{k=j}^{m} \binom{k}{j} 2^{k-j} N_k^\sigma \\
&= \sum_{j=0}^{m} (-1)^j T_j^\sigma
\end{aligned}
$$