

Minimal Overlapping Patterns

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Given a sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, let $\text{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i . For example, if $\sigma = 2\ 7\ 5\ 4$, then $\text{red}(\sigma) = 1\ 4\ 3\ 2$.

Given a permutation τ in the symmetric group S_j , define a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ to have a **τ -match at place i** provided $\text{red}(\sigma_i \cdots \sigma_{i+j-1}) = \tau$. Let **τ -mch**(σ) be the number of τ -matches in the permutation σ .

A permutation is called **τ -avoiding** if there are no indices $i_1 < \cdots < i_j$ such that $\text{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$.

τ -matches

Let $\tau\text{-mch}(\sigma)$ be the number of τ -matches in the permutation σ .

$$A_\tau(t) = \sum_{n \geq 0} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}| \text{ and} \quad (1)$$

$$P_\tau(x, t) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)}. \quad (2)$$

Theorem 0.1. *Goulden and Jackson (1983)*

If $\tau = 1\ 2 \cdots k$, then

$$A_\tau(t) = \frac{1}{\sum_{i \geq 0} \frac{x^{ki}}{(ki)!} - \frac{x^{ki+1}}{(ki+1)!}} \quad (3)$$

Theorem 0.2. *Elizalde and Noy (2003)*

$$P_{132}(t, u) = \frac{1}{1 - \int_0^t e^{\frac{(u-1)z^2}{2}} dz}, \text{ and}$$
$$P_{1342}(t, u) = \frac{1}{1 - \int_0^t e^{\frac{(u-1)z^3}{6}} dz}.$$

Theorem 0.3. *Kitaev (2005)*

Let $\tau = 12 \cdots a\sigma(a+1)$, where σ is a permutation of $\{a+2, a+3, \dots, k+1\}$, then

$$A_\tau(t) = \frac{1}{1 - t + \sum_{i \geq 1} \frac{(-1)^{i+1} x^{ki+1}}{(ki+1)!} \prod_{j=2}^i \binom{jk-a}{k-a}}. \quad (4)$$

c-Wilf equivalence and strong c-Wilf equivalence

Given permutations $\alpha, \beta \in S_n$, we say that α is **c-Wilf equivalent** to β if $A_\alpha(t) = A_\beta(t)$.

Given permutations $\alpha, \beta \in S_n$, we say that α is **strongly c-Wilf equivalent** to β if $P_\alpha(u, t) = P_\beta(u, t)$.

If $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$, we let

$\sigma^r = \sigma_n \cdots \sigma_2 \sigma_1$ be the reverse of σ and

$\sigma^c = (n+1-\sigma_1) (n+1-\sigma_2) \cdots (n+1-\sigma_n)$ denote the complement of σ .

It is easy to see that c-Wilf equivalence classes and strong c-Wilf equivalence classes are closed under complement and reverse.

Generating Functions for the maximum number of non-overlapping patterns

Let $\tau\text{-nlap}(\sigma)$ be the maximum number of non-overlapping τ -matches in σ where two τ -matches are said to overlap if they contain any of the same integers.

Kitaev's Theorem (2003)

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{A(t)}{(1-x) + x(1-t)A(t)} \quad (5)$$

where $A(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}|$.

Υ -matches

Suppose $\Upsilon \subseteq S_j$.

We say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has an Υ -match at place i provided $\text{red}(\sigma_i \cdots \sigma_{i+j-1}) \in \Upsilon$.

Let $\Upsilon\text{-mch}(\sigma)$ and $\Upsilon\text{-nlap}(\sigma)$ be the number of Υ -matches and non-overlapping Υ matches in σ , respectively.

Theorem 0.4. (*Mendes-Remmel*)

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\Upsilon - n \text{lap}(\sigma)} q^{\text{inv}(\sigma)} = \frac{A_q^{\Upsilon}(t)}{(1-x) + x(1-t)A_q^{\Upsilon}(t)} \quad (6)$$

where $A_q^{\Upsilon}(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n: \Upsilon\text{-mch}(\sigma)=0} q^{\text{inv}(\sigma)}$.

Mendes and Remmel (2006) developed a general method for computing $A_\tau(t)$.

They proved $A_\tau(t)$ is of the form

$$A_\tau(t) = \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\tau, ||w||=n} (-1)^{\bar{\ell}(w)} |\mathcal{P}_w^\tau|}. \quad (7)$$

where $||w|| = |\tau| + w_1 + \cdots + w_n$,

J_τ is a certain collection of words associated with τ , and

\mathcal{P}_w^τ is a certain collection of permutations $\sigma \in S_{||w||}$ if $w = w_1 \dots w_n$.

Liese and Remmel (2009) showed how to use the Mendes-Remmel result to compute $A_\tau(t)$ for various classes of permutations which are shuffles of the permutation $1\ 2\ \cdots\ n$ with an arbitrary permutation $\sigma = \sigma_1\sigma_2\ \cdots\ \sigma_m$ of the numbers $\{n + 1, \dots, n + m\}$.

Partition $1\ 2\ \cdots\ n$ into $\alpha^{(1)}\alpha^{(2)}\ \cdots\ \alpha^{(k+1)}$ where each $\alpha^{(j)}$ is of size $r_j > 0$ and we partition $\sigma = \beta^{(1)}\beta^{(2)}\ \cdots\ \beta^{(k)}$ where each $\beta^{(j)}$ is of size $s_j > 0$ and $\text{red}(\beta^{(1)}), \text{red}(\beta^{(2)}), \dots, \text{red}(\beta^{(k)})$ are pairwise distinct.

$$\tau_{r_1, \beta^{(1)}, r_2, \beta^{(2)}, \dots, r_k, \beta^{(k)}, r_{k+1}} = \alpha^{(1)}\beta^{(1)}\alpha^{(2)}\beta^{(2)}\ \cdots\ \alpha^{(k)}\beta^{(k)}\alpha^{(k+1)}. \quad (8)$$

Minimal overlapping permutations

We say that a permutation $\tau \in S_j$ is a **minimal overlapping permutation** if the smallest n such that there is $\sigma \in S_n$ where $\tau\text{-mch}(\sigma) = 2$ is $2j - 1$.

Examples.

$\tau_1 = 1234$, $\tau_2 = 1324$ are not minimal overlapping permutations.

$\tau_3 = 132$, $\tau_4 = 1243$, $\tau = 12 \cdots (n - 2)n(n - 1)$ are minimal overlapping.

$\tau = 23154$ is the smallest minimal overlapping permutation such neither it, its reverse, its complement, or its reverse-complement starts with 1.

Maximum packings

$$\tau = 1342$$

$$\sigma = 13829114613510127$$



τ -matches

We let $MP_{\tau, j+s(j-1)}$ be the number of $\sigma \in \mathcal{S}_{j+s(j-1)}$ such that σ is a maximum packing for τ

Theorem 0.5. *Suppose that τ is a minimal overlapping permutation. Then*

$$\begin{aligned}
 P_\tau(x, t) &= \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} \\
 &= \frac{1}{1 - (t + \sum_{s \geq 0} \frac{t^{j+s(j-1)}}{(j+s(j-1))!} (x-1)^{s+1} MP_{\tau, j+s(j-1)})}.
 \end{aligned} \tag{9}$$

Theorem 0.6. *If $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are minimal overlapping permutations in S_j and $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$, then*

(1) *for all $s \geq 0$, $MP_{\alpha, j+s(j-1)} = MP_{\beta, j+s(j-1)}$. and, hence,*

$$(2) P_\alpha(x, t) = P_\beta(x, t).$$

This solves a conjecture of Elizalde.

This has also been proved independently by Vladimir Dotsenko and Anton Khoroshkin.

$\tau = 24153$

$S_1 =$	$S_2 =$	$S_3 =$	$S_4 =$
$\{2,5,6,7,10\}$	$\{1,8,9,15\}$	$\{3,11,13,14\}$	$\{4,12,16,17\}$

5	7	2	10	6															
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5	7	2	10	6	9	1	15	8												
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5	7	2	10	6	9	1	15	8	13	3	14	11								
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5	7	2	10	6	9	1	15	8	13	3	14	11	16	4	17	12					
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We are interested on the expansions.

$$h_{\mu}(\bar{x}) = \sum_{\mu \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda, \mu} e_{\lambda}(\bar{x})$$

$$p_{\mu}(\bar{x}) = \sum_{\mu \vdash n} (-1)^{n-\ell(\lambda)} w(B)_{\lambda, \mu} e_{\lambda}(\bar{x})$$

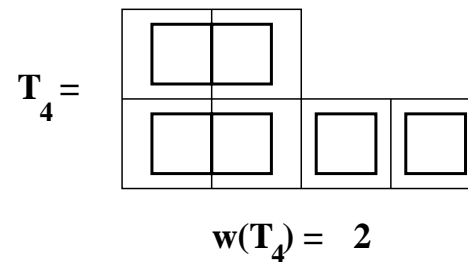
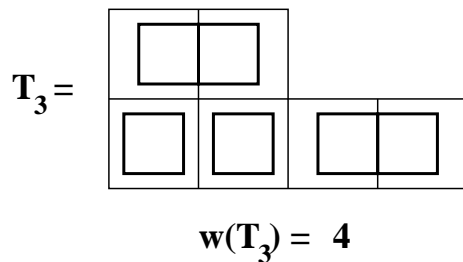
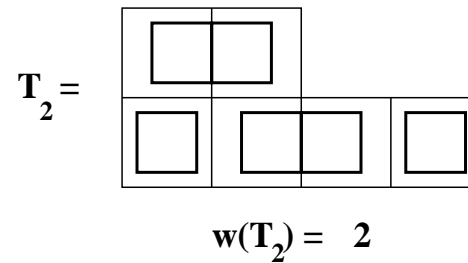
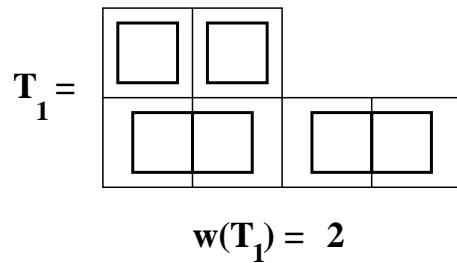
(Egecioglu and Remmel 1991)

λ -Brick Tabloids and Weighted λ -Brick Tabloids.

Suppose that $\lambda = (1, 1, 2, 2)$ and $\mu = (2, 4)$.

$$B_{\lambda, \mu} = 4 \text{ and } w(B_{\lambda, \mu}) = 10$$

λ -bricks 



Let $f_1 : \{0, 1, \dots\} \rightarrow \mathbb{Q}[x]$ such that:

$$f_1(n) = \begin{cases} 1 & \text{if } n = 0, 1 \\ (x-1)^{s+1} MP_{\tau, j+s(j-1)} & \text{if } n = j + s(j-1) \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

Define $\xi^{f_1} : \Lambda \rightarrow \mathbb{Q}[x]$ as a homomorphism such that

$$\xi^{f_1}(e_n) = \frac{(-1)^{n-1}}{n!} f_1(n).$$

Theorem.

$$n! \xi^{f_1}(h_n) = \sum_{\sigma \in S_n} x^{\tau - \text{mch}(\sigma)}.$$

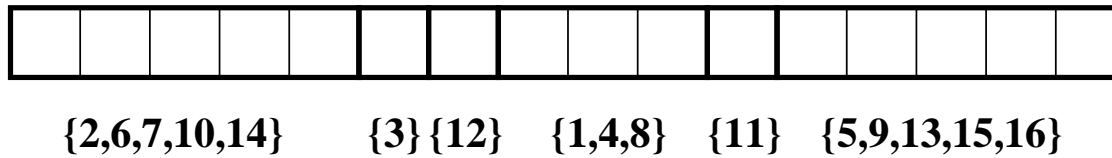
Proof.

$$\begin{aligned}
 n! \xi^{f_1}(h_n) &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \xi^{f_1}(e_\lambda) \\
 &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{\lambda_i-1}}{\lambda_i!} f_1(\lambda_i) \\
 &= \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \dots, \lambda_\ell} B_{\lambda,n} f_1(\lambda_1) \cdots f_1(\lambda_\ell).
 \end{aligned}$$

We have

$$n! \xi^{f_1}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \dots, \lambda_\ell} B_{\lambda, n} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$$

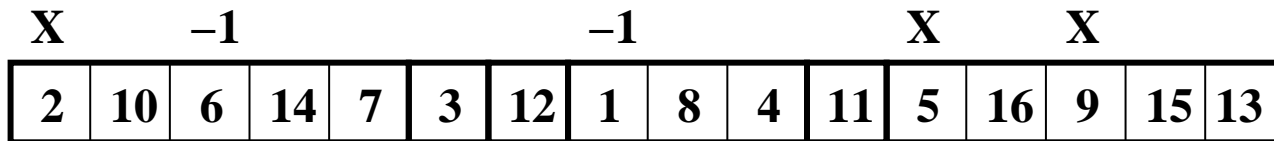
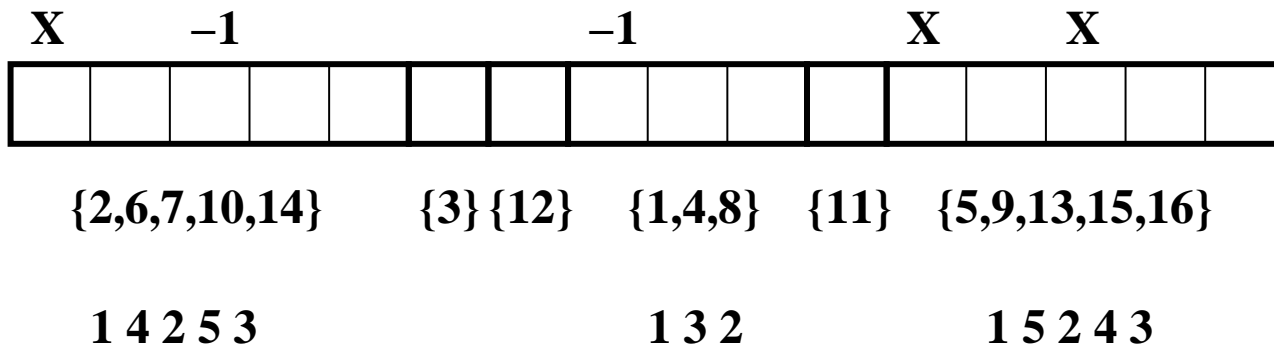
from which we create the following objects:



We have

$$n! \xi^{f_1}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \dots, \lambda_\ell} B_{\lambda, n} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$$

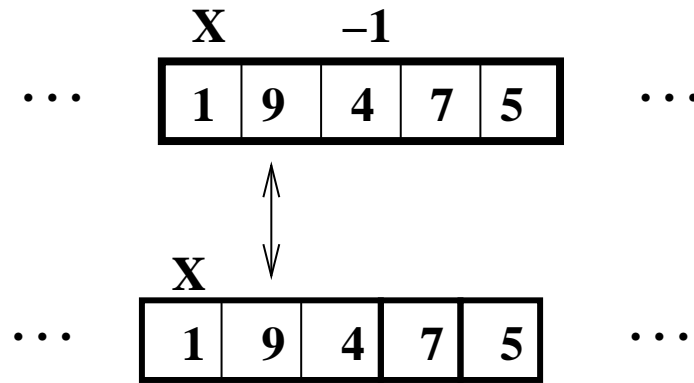
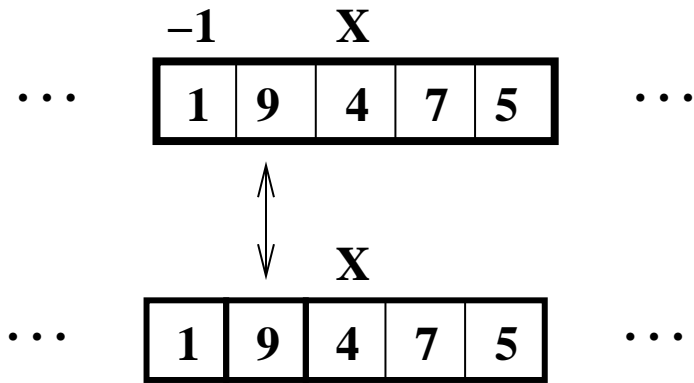
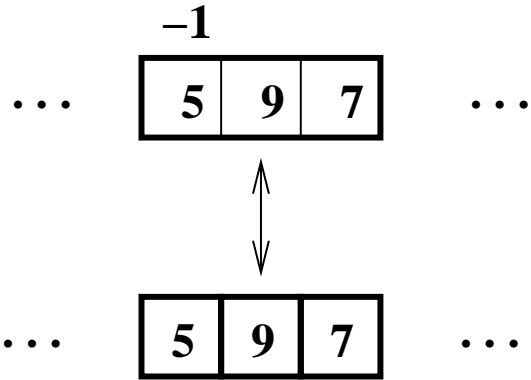
$$f_1(n) = \begin{cases} 1 & \text{if } n = 0, 1 \\ (x-1)^{s+1} MP_{\tau, j+s(j-1)} & \text{if } n = j + s(j-1) \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

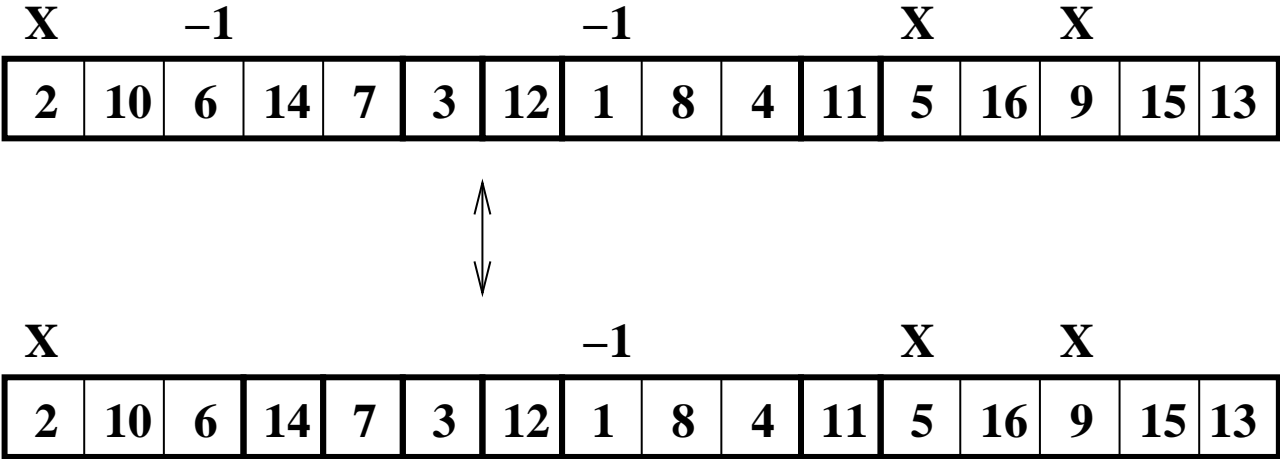


The weight of $T \in \mathcal{T}_{f_1}$, $w(T)$, is the product of x labels and the sign of $T \in \mathcal{T}_{f_1}$, $sgn(T)$ is the product of the -1 labels.

$$n! \xi^{f_1}(h_n) = \sum_{T \in \mathcal{T}_{f_1}} sgn(T) w(T).$$

Next we define an involution will rid us of all $T \in \mathcal{T}_{f_1}$ with negative weights.





What are the fixed points?

(1) no -1 s.

(2) Each brick has only x 's so that we are counting

$$x^{\text{no. of } \tau\text{-matches}}$$

in each brick

(3) there can be no τ -matches that are not part of a brick.

Thus

$$n! \xi^{f_1}(h_n) = \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)}.$$

This gives a generating function:

$$\begin{aligned}
\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau - \text{mch}(\sigma)} &= \xi^{f_1} \left(\sum_{n \geq 0} h_n t^n \right) \\
&= \xi^{f_1} \left(\sum_{n \geq 0} e_n (-t)^n \right)^{-1} \\
&= \frac{1}{1 - \left(t + \sum_{s \geq 0} \frac{t^{j+s(j-1)}}{(j+s(j-1))!} (x-1)^{s+1} MP_{\tau, j+s(j-1)} \right)}.
\end{aligned}$$

Let

$$MP_{\tau, j+s(j-1)}(q) = \sum_{\sigma \in \mathcal{MP}_{\tau, j+s(j-1)}} q^{\text{inv}(\sigma)}.$$

Theorem 0.7.

$$\begin{aligned} & \sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\tau - \text{mch}(\sigma)} q^{\text{inv}(\sigma)} \\ &= \frac{1}{1 - (t + \sum_{s \geq 0} \frac{t^{j+s(j-1)}}{[j+s(j-1)]_q!} (x-1)^{s+1} MP_{\tau, j+s(j-1)}(q))}. \end{aligned}$$

Can we compute $MP_{\tau, j+s(j-1)}(q)$?

Answer: Yes for all minimal overlapping patterns that start with 1.

Suppose that $\tau = \tau_1 \dots \tau_j$ where $\tau_1 = 1$ and $\tau_j = k$. Then

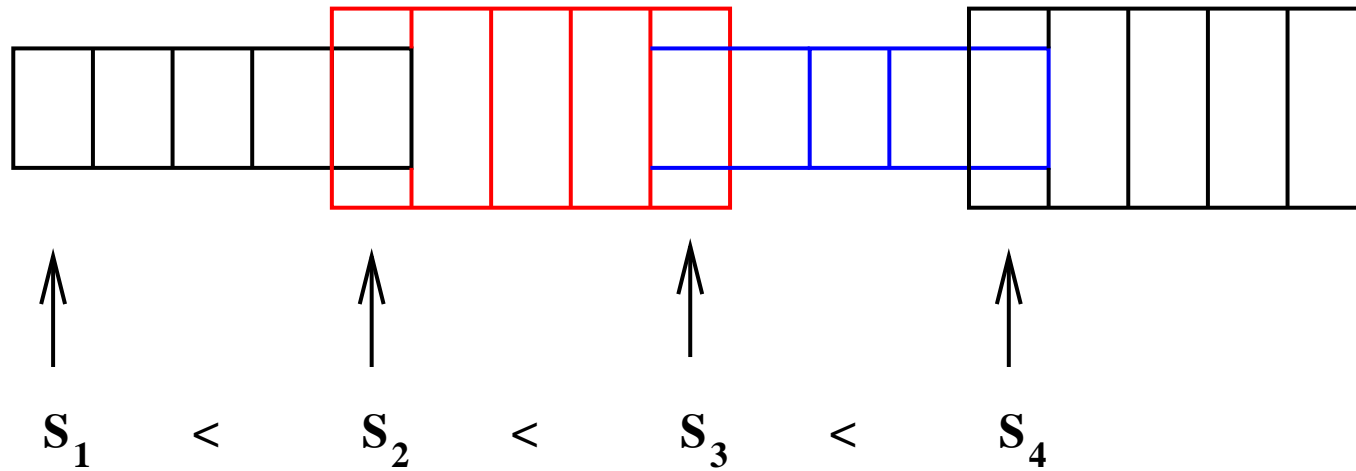
$$MP_{\tau, j+s(j-1)}(q) = q^{inv(\tau)}$$

and

$$\begin{aligned} & MP_{\tau, j+s(j-1)}(q) \\ &= q^{inv(\tau)} \begin{bmatrix} j + s(j-1) - k \\ j - k \end{bmatrix} MP_{\tau, j+(s-1)(j-1)}(q) \end{aligned}$$

so that for $s \geq 1$,

$$MP_{\tau, j+s(j-1)} = \left(q^{inv(\tau)} \right)^s \prod_{i=1}^s \begin{bmatrix} j + i(j-1) - k \\ j - k \end{bmatrix}.$$



(A) $S_1 = 1$

(B) $S_2 > k-1$

(C) $S_2 > k$ is impossible

(D) $S_2 = k$ and $1, 2, \dots, k$ must be in the first black box

We have shown that for all minimal overlapping permutations $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ that $P_\alpha(x, t) = P_\beta(x, t)$ if $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$.

Can we prove this bijectively?

Problem: α and β can overlap.

$$\alpha = 3 \ 1 \ 5 \ 7 \ 2 \ 4 \ 6$$

$$\beta = 3 \ 5 \ 1 \ 2 \ 4 \ 7 \ 6$$

Extensions.

(1) Words.

(2) You can use the idea of minimal overlapping patterns plus a bijection to find explicit generating functions for the number of occurrences of $0^{x_1} 10^{x_2} 1 \dots 10^{x_{k-1}} 10^{x_k}$ in words in $\{0, 1\}^*$ if $x_1, x_k > x_2, \dots, x_{k-1}$.

(3) patterns in cycle structures.

(4) $C_k \wr S_n$