Minimal Overlapping Patterns

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Given a sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, let $\mathrm{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\mathrm{red}(\sigma) = 1\ 4\ 3\ 2$.

Given a permutation τ in the symmetric group S_j, define a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ to have a **τ-match at place i** provided $\mathrm{red}(\sigma_i \cdots \sigma_{i+j-1}) = \tau$. Let $\tau\text{-mch}(\sigma)$ be the number of τ-matches in the permutation σ.

A permutation is called **τ-avoiding** if there are no indices $i_1 < \cdots < i_j$ such that $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$.

τ-matches

Let $\tau\text{-mch}(\sigma)$ be the number of τ-matches in the permutation σ.

$$A_\tau(t) = \sum_{n\geq 0} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}| \text{ and } \quad (1)$$

$$P_\tau(x,t) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in S_n} x^{\tau\text{-mch}(\sigma)}. \quad (2)$$

Theorem 0.1. *Goulden and Jackson (1983)*
If $\tau = 1\ 2 \cdots k$, *then*

$$A_\tau(t) = \frac{1}{\sum_{i\geq 0} \frac{x^{ki}}{(ki)!} - \frac{x^{ki+1}}{(ki+1)!}} \quad (3)$$

Theorem 0.2. *Elizalde and Noy (2003)*

$$
\begin{aligned}
P_{132}(t,u) &= \frac{1}{1-\int_0^t e^{\frac{(u-1)z^2}{2}}\,dz}, \quad \textit{and} \\
P_{1342}(t,u) &= \frac{1}{1-\int_0^t e^{\frac{(u-1)z^3}{6}}\,dz}.
\end{aligned}
$$

Theorem 0.3. *Kitaev (2005)*
Let $\tau = 12\cdots a\sigma(a+1)$, *where* σ *is a permutation of* $\{a+2, a+3, \ldots, k+1\}$, *then*

$$A_\tau(t) = \frac{1}{1 - t + \sum_{i\geq 1} \frac{(-1)^{i+1}x^{ki+1}}{(ki+1)!} \prod_{j=2}^{i} \binom{jk-a}{k-a}}. \tag{4}$$

c-Wilf equivalence and strong c-Wilf equivalence

Given permutations $\alpha, \beta \in S_n$, we say that α is **c-Wilf equivalent** to β if $A_\alpha(t) = A_\beta(t)$.

Given permutations $\alpha, \beta \in S_n$, we say that α is **strongly c-Wilf equivalent** to β if $P_\alpha(u,t) = P_\beta(u,t)$.

If $\sigma = \sigma_1\ \sigma_2 \cdots \sigma_n \in S_n$, we let

$\sigma^r = \sigma_n \cdots \sigma_2\ \sigma_1$ be the reverse of σ and
$\sigma^c = (n+1-\sigma_1)\ (n+1-\sigma_2) \cdots (n+1-\sigma_n)$ denote the complement of σ.

It is easy to see that c-Wilf equivalences classes and strong c-Wilf equivalence classes are closed under complement and reverse.

Generating Functions for the maximum number of non-overlapping patterns

Let τ-nlap(σ) be the maximum number of non-overlapping τ-matches in σ where two τ-matches are said to overlap if they contain any of the same integers.

Kitaev's Theorem (2003)

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{A(t)}{(1-x) + x(1-t)A(t)} \tag{5}$$

where $A(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}|$.

Υ-matches

Suppose $\Upsilon \subseteq S_j$.

We say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has an Υ-match at place i provided $\mathrm{red}(\sigma_i \cdots \sigma_{i+j-1}) \in \Upsilon$.

Let $\Upsilon\text{-mch}(\sigma)$ and $\Upsilon\text{-nlap}(\sigma)$ be the number of Υ-matches and non-overlapping Υ matches in σ, respectively.

Theorem 0.4. *(Mendes-Remmel)*

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\Upsilon - nlap(\sigma)} q^{inv(\sigma)} = \frac{A_q^{\Upsilon}(t)}{(1-x) + x(1-t)A_q^{\Upsilon}(t)} \qquad (6)$$

where $A_q^{\Upsilon}(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n : \Upsilon\text{-mch}(\sigma)=0} q^{inv(\sigma)}$.

Mendes and Remmel (2006) developed a general method for computing $A_\tau(t)$.

They proved $A_\tau(t)$ is of the form

$$A_\tau(t) = \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|}. \qquad (7)$$

where $||w|| = |\tau| + w_1 + \cdots + w_n$,

J_τ is a certain collection of words associated with τ, and

$\mathcal{P}_w^\tau$ is a certain collection of permutations $\sigma \in S_{||w||}$ if $w = w_1 \ldots w_n$.

Liese and Remmel (2009) showed how to use the Mendes-Remmel result to compute $A_\tau(t)$ for various classes of permutations which are shuffles of the permutation $1\ 2 \cdots n$ with an arbitrary permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ of the numbers $\{n+1,\ldots,n+m\}$.

Partition $1\ 2\cdots n$ into $\alpha^{(1)}\alpha^{(2)}\cdots\alpha^{(k+1)}$ where each $\alpha^{(j)}$ is of size $r_j > 0$ and we partition $\sigma = \beta^{(1)}\beta^{(2)}\cdots\beta^{(k)}$ where where each $\beta^{(j)}$ is of size $s_j > 0$ and $\mathrm{red}(\beta^{(1)}), \mathrm{red}(\beta^{(2)}), \ldots, \mathrm{red}(\beta^{(k)})$ are pairwise distinct.

$$\tau_{r_1,\beta^{(1)},r_2,\beta^{(2)},\ldots,r_k,\beta^{(k)},r_{k+1}} = \alpha^{(1)}\beta^{(1)}\alpha^{(2)}\beta^{(2)}\cdots\alpha^{(k)}\beta^{(k)}\alpha^{(k+1)}. \tag{8}$$

Minimal overlapping permutations

We say that a permutation $\tau \in S_j$ is a **minimal overlapping permutation** if the smallest n such that there is $\sigma \in S_n$ where $\tau\text{-mch}(\sigma) = 2$ is $2j - 1$.

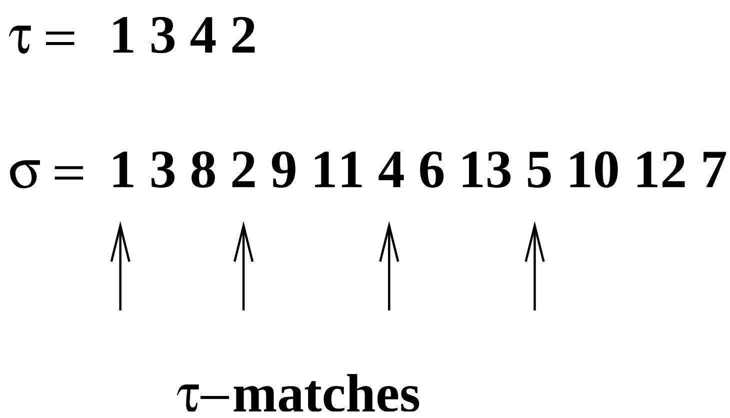
Examples.

$\tau_1 = 1234$, $\tau_2 = 1324$ are not minimal overlapping permutations.

$\tau_3 = 132$, $\tau_4 = 1243$, $\tau = 12\cdots(n-2)n(n-1)$ are minimal overlapping.

$\tau = 23154$ is the smallest minimal overlapping permutation such neither it, its reverse, its complement, or its reverse-complement starts with 1.

Maximum packings



We let $MP_{\tau,j+s(j-1)}$ be the number of $\sigma \in S_{j+s(j-1)}$ such that σ is a maximum packing for τ

Theorem 0.5. *Suppose that τ is a minimal overlapping permutation. Then*

$$\begin{aligned} P_\tau(x,t) &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} \qquad (9) \\ &= \frac{1}{1-(t+\sum_{s\geq 0} \frac{t^{j+s(j-1)}}{(j+s(j-1)!)}(x-1)^{s+1} MP_{\tau,j+s(j-1)})}. \end{aligned}$$

Theorem 0.6. *If $\alpha = \alpha_1 \ldots \alpha_j$ and $\beta = \beta_1 \ldots \beta_j$ are minimal overlapping permutations in S_j and $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$, then (1) for all $s \geq 0$, $MP_{\alpha,j+s(j-1)} = MP_{\beta,j+s(j-1)}$. and, hence,*

(2) $P_\alpha(x,t) = P_\beta(x,t)$.

This solves a conjecture of Elizalde.

This has also been proved independently by Vladimir Dotsenko and Anton Khoroshkin.

$\tau =$ **2 4 1 5 3**

$S_1 =$ {2,5,6,7,10} $S_2 =$ {1,8,9,15} $S_3 =$ {3,11,13,14} $S_4 =$ {4,12,16,17}

5	7	2	10	6												
5	7	2	10	6	9	1	15	8								
5	7	2	10	6	9	1	15	8	13	3	14	11				
5	7	2	10	6	9	1	15	8	13	3	14	11	16	4	17	12

We are interested on the expansions.

$$h_\mu(\overline{x}) = \sum_{\mu \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,\mu} e_\lambda(\overline{x})$$

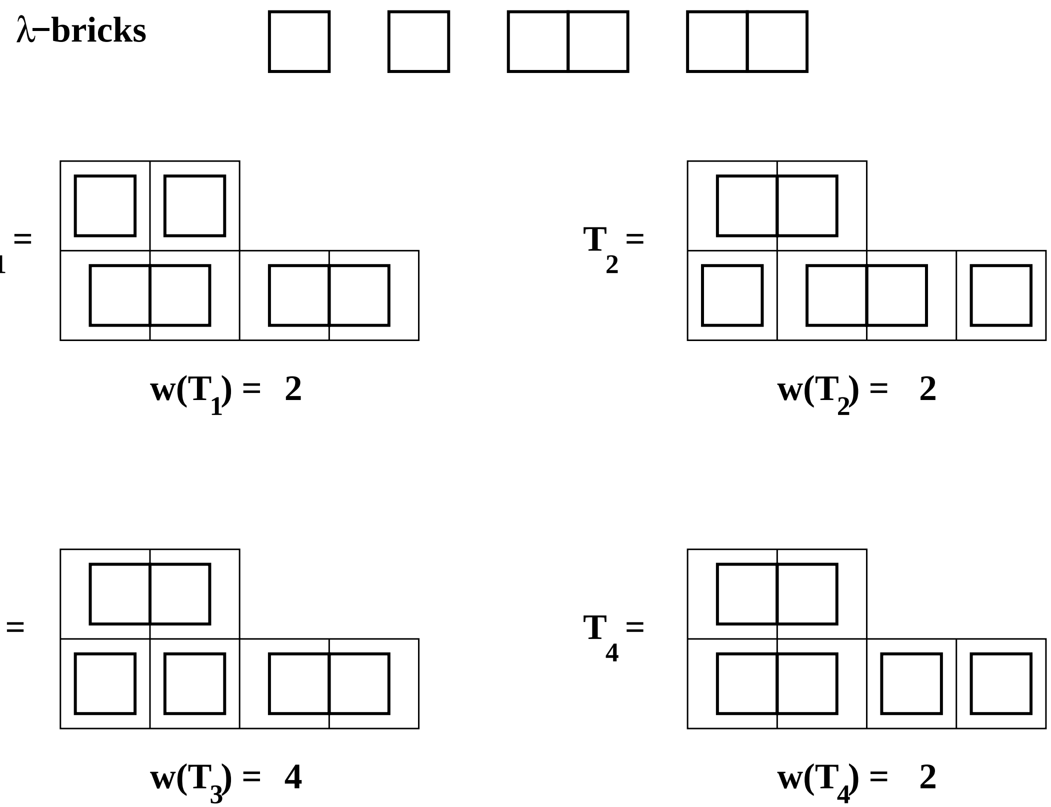
$$p_\mu(\overline{x}) = \sum_{\mu \vdash n} (-1)^{n-\ell(\lambda)} w(B)_{\lambda,\mu} e_\lambda(\overline{x})$$

(Egecioglu and Remmel 1991)

λ-Brick Tabloids and Weighted λ-Brick Tabloids.

Suppose that $\lambda = (1,1,2,2)$ and $\mu = (2,4)$.

$B_{\lambda,\mu} = 4$ and $w(B_{\lambda,\mu}) = 10$



Let $f_1 : \{0, 1, \ldots\} \to \mathbb{Q}[x]$ such that:

$$f_1(n) = \begin{cases} 1 & \text{if } n = 0, 1 \\ (x-1)^{s+1} MP_{\tau, j+s(j-1)} & \text{if } n = j + s(j-1) \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

Define $\xi^{f_1} : \Lambda \to \mathbb{Q}[x]$ as a homomorphism such that

$$\xi^{f_1}(e_n) = \frac{(-1)^{n-1}}{n!} f_1(n).$$

Theorem.

$$n! \xi^{f_1}(h_n) = \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)}.$$

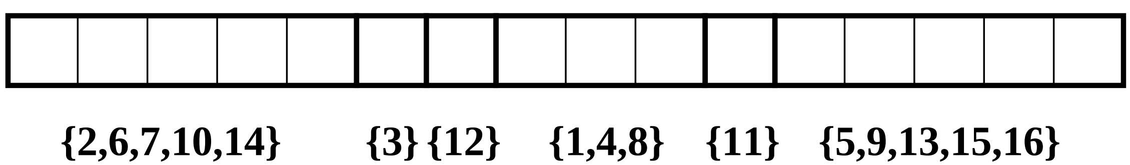
Proof.

$$
\begin{aligned}
n!\xi^{f_1}(h_n) &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \xi^{f_1}(e_\lambda) \\
&= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{\lambda_i - 1}}{\lambda_i!} f_1(\lambda_i) \\
&= \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} f_1(\lambda_1) \cdots f_1(\lambda_\ell).
\end{aligned}
$$

We have

$$n!\xi^{f_1}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$$

from which we create the following objects:



We have

$$n!\xi^{f_1}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$$

$$f_1(n) = \begin{cases} 1 & \text{if } n = 0, 1 \\ (x-1)^{s+1} MP_{\tau, j+s(j-1)} & \text{if } n = j + s(j-1) \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

X		−1					−1				X		X		

{2,6,7,10,14} {3} {12} {1,4,8} {11} {5,9,13,15,16}

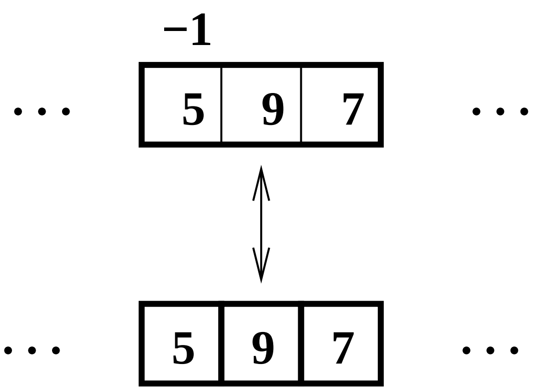
1 4 2 5 3 1 3 2 1 5 2 4 3

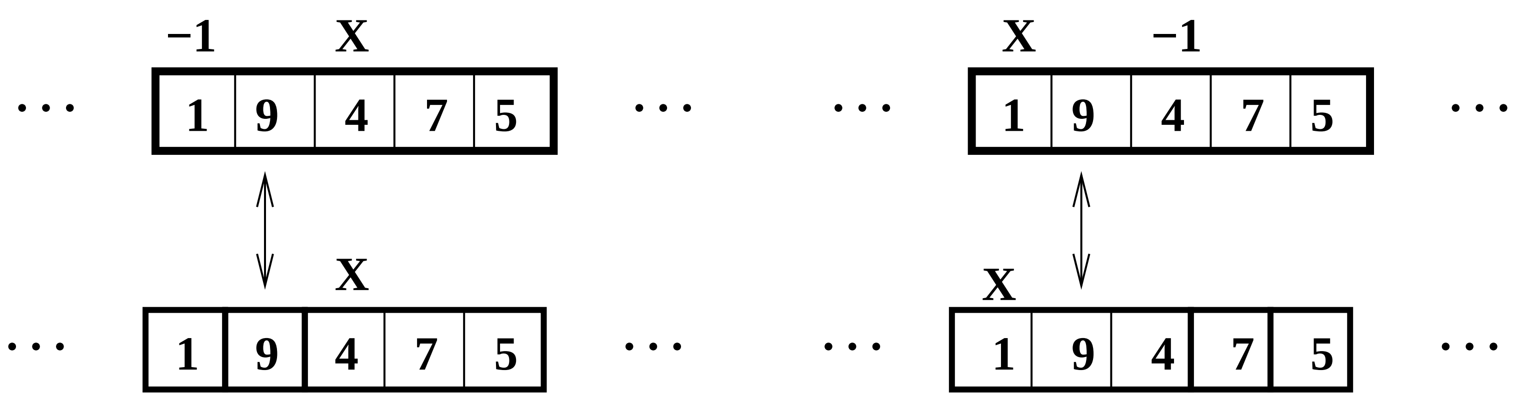
X		−1					−1				X		X		
2	10	6	14	7	3	12	1	8	4	11	5	16	9	15	13

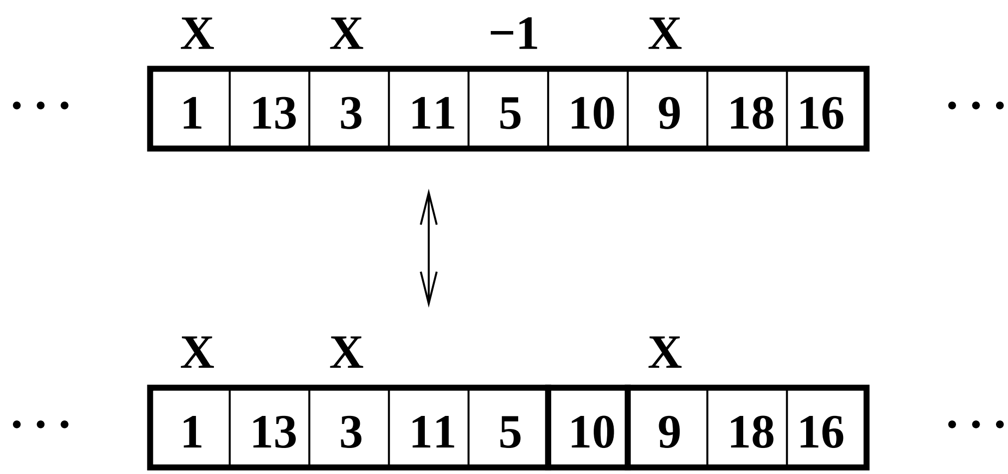
The weight of $T \in \mathcal{T}_{f_1}$, $w(T)$, is the product of x labels and the sign of $T \in \mathcal{T}_{f_1}$, $sgn(T)$ is the product of the -1 labels.

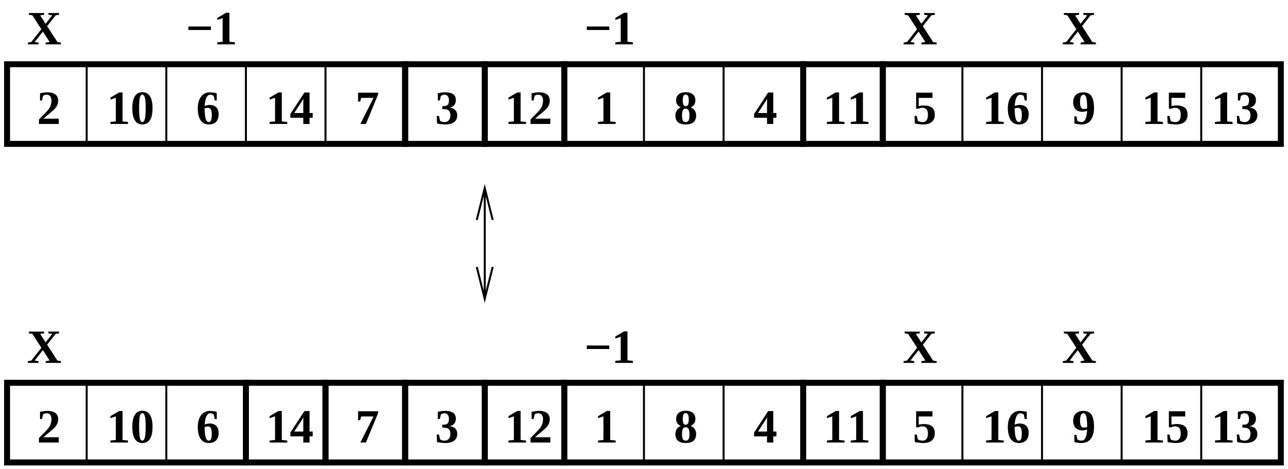
$$n!\xi^{f_1}(h_n) = \sum_{T \in \mathcal{T}_{f_1}} sgn(T)w(T).$$

Next we define an involution will rid us of all $T \in \mathcal{T}_{f_1}$ with negative weights.


−1
5 9 7
5 9 7


−1 X
1 9 4 7 5
X
1 9 4 7 5
X −1
1 9 4 7 5
X
1 9 4 7 5


X X −1 X
· · · 1 13 3 11 5 10 9 18 16 · · ·
X X X
· · · 1 13 3 11 5 10 9 18 16 · · ·


X −1 −1 X X
2 10 6 14 7 3 12 1 8 4 11 5 16 9 15 13
X −1 X X
2 10 6 14 7 3 12 1 8 4 11 5 16 9 15 13

What are the fixed points?

(1) no -1s.

(2) Each brick has only x's so that we are counting

$$x^{\text{no. of } \tau\text{-matches}}$$

in each brick

(3) there can be no τ-matches that are not part of a brick.

Thus

$$n!\xi^{f_1}(h_n) = \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)}.$$

This gives a generating function:

$$\begin{aligned}\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in S_n}x^{\tau\text{-mch}(\sigma)} &= \xi^{f_1}\left(\sum_{n\geq 0}h_n t^n\right)\\ &= \xi^{f_1}\left(\sum_{n\geq 0}e_n(-t)^n\right)^{-1}\\ &= \frac{1}{1-\left(t+\sum_{s\geq 0}\frac{t^{j+s(j-1)}}{(j+s(j-1))!}(x-1)^{s+1}MP_{\tau,j+s(j-1)}\right)}.\end{aligned}$$

Let

$$MP_{\tau,j+s(j-1)}(q) = \sum_{\sigma \in \mathcal{MP}_{\tau,j+s(j-1)}} q^{inv(\sigma)}.$$

Theorem 0.7.

$$\sum_{n\geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} q^{inv(\sigma)}$$

$$= \frac{1}{1-(t+\sum_{s\geq 0} \frac{t^{j+s(j-1)}}{[j+s(j-1)]_q!}(x-1)^{s+1} MP_{\tau,j+s(j-1)}(q))}.$$

Can we compute $MP_{\tau,j+s(j-1)}(q)$?

Answer: Yes for all minimal overlapping patterns that start with 1.

Suppose that $\tau = \tau_1 \ldots \tau_j$ where $\tau_1 = 1$ and $\tau_j = k$. Then

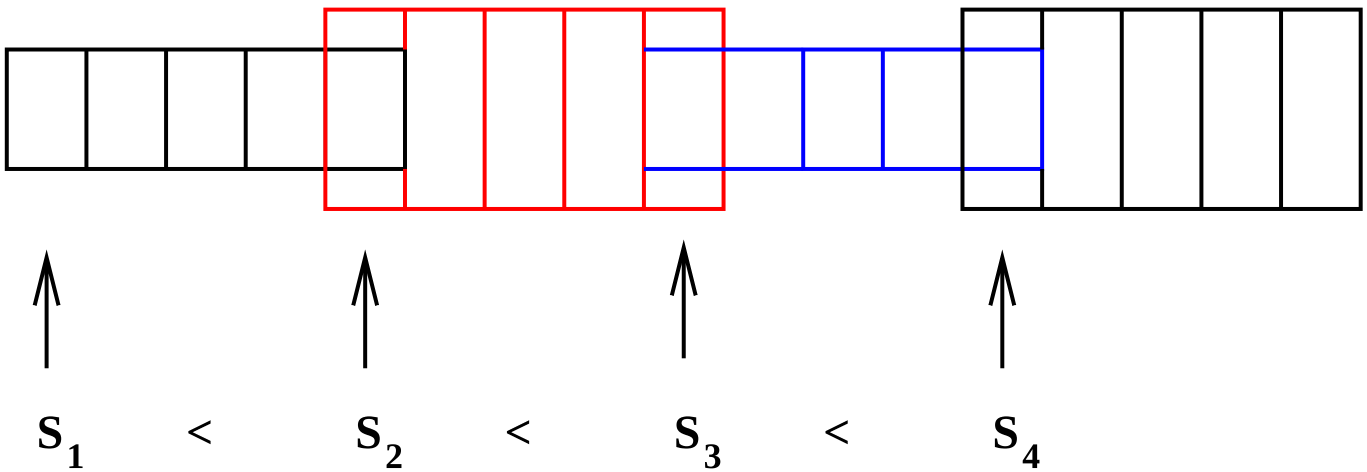
$$MP_{\tau,j+s(j-1)}(q) = q^{inv(\tau)}$$

and

$$\begin{aligned} &MP_{\tau,j+s(j-1)}(q) \\ &= q^{inv(\tau)} \begin{bmatrix} j+s(j-1)-k \\ j-k \end{bmatrix} MP_{\tau,j+(s-1)(j-1)}(q) \end{aligned}$$

so that for $s \geq 1$,

$$MP_{\tau,j+s(j-1)} = \left(q^{inv(\tau)}\right)^s \prod_{i=1}^{s} \begin{bmatrix} j+i(j-1)-k \\ j-k \end{bmatrix}.$$



(A) $S_1 = 1$

(B) $S_2 > k-1$

(C) $S_2 > k$ is impossible

(D) $S_2 = k$ and 1, 2, ... , k must be in the first black box

We have shown that for all minimal overlapping permutations $\alpha = \alpha_1 \ldots \alpha_j$ and $\beta = \beta_1 \ldots \beta_j$ that $P_\alpha(x,t) = P_\beta(x,t)$ if $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$.

Can we prove this bijectively?

Problem: α and β can overlap.

$$\alpha = 3\ 1\ 5\ 7\ 2\ 4\ 6$$
$$\beta = 3\ 5\ 1\ 2\ 4\ 7\ 6$$

Extensions.

(1) Words.

(2) You can use the idea of minimal overlapping patterns plus a bijection two find explicit generating functions for the number of occurrences of $0^{x_1}10^{x_2}1\dots10^{x_{k-1}}10^{x_k}$ in words in $\{0,1\}^*$ if $x_1, x_k > x_2, \dots, x_{k-1}$.

(3) patterns in cycle structures.

(4) $C_k \wr S_n$