Consecutive Patterns: From Permutations to Column-Convex Polyominoes and Back

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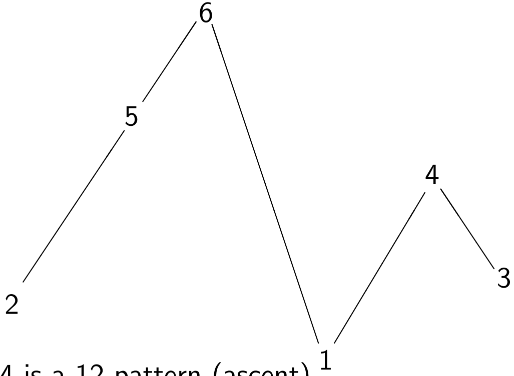
Main Result

$$\mathcal{PS} \subset \mathcal{PC} \subset \mathcal{PCCP} \subset \mathcal{PW}.$$

Remark: Our perspective allows powerful methods from the contexts of compositions, column-convex polyominoes, and of words to be applied directly to the enumeration of permutations by consecutive patterns.

Consecutive Patterns in Permutations

$\sigma = 2\,5\,6\,1\,4\,3 =$



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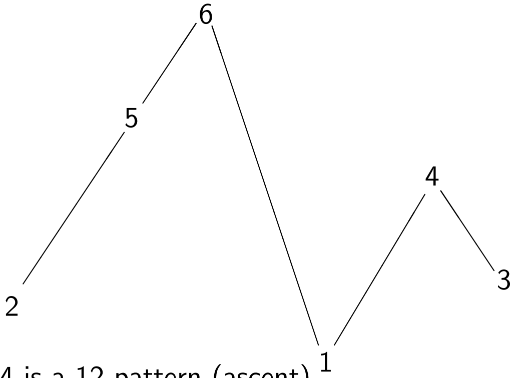

Examples: $\sigma_4\sigma_5 = 1\,4$ is a 12-pattern (ascent)
$\sigma_2\sigma_3\sigma_4 = 5\,6\,1$ is a 231-pattern

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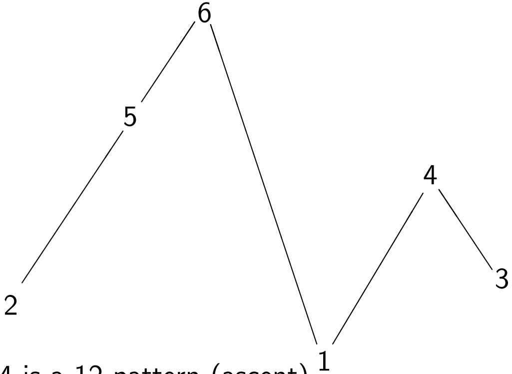

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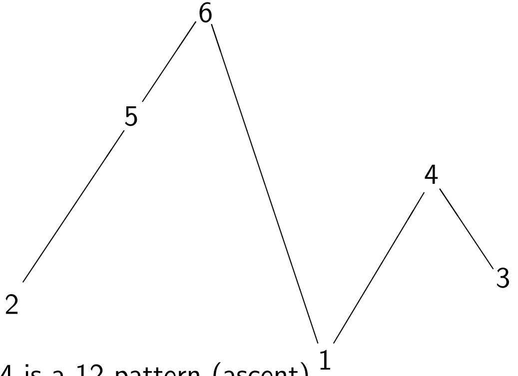

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Definition: For a pattern set $P \subseteq \bigcup_{m \geq 1} S_m$,

- $P(\sigma)$ = the total number of times elements of P occur in σ and
- $P_{no}(\sigma)$ = the maximum number of non-overlapping times elements of P occur in σ.

Consecutive Patterns in Compositions

Notation: $K_n = \{w = w_1 w_2 \ldots w_n : w_1, w_2, \ldots, w_n \text{ are positive integers}\}$

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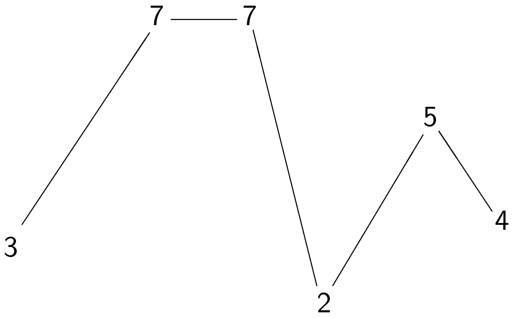
$w = 3\,7\,7\,2\,5\,4 =$



Examples: $w_2 w_3 = 7\,7$ is a level

$w_2 w_3 w_4 = 7\,7\,2$ is a peak ($w_2 \leq w_3 > w_4$)

Inverse of Fedou's Insertion-Shift Bijection

Definitions: For $\sigma \in S_n$, set $\mathrm{inv}_i \sigma = |\{k : i < k \leq n, \sigma_i > \sigma_k\}|$.
Put $\Lambda_n = \{w \in K_n : w_1 \leq w_2 \leq ... \leq w_n\}$.

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The inverse of Fedou's insertion-shift bijection $\nabla_n : S_n \times \Lambda_n \to K_n$ is given by the rule $\nabla_n(\sigma, \lambda) = w$ where $w_i = \mathrm{inv}_i\sigma + \lambda_{\sigma_i}$.

$$\nabla_6(2\,5\,6\,1\,4\,3,\ 2\,2\,4\,4\,4\,4) = 3\,7\,7\,2\,5\,4.$$

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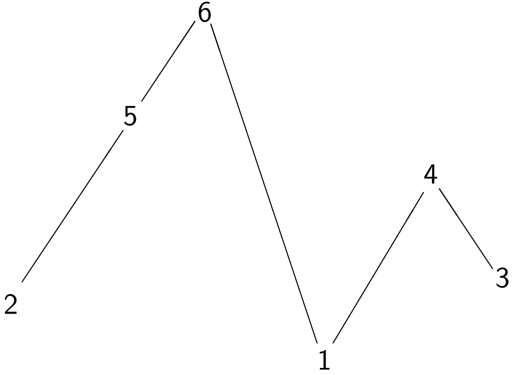

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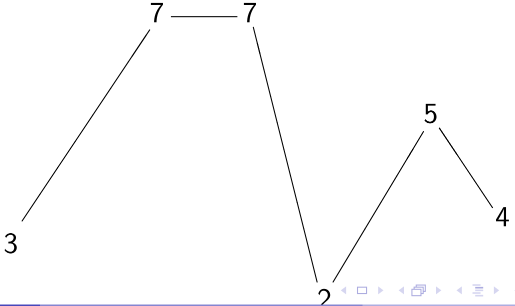

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Key Properties: If $\nabla_n(\sigma, \lambda) = w$, then

(1) $\mathrm{inv}\,\sigma + \mathrm{sum}\,\lambda = \mathrm{sum}\,w$ and, for $i < m$,

(2) $\sigma_i < \sigma_m$ if and only if $w_i \leq w_m + |\{j : i < j < m,\ \sigma_i > \sigma_j\}|$.

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Remark: ∇_n preserves general shape. As illustrations, (1) peaks are preserved as $\sigma_k < \sigma_{k+1} > \sigma_{k+2}$ if and only if $w_k \le w_{k+1} > w_{k+2}$ and (2) up-down permutations coincide with up-down compositions.

Definition: For $P \subseteq \bigcup_{m \ge 1} S_m$ and $w \in K_n$, set $P(w) = P(\sigma)$ where σ is the unique permutation satisfying $w = \nabla_n(\sigma, \lambda)$.

Theorem $\mathcal{PS} \subset \mathcal{PC}$

If $P \subseteq \bigcup_{m\geq 1} S_m$ and if $B_n \subseteq S_n$, then

$$\sum_{n\geq 0} \sum_{\sigma\in B_n} q^{\mathrm{inv}\,\sigma} \left(\prod_{p\in P} y_p^{p(\sigma)} \right) \frac{z^n}{(q;q)_n}$$
$$= \sum_{n\geq 0} \sum_{w\in\phi_n(B_n,\Lambda_n)} q^{\mathrm{sum\,w}} \left(\prod_{p\in P} y_p^{p(w)} \right) (z/q)^n.$$

Moreover, the above equality remains true if $y_p^{p(\sigma)}$ and $y_p^{p(w)}$ are respectively replaced by $y_p^{p_{no}(\sigma)}$ and $y_p^{p_{no}(w)}$ for some or all $p \in P$.

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$$\text{Gessel :} \quad \sum_{n \geq 0} \sum_{\sigma \in \mathrm{UD} S_n} \frac{q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \sec_q z + \tan_q z$$

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$$\text{Mendes, Remmel}: \sum_{n\geq 0} \sum_{\sigma \in S_n} \frac{y^{\mathrm{pic}\,(\sigma)} q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \frac{\sqrt{y-1}}{\sqrt{y-1} - \tan_q(z\sqrt{y-1})}$$

$$\text{Heubach, Mansour}: \sum_{n\geq 0} \sum_{w \in K_n} y^{\mathrm{pic}\,(w)} q^{\mathrm{sum}\,w} (z/q)^n = \frac{\sqrt{y-1}}{\sqrt{y-1} - \tan_q(z\sqrt{y-1})}$$

3^{rd} Ex of Theorem $\mathcal{PS} \subset \mathcal{PC}$: For $P \subseteq S_m$ with $m > 1$,

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} \frac{q^{\mathrm{inv}\,\sigma} y^{P_{no}(\sigma)} z^n}{(q;q)_n} = \frac{\mathcal{K}_q(z)}{1 - y + y\left(1 - z(1-q)^{-1}\right)\mathcal{K}_q(z)}$$

where $\mathcal{K}_q(z) = \sum_{n \geq 0} \left(\sum_{\sigma \in S_n} q^{\mathrm{inv}\,\sigma} 0^{P(\sigma)} \right) z^n/(q;q)_n$ is the q-exponential generating function for permutations that avoid P.

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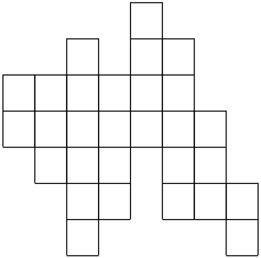
$$\sum_{n\geq 0} \sum_{w \in K_n} y^{P_{no}(w)} q^{\mathrm{sum}\,w} z^n = \frac{L_q(z)}{1 - y + y\left(1 - zq(1-q)^{-1}\right) L_q(z)}$$

where $L_q(z) = \sum_{n\geq 0} \left(\sum_{w \in K_n} q^{\mathrm{sum}\,w} 0^{P(w)}\right) z^n$ is the generating function for compositions that avoid P.

Remark: The latter is more and less general than a result due to Heubach, Kitaev, and Mansour; for a pattern set of cardinality 1, their result holds for an arbitrary alphabet of positive integers.

Consecutive Patterns in Column-Convex Polyominoes



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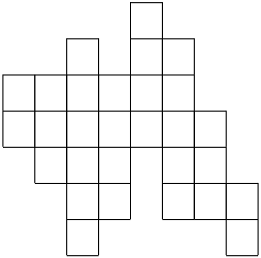

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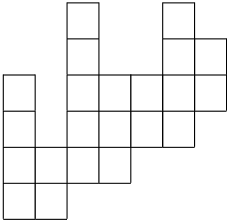
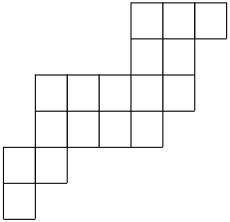
Remark 2: The simplest ridge patterns are formed between two adjacent columns. The two-column ridge patterns may be used to characterize many of the common classes of CCPs. For instance, a CCP with no lower descents is known as a directed column-convex polyomino (DCCP).

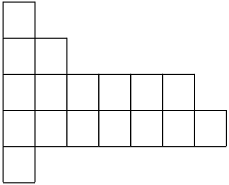
Common Classes of CCPs.



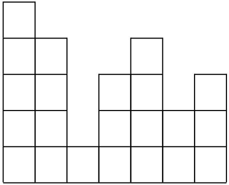
Directed Column–Convex Polyomino (DCCP): No lower descents



Parallelogram Polyomino: No lower or upper descents



Stack Polyomino: No upper ascents and no lower descents



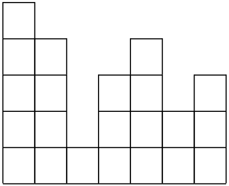
Wall Polyomino: No lower ascents and no lower descents

Compositions and Wall Polyominoes

Notation: Let WP_n= the set of wall polyominoes with n columns.

Bijection: Let γ_n denote the "natural" bijection from K_n to WP_n.

Example: γ_7 maps the composition $w = 5\,4\,1\,3\,4\,2\,3$ to



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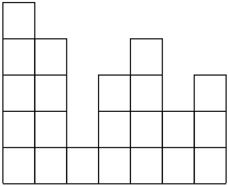

Properties: If $\gamma_n(w) = Q$, then

$$\mathrm{area}\, Q = \mathrm{sum}\, w \;\text{ and }\; \mathrm{per}\, Q - 2\mathrm{col}\, Q = \mathrm{var}\, w$$

where variation of w is $\mathrm{var}\, w = \sum_{k=0}^{n} |w_{k+1} - w_k|$ with the convention that $w_0 = 0 = w_{n+1}$.

Fact $\mathcal{PC} \subset \mathcal{PCCP}$

If P is a pattern set defined for compositions and if $B_n \subseteq K_n$, then

$$\sum_{n\geq 0}\sum_{w\in B_n} c^{\mathrm{var}\, w} q^{\mathrm{sum}\, w} z^n \prod_{p\in P} y_p^{p(w)} = \sum_{n\geq 0}\sum_{Q\in\gamma_n(B_n)} c^{\mathrm{per}\, Q} q^{\mathrm{area}\, Q} (z/c^2)^n \prod_{p\in P} y_p^{p(Q)}.$$

Example of Fact $\mathcal{PC} \subset \mathcal{PCCP}$

$$\mathrm{Raw, Tief}: \sum_{Q \in \mathrm{DCCP}} a_u^{\mathrm{uasc}\, Q} a_l^{\mathrm{lasc}\, Q} b_u^{\mathrm{ulev}\, Q} b_l^{\mathrm{llev}\, Q} c^{\mathrm{per}\, Q} d^{\mathrm{udes}\, Q} h^{\mathrm{relh}\, Q} q^{\mathrm{area}\, Q} z^{\mathrm{col}\, Q}$$

$$= \frac{c^2 h \sum_{n\geq 0} \frac{(c^2qz)^{n+1}}{1-c^2hq^{n+1}} \prod_{k=1}^{n} \left(b_l + \frac{a_l c^2 hq^k}{1-c^2hq^k} \right) \left(b_u + \frac{c^2dq^k}{1-c^2q^k} - \frac{a_u}{1-q^k} \right)}{1 - a_u \sum_{n\geq 1} \frac{(c^2qz)^n}{1-q^n} \prod_{k=1}^{n} \left(b_l + \frac{a_l c^2 hq^k}{1-c^2hq^k} \right) \prod_{k=1}^{n-1} \left(b_u + \frac{c^2dq^k}{1-c^2q^k} - \frac{a_u}{1-q^k} \right)}.$$

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Setting $a_l = 0$, $a_u = a$, $b_u = b$, $b_l = h = 1$, and replacing z by z/c^2 gives the gen func for compositions by ascents, levels, descents, and variation. ($c = 1$ is a classic result due to Carlitz)

$$\sum_{n\geq 0} \sum_{w \in K_n} a^{\text{asc}\,w} b^{\text{lev}\,w} d^{\text{des}\,w} c^{\text{var}\,w} q^{\text{sum}\,w} z^n$$

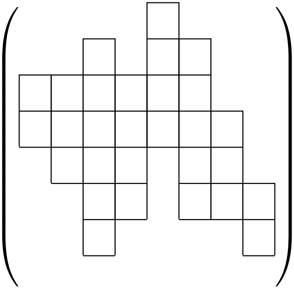
$$= 1 + \frac{c^2 \sum_{n\geq 0} \frac{(qz)^{n+1}}{1-c^2q^{n+1}} \prod_{k=1}^{n} \left(b + \frac{c^2dq^k}{1-c^2q^k} - \frac{a}{1-q^k} \right)}{1 - a \sum_{n\geq 1} \frac{(qz)^n}{1-q^n} \prod_{k=1}^{n-1} \left(b + \frac{c^2dq^k}{1-c^2q^k} - \frac{a}{1-q^k} \right)}.$$

The Inclusion $\mathcal{PCCP} \subset \mathcal{PW}$

Let $X = \left\{ \binom{j}{m} : j, m \text{ are positive integers} \right\}$ and let

$$\mathcal{Y} = \bigcup_{n \geq 0} \left\{ \begin{pmatrix} j_1 \; j_2 \; \cdots \; j_n \\ m_1 \; m_2 \ldots m_n \end{pmatrix} \in X^n : m_n = 1 \text{ and } j_k + j_{k+1} > m_k \text{ for } 1 \leq k < n \right\}.$$

For a CCP Q with n columns, define $\delta(Q) = \binom{j_1 \; j_2 \; \cdots \; j_n}{m_1 \; m_2 \ldots m_n}$ where j_k is the number cells in Q_k, $m_n = 1$, and, for $1 \leq k < n$, m_k is the change in the y-ordinate from the bottom edge of $(k+1)^{st}$ column of Q to the top edge of the k^{th} column of Q.

δ  $= \binom{2\,3\,6\,4\,4\,5\,3\,2}{3\,5\,5\,2\,6\,5\,4\,1}$

Factors and Consecutive Patterns in Words

Let X^* denote the free moniod generated by the alphabet X.

Definition: For $\mathcal{F} \subseteq X^+$, a factor f of a word w is a said to be a consecutive $\mathcal{F}$-pattern in w if $f \in \mathcal{F}$. The number of consecutive $\mathcal{F}$-patterns in w is denoted by $\mathcal{F}(w)$.

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Definition: For $\mathcal{F} \subset X^+$, an $\mathcal{F}$-cluster is a triple (w, ν, β) in which

$$w = w_1 w_2 \ldots w_{\mathrm{len}\, w} \in X^+,$$
$$\nu = (f_{(1)}, f_{(2)}, \ldots, f_{(k)}) \text{ for some } k \geq 1 \text{ with each } f_{(i)} \in \mathcal{F}, \text{ and}$$
$$\beta = (b_1, b_2, \ldots, b_k) \text{ with each } b_i \text{ being a positive integer}$$

where $f_{(i)} = w_{b_i} w_{b_i+1} \ldots w_{b_i+\mathrm{len}\, f_{(i)}-1}$, each $w_i w_{i+1}$ is a factor of some $f_{(j)}$, $b_1 \leq b_2 \leq \cdots \leq b_k$, and if $b_i = b_{i+1}$, then $\mathrm{len}\, f_{(i)} < \mathrm{len}\, f_{(i+1)}$.

Factors and Consecutive Patterns in Words

Let X^* denote the free moniod generated by the alphabet X.

Definition: For $\mathcal{F} \subseteq X^+$, a factor f of a word w is a said to be a consecutive $\mathcal{F}$-pattern in w if $f \in \mathcal{F}$. The number of consecutive $\mathcal{F}$-patterns in w is denoted by $\mathcal{F}(w)$.

Definition: For $\mathcal{F} \subset X^+$, an $\mathcal{F}$-cluster is a triple (w, ν, β) in which

$$w = w_1 w_2 \ldots w_{\operatorname{len} w} \in X^+,$$
$$\nu = (f_{(1)}, f_{(2)}, \ldots, f_{(k)}) \text{ for some } k \geq 1 \text{ with each } f_{(i)} \in \mathcal{F}, \text{ and}$$
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where $f_{(i)} = w_{b_i} w_{b_i+1} \ldots w_{b_i + \operatorname{len} f_{(i)} - 1}$, each $w_i w_{i+1}$ is a factor of some $f_{(j)}$, $b_1 \leq b_2 \leq \cdots \leq b_k$, and if $b_i = b_{i+1}$, then $\operatorname{len} f_{(i)} < \operatorname{len} f_{(i+1)}$.

Definition: The cluster generating function over a subset $W \subset X^*$ is the formal series

$$C_{\mathcal{F}}(\mathbf{y}, W) = \sum_{(w,\nu,\beta) \in C_{\mathcal{F}},\ w \in W} \left(\prod_{f \in \mathcal{F}} y_f^{f(\nu)} \right) w$$

Words by Factors Theorem (Goulden and Jackson)

If, for nonempty $L, R \subseteq X$ and a nonempty $\mathcal{F} \subseteq X^+$, we define

$$\mathcal{L}(\mathbf{y}) = \sum_{l \in L} l + C_{\mathcal{F}}(\mathbf{y}, LX^*), \quad \mathcal{R}(\mathbf{y}) = \sum_{r \in R} r + C_{\mathcal{F}}(\mathbf{y}, X^*R), \quad \text{and}$$

$$\mathcal{X}(\mathbf{y}) = \sum_{x \in X} x + C_{\mathcal{F}}(\mathbf{y}, X^*)$$

and if the result of replacing y_f in $\mathbf{y}$ by $y_f - 1$ is denoted by $\mathbf{y} - \mathbf{1}$, then

$$\sum_{w \in X^*} \left(\prod_{f \in \mathcal{F}} y_f^{f(w)} \right) w = (1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1},$$

$$\sum_{w \in LX^*} \left(\prod_{f \in \mathcal{F}} y_f^{f(w)} \right) w = \mathcal{L}(\mathbf{y} - \mathbf{1})(1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1},$$

$$\sum_{w \in X^*R} \left(\prod_{f \in \mathcal{F}} y_f^{f(w)} \right) w = (1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1} \mathcal{R}(\mathbf{y} - \mathbf{1}), \quad \text{and}$$

$$\sum \left(\prod y_f^{f(w)} \right) w = C_{\mathcal{F}}(\mathbf{y} - \mathbf{1}, LX^*R) + \mathcal{L}(\mathbf{y} - \mathbf{1})(1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1} \mathcal{R}(\mathbf{y}$$

Application of Words by Factors Theorem to Permutations

Consider the alphabet $N = \{1, 2, 3, \ldots\}$, let $P \subseteq \bigcup_{m \geq 1} S_m$, and put

$$D_P(\mathbf{y}; z) = \sum_{(w,\nu,\beta) \in C_{\mathcal{F}_P}} \left(\prod_{p \in P} y_p^{p(\nu)} \right) q^{\mathrm{sum}\, w} z^{\mathrm{len}\, w} \quad \text{where} \quad p(\nu) = \sum_{f \in \mathcal{F}_p} f(\nu).$$

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Replacement of w by $q^{\mathrm{sum}\, w} z^{\mathrm{len}\, w}$ in the first identity of the Words by Factors Theorem and application of Fedou's bijection implies

Extension of Rawlings' Theorem: If $P \subseteq \bigcup_{m\geq 1} S_m$, then

$$\sum_{n\geq 0} \sum_{\sigma\in S_n} \left(\prod_{p\in P} y_p^{p(\sigma)} \right) \frac{q^{\mathrm{inv}\, \sigma} z^n}{(q; q)_n} = \left(1 - z(1-q)^{-1} - D_P(\mathbf{y} - \mathbf{1}; z/q) \right)^{-1}.$$

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Non-overlapping Version: For $P \subseteq S_m$ with $m > 1$, then

$$\sum_{n\geq 0} \sum_{\sigma\in S_n} y^{P_{no}(\sigma)} \frac{q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - (1-y) D_P(-\mathbf{1})\right)^{-1}.$$

Extension of Rawlings' Theorem

If $P \subseteq \bigcup_{m \geq 1} S_m$, then

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\left(\prod_{p\in P} y_p^{p(\sigma)}\right)\frac{q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - D_P(\mathbf{y}-\mathbf{1}; z/q)\right)^{-1}.$$

Ex 1: Permutations by Peaks

Let $\mathrm{pic} = \{132, 231\}$. As the pic-clusters are in 1-to-1 correspondence with the up-down compositions of odd length > 1,

$$\frac{z}{1-q} + D_{\mathrm{pic}}(y; z/q) = \frac{1}{\sqrt{y}} \sum_{n\geq 0} \sum_{w \in \mathrm{UD}K_{2n+1}} q^{\mathrm{sum}\,w}\left(\frac{z\sqrt{y}}{q}\right)^{2n+1} = \frac{\tan_q(z\sqrt{y})}{\sqrt{y}}.$$

Thus,

$$\text{Mendes, Remmel}: \sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{y^{\mathrm{pic}(\sigma)} q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \frac{\sqrt{y-1}}{\sqrt{y-1} - \tan_q(z\sqrt{y-1})}$$

Ex 2: Permutations by Peaks and Twin Peaks

Let $\mathrm{tpic} = \{p \in S_5 : p_1 < p_2 > p_3 < p_4 > p_5\}$.

Result using extension of Rawlings' Theorem:

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{q^{\mathrm{inv}\,\sigma}x^{\mathrm{pic}\,(\sigma)}y^{\mathrm{tpic}\,(\sigma)}z^n}{(q;q)_n} = \frac{1}{1-\frac{z}{1-q}-\sum_{n\geq 1}A_n(x-1,y-1)B_n(q)(z/q)^{2n+1}}$$

where $A_n(x,y) = (xz+yz^2+xyz^2)(1-xz-xyz-xyz^2-yz-yz^2)^{-1}\big|_{z^n}$ and $B_n(q) = \tan_q z\big|_{z^{2n+1}}$.

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Alternative Result using Pattern Algebra of Goulden and Jackson:

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{q^{\mathrm{inv}\,\sigma}x^{\mathrm{pic}\,(\sigma)}y^{\mathrm{tpic}\,(\sigma)}z^n}{(q;q)_n}$$

$$= \left(1 - \frac{s_+\sin_q(z\sqrt{r_+})}{2\sqrt{r_+}\cos_q(z\sqrt{r_+})} - \frac{s_-\sin_q(z\sqrt{r_-})}{2\sqrt{r_-}\cos_q(z\sqrt{r_-})}\right)^{-1}$$

where $r_\pm = (xy-1\pm\sqrt{D})/2$, $s_\pm = 1\pm(2x-xy-1)/\sqrt{D}$, and $D = (xy+1)^2-4x$.

q-Olivier functions

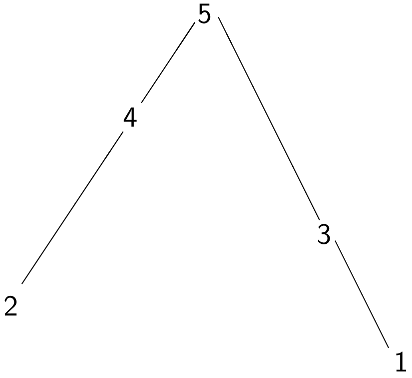
$$\Phi_{j,k}(z) = \sum_{n \geq 0} \frac{z^{jn+k}}{(q;q)_{jn+k}}$$

Examples: $\Phi_{1,0}(z) = e_q(z)$, $\quad \Phi_{2,0}(iz) = \cos_q z$, $\quad$ and $\Phi_{2,1}(iz) = i \sin_q z$.

Ex 3: Permutations by (i,d)-peaks and Inversions

Let $P_{i,d} = \{p \in S_{i+d-1} : p_1 < p_2 < \cdots < p_i > p_{i+1} > \cdots > p_{i+d-1}\}$.

Example of a (3,3)-peak:

$p = 2\,4\,5\,3\,1 =$ 

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Ex 3: Permutations by (i,d)-peaks and Inversions

If, for $i, j, d \geq 2$, we set $\mu = i + d - 2$ and $\xi_m = \sqrt[m]{-1}$, then

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} \frac{y^{P_{i,d}(\sigma)} q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - \frac{K_{i,i,d;1}(\sqrt[\mu]{y-1}\, z)}{\sqrt[\mu]{y-1}}\right)^{-1}$$

where, for $k \geq 1$,

$$K_{i,j,d;k}(z) = \sum_{m \geq 0} \sum_{w \in K_{i,d;(j,d)^m;k}} q^{\mathrm{sum}\, w} z^{\mathrm{len}\, w}.$$

Ex 3: Permutations by (i,d)-peaks and Inversions

If, for $i, j, d \geq 2$, we set $\mu = i + d - 2$ and $\xi_m = \sqrt[m]{-1}$, then

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where, for $k \geq 1$,

$$K_{i,j,d;k}(z) = \sum_{m\geq 0}\sum_{w\in K_{i,d;(j,d)^m;k}} q^{\mathrm{sum}\,w} z^{\mathrm{len}\,w}.$$

Moreover, $K_{i,j,d;k}(z)$ satisfies, for $d \geq 3$ and $\nu = j + d - 2$, the recurrence

$$K_{i,j,d;k}(z) = \frac{\xi_\nu^{-\mu}K_{i,j+1,d-1;1}(\xi_\nu z)\big(z^k(q;q)_k^{-1} + \xi_\nu^{-k}K_{j,j+1,d-1;k+1}(\xi_\nu z)\big)}{1 + K_{j,j+1,d-1;1}(\xi_\nu z)}$$

$$- \xi_\nu^{-\mu-k}K_{i,j+1,d-1;k+1}(\xi_\nu z)$$

with the initial condition

$$K_{i,j,2;k}(z) = \xi_j^{-i-k}\left[-\Phi_{j,i+k}(\xi_j z) + \Phi_{j,i}(\xi_j z)\Phi_{j,k}(\xi_j z)/\Phi_{j,0}(\xi_j z)\right].$$

Permutations by (3,3)-peaks and Inversions

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{y^{P_{3,3}(\sigma)}q^{\mathrm{inv}\,\sigma}z^n}{(q;q)_n}=\left(1-z(1-q)^{-1}-\frac{K_{3,3,3;1}(\sqrt[4]{y-1}\,z)}{\sqrt[4]{y-1}}\right)^{-1}$$

where

$$K_{3,3,3;1}(z)=\frac{-K_{3,4,2;1}(\xi_4 z)\left(z(1-q)^{-1}+\xi_4^{-1}K_{3,4,2;2}(\xi_4 z)\right)}{1+K_{3,4,2;1}(\xi_4 z)}$$

$$+\xi_4^{-1}K_{3,4,2;2}(\xi_4 z)$$

with

$$K_{3,4,2;1}(z)=-\frac{\Phi_{4,3}(\xi_4 z)\Phi_{4,1}(\xi_4 z)}{\Phi_{4,0}(\xi_4 z)}+\Phi_{4,4}(\xi_4 z)\quad\text{and}$$

$$K_{3,4,2;2}(z)=\xi_4^{-1}\left[-\frac{\Phi_{4,3}(\xi_4 z)\Phi_{4,2}(\xi_4 z)}{\Phi_{4,0}(\xi_4 z)}+\Phi_{4,5}(\xi_4 z)\right].$$

Ex 4: Permutations by m-Peak Ranges of (i,d)-Peaks

Corollary

If $i, d \geq 2$, $m \geq 1$, and $\nu = i + d - 2$, then the generating function for permutations by uniform m-peak ranges and inversions is

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{y^{P_{(i,d)^m}(\sigma)}q^{\mathrm{inv}\,\sigma}z^n}{(q;q)_n}=\left(1-\frac{z}{1-q}-\sum_{n\geq m}A_{n,m}(y-1)B_n(q)z^{n\nu+1}\right)^{-1}$$

where

$$A_{n,m}(y)=\left.\frac{yz^m(1-z)}{1-z-yz(1-z^m)}\right|_{z^n}\quad\text{and}\quad B_n(q)=K_{i,i,d;1}(z)|_{z^{n\nu+1}}$$

with $K_{i,i,d;1}(z)$ as determined earlier.

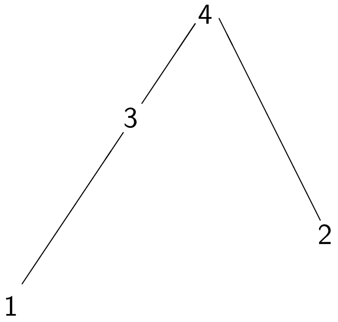
Remark: For $i = d = 2$ with $y = 0$, the above provides a solution to a problem posed by Kitaev of counting permutations that avoid $(2m+1)$-reverse-alternating patterns.

Ex 5: Permutations by (i,m)-Maxima and Inversions

Let $p_{(m)} \in S_{i+1}$ with $p_{(m)1} < p_{(m)2} < \cdots < p_{(m)i}$ and $p_{(m)i+1} = i + 1 - m$.

Example: The (3,2)-maxima pattern in S_{3+1}:

$p_{(2)} = 1\,3\,4\,2 =$



Remark: Carlitz and Scoville referred to $(2, 1)$ and $(2, 2)$-maxima as rising and falling maxima.

Ex 5: Permutations by (i,m)-Maxima and Inversions

Corollary (Words by Factors via extension of Rawlings' Theorem): If $i \geq 2$, $1 \leq m \leq i$ and $\xi_i = \sqrt[i]{-1}$, then

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} \left(\prod_{m=1}^{i} y_m^{p_{(m)}(\sigma)} \right) \frac{q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \left(1 - \frac{\Phi_{i,1}(\mathbf{y}-\mathbf{1}; \xi_i z)}{\xi_i \Phi_{i,0}(\mathbf{y}-\mathbf{1}; \xi_i z)} \right)^{-1} \text{where}$$

$$\Phi_{i,k}(y_1, \ldots, y_i; z) = \sum_{n \geq 0} \frac{z^{in+k}}{(q;q)_{in+k}} \prod_{j=0}^{n-1} \left(y_i + \sum_{m=1}^{i-1} (y_i - y_m) q^m \begin{bmatrix} ij+k+m-1 \\ m \end{bmatrix} \right).$$

For $i = 2$, $y_1 = y$, and $y_2 = 1$, above gives Mendes and Remmel's q-analog of Elizalde and Noy's result for permutations by $p = 132$:

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} \frac{y^{132(\sigma)} q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \left(1 - \sum_{n \geq 0} \frac{(y-1)^n q^n z^{2n+1}}{(q^2;q^2)_n (1-q^{2n+1})(1-q)^n} \right)^{-1}.$$

An Aside

Proof of the generating function for permutations by (i, m)-maxima of the previous slide reveals the generating function for up-down permutations of type $(i, i, 2; 1)$ by (i, m)-maxima:

$$\frac{z}{1-q} + \sum_{\sigma \in \mathrm{UD}S_{i,i,2;1}} \left(\prod_{m=1}^{i} y_m^{p_{(m)}} \right) \frac{q^{\mathrm{inv}\,\sigma} z^{\mathrm{len}\,\sigma}}{(q;q)_{\mathrm{inv}\,\sigma}} = \frac{\Phi_{i,1}(y_1, \ldots, y_{i-1}, 1; \xi_i z)}{\xi_i \Phi_{i,0}(y_1, \ldots, y_{i-1}, 1; \xi_i z)}.$$

Setting $y_1 = y_2 = \ldots = y_i = 1$, replacing z by $(1-q)z$, and letting $q \to 1$ gives a result of Carlitz's .

Ex 5: Perms by (i,m)-Maxima using the Temperley Method

If $i \geq 2$ and $1 \leq m \leq i$, then the generating function for permutations by (i,m)-maxima and inversions is also given by

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\left(\prod_{m=1}^{i} y_m^{p_{(m)}(\sigma)}\right)\frac{q^{\mathrm{inv}\,\sigma}z^n}{(q;q)_n}$$

$$=\left(1-\frac{\displaystyle\sum_{n\geq 0}\frac{z^{in+1}}{1-q^{in+1}}\prod_{k=0}^{n-1}T(q^{ik})}{1-\displaystyle\frac{y_i-1}{(q;q)_{i-1}}\sum_{n\geq 1}\frac{z^{in}}{1-q^{in}}\prod_{k=1}^{n-1}T(q^{ik-1})}\right)^{-1}$$

where

$$T(b)=\sum_{m=1}^{i-1}\frac{(y_m-1)q^m}{(q;q)_m(q^{m+1}b;q)_{i-m}}-\frac{y_i-1}{(q;q)_{i-1}(1-qb)}.$$

Ex 6: Permutations by maximal number of non-overlapping $P = \{1243, 1342, 1432, 2341, 2431, 3421\}$

Corollary

$$\sum_{n\geq 0}\sum_{\sigma\in S_n} \frac{y^{P_{no}(\sigma)} q^{\mathrm{inv}\,\sigma} z^n}{(q;q)_n} = \frac{\mathcal{K}}{1-y+y(1-\frac{z}{1-q})\mathcal{K}}$$

where

$$\mathcal{K} = \frac{\begin{array}{r} e_q(z)e_q(-z) + \cos_q^2 z + \sin_q^2 z + 2e_q(-z)\cos_q z \\ +(e_q(z)+e_q(-z))\sin_q z \end{array}}{4e_q(-z)\cos_q z}.$$

Remark: $\mathcal{K}$ is a q-analog of Kitaev's generating function that enumerates permutations that avoid $P = \{4312, 4213, 4123, 3214, 3124, 2134\}$.

Ex 7: DCCPs by All Five Two-Column Statistics

$$\text{Raw, Tief}: \sum_{Q\in\text{DCCP}} a_u^{\text{uasc}\,Q} a_l^{\text{lasc}\,Q} b_u^{\text{ulev}\,Q} b_l^{\text{llev}\,Q} c^{\text{per}\,Q} d^{\text{udes}\,Q} h^{\text{relh}\,Q} q^{\text{area}\,Q} z^{\text{col}\,Q}$$

$$= \frac{c^2 h \sum_{n\geq 0} \frac{(c^2qz)^{n+1}}{1-c^2hq^{n+1}} \prod_{k=1}^{n} \left(b_l + \frac{a_l c^2 hq^k}{1-c^2hq^k}\right)\left(b_u + \frac{c^2dq^k}{1-c^2q^k} - \frac{a_u}{1-q^k}\right)}{1 - a_u \sum_{n\geq 1} \frac{(c^2qz)^n}{1-q^n} \prod_{k=1}^{n} \left(b_l + \frac{a_l c^2 hq^k}{1-c^2hq^k}\right) \prod_{k=1}^{n-1} \left(b_u + \frac{c^2dq^k}{1-c^2q^k} - \frac{a_u}{1-q^k}\right)}.$$

Ex 8: CCPs by All Six Two-Column Statistics (Temperley)

If we set $F(x)$ equal to

$$\sum_{Q\in\mathrm{CCP}} a_u^{\mathrm{uasc}\,Q} b_u^{\mathrm{ulev}\,Q} d_u^{\mathrm{udes}\,Q} a_l^{\mathrm{lasc}\,Q} b_l^{\mathrm{llev}\,Q} d_l^{\mathrm{ldes}\,Q} c^{\mathrm{per}\,Q} q^{\mathrm{area}\,Q} h^{\mathrm{relh}\,Q} x^{\alpha(Q)} z^{\mathrm{col}\,Q}$$

where $\alpha(Q)$ denotes the area of the last column in Q, then

$$F(x) = \frac{\begin{vmatrix} R(x) & S(x) & T(x) \\ R(1) & S(1)-1 & T(1) \\ R(\frac{1}{h}) & S(\frac{1}{h}) & T(\frac{1}{h})-1 \end{vmatrix}}{\begin{vmatrix} S(1)-1 & T(1) \\ S(\frac{1}{h}) & T(\frac{1}{h})-1 \end{vmatrix}},$$

where

$$\begin{aligned} R(x) &= \sum_{n\geq 0} z^{n+1} y(x)y(qx)\ldots y(q^{n-1}x)r(q^n x), \\ S(x) &= \sum_{n\geq 0} z^{n+1} y(x)y(qx)\ldots y(q^{n-1}x)s(q^n x), \\ T(x) &= \sum z^{n+1} y(x)y(qx)\ldots y(q^{n-1}x)t(q^n x), \text{ and} \ldots \end{aligned}$$

Ex 8 Continued : The Rest of the Formula

$$\begin{aligned}
r(x) &= \frac{qxc^4h}{1-qxc^2h}, \\
s(x) &= \frac{q^2x^2c^4ha_ua_l}{(1-qx)(1-qxc^2h)} + \frac{qxc^2a_lb_u}{1-qx} - \frac{qxc^2d_ua_l}{(1-h)(1-qx)}, \\
t(x) &= \frac{qxc^2hd_ub_l}{1-qxh} + \frac{q^2x^2c^4hd_ud_l}{(1-qxh)(1-qxc^2)} + \frac{qxc^2h^2d_ua_l}{(1-h)(1-qxh)}, \\
y(x) &= \frac{qxc^4ha_ub_l}{1-qxc^2h} + \frac{q^2x^2c^6ha_ud_l}{(1-qxc^2)(1-qxc^2h)} - \frac{qxc^4ha_ua_l}{(1-qx)(1-qxc^2h)} \\
&\quad + c^2b_ub_l + \frac{qxc^4b_ud_l}{1-qxc^2} - \frac{c^2b_ua_l}{1-qx} - \frac{c^2d_ub_l}{1-qxh} \\
&\quad - \frac{qxc^4d_ud_l}{(1-qxh)(1-qxc^2)} + \frac{c^2d_ua_l}{(1-qx)(1-qxh)}.
\end{aligned}$$

Ex 9: DCCPs by Upper Valleys

A column-segment $Q_k Q_{k+1} Q_{k+2}$ in a column-convex polyomino Q is said to be a valley provided that $Q_k Q_{k+1}$ is an upper descent and $Q_{k+1} Q_{k+2}$ is an upper ascent or an upper level.

Corollary of Words by Factors

$$\sum_{Q \in \mathrm{DCCP}} y^{\mathrm{val}(Q)} q^{\mathrm{area}\, Q} z^{\mathrm{col}\, Q} = \frac{\displaystyle\sum_{n \geq 0} \frac{(1-y)^n q^{(n+1)(2n+1)} z^{2n+1}}{(q;q)_{2n+1}(q;q)_{2n}}}{\displaystyle\sum_{n \geq 0} \frac{(1-y)^n q^{n(2n+1)} z^{2n}}{(q;q)_{2n}^2} - \sum_{n \geq 0} \frac{(1-y)^n q^{(n+1)(2n+1)} z^{2n+1}}{(q;q)_{2n+1}^2}}.$$

Ex 10: CCPs by Peaks, Area, and Column Number (Temperley)

$$\sum_{Q\in \mathrm{CCP}} y^{\mathrm{pic}\,(Q)} q^{\mathrm{area}\,Q} z^{\mathrm{col}\,Q} = \frac{\left(\frac{zq}{1-q} + \frac{2z^2q^3}{(1-q)^3}\right)\left(1 + \frac{2zq}{(1-q)^2}\right)}{\left(1 - \frac{zq^2}{(1-q)^2}\right)\left(1 + \frac{zq}{(1-q)^2}\right) - \frac{2yz^2q^3}{(1-q)^4}}.$$