

Pattern Avoiding Colored Partitions

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- 1 History and Definitions
- 2 Colored Partitions and Avoidance
- 3 A Flavor of the Proofs
- 4 Summary and Future Ideas

Outline

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Who and What

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- The notion of signed set partitions was considered by Anders Björner and Michelle Wachs [1] from a poset and homological perspective.
- Now, we consider Pattern Avoidance in Colored Set Partitions. (Boom?)

Set Partition Definition

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Example

$$137/25/46 \vdash [7]$$

Canonical Words

Definition

*Given any word $w \in [k]^n$ we may **canonicalize** w by replacing all occurrences of the first letter of w by 1, all occurrences of the next occurring letter by 2, etc.*

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Definition

Given any word $w \in [k]^n$ we may *canonicalize* w by replacing all occurrences of the first letter of w by 1, all occurrences of the next occurring letter by 2, etc.

Example

The canonicalized form of 47411477 is 12133122.

There is a bijection between all canonicalized words of length n and partitions of $[n]$.

Canonical Words

To each set partition is associated a canonical word $a_1 a_2 \dots a_n$ where $a_i = j$ if $i \in B_j$.

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$137/25/46$ corresponds to 1213231.

We will say that a partition, π is of length n , $\ell(\pi) = n$, if its associated canonical word has n letters.

From now on we will refer to these canonical words as partitions.

Pattern Avoidance

Definition

Let σ be a partition with $\ell(\sigma) = n$ and π be a partition with $\ell(\pi) = k$. We say that σ **contains** π if there is a subsequence of σ of length k whose canonization is π . Otherwise we say that σ **avoids** π .

Example

Let $\sigma = 1213431$.

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Example

Let $\sigma = 1213431$.

σ contains a copy of 112 namely 1213431 or 1213431. However, σ avoids 1112.

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Colored Partitions

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Definition

Denote the set of all k -colored set partitions of $[n]$ by $\Pi_n \wr C_k$.

Example

Consider $\sigma = 1213431 \in \Pi_7$ from the previous slide. We can make σ an element of $\Pi_7 \wr C_3$ simply by choosing one of three colors for each of the elements. So $1211341 \in \Pi_7 \wr C_3$.

Avoiding Colored Partitions

Definition

We say that $\sigma \in \Pi_n \setminus C_k$ *contains* a copy of $\pi \in \Pi_m \setminus C_j$ if

- 1 the uncolored version of σ contains a copy of the uncolored version of π and

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Example

Consider $\sigma = 1211341$. Then σ contains a copy of 122 , but σ avoids 122 .

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Example

Consider $\sigma = 1211341$. Then σ contains a copy of 122 , but σ avoids 122 .

Definition

For a set of colored set partitions S let $\Pi_n \setminus C_k(S)$ be the set of partitions in $\Pi_n \setminus C_k$ that avoid every pattern in S .

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Friendly Results

Theorem

For $n \geq 1$ and $c \geq 2$, $|\Pi_n \setminus C_2(\mathbf{11}, \mathbf{11})| = \sum_{i=1}^n 2^i S(n, i)$
 (OEIS A001861)

Theorem

For $n \geq 1$ and $c \geq 2$, $|\Pi_n \setminus C_2(\mathbf{12})| = (B_{n+1} - B_{n+1}) - (B_{n+1} - B_n)$
 (OEIS A011965)

$$|\Pi_n \wr C_k(112)|$$

Theorem

For $n \geq 1$ and $c \geq 2$, $|\Pi_n \wr C_k(112)| =$

$$B(n)(k-1)^n + \sum_{m=1}^n \sum_{j=1}^m \binom{n}{m} \binom{m}{j} B(n-j)(k-1)^{n-j} +$$

$$\sum_{1 \leq i < j \leq n} \sum_{a,b} \sum_{d,e} \sum_{f,g} \sum_m \sum_{p,q} \sum_{\ell} \binom{i-1}{a,b} \binom{j-i-1}{d,e} \binom{n-j}{f,g}.$$

$$\binom{i-a-b-1}{m} \binom{j-i-d-e-1}{p} \binom{n-a-b-f-g-j+i-m-1}{q}.$$

$$S(p+q, \ell) m^{\ell} B(n-a-b-d-e-f-g-m-p-q-2).$$

$$((k-1)^b + b(k-1)^{b-1})((k-1)^a + a(k-1)^{a-1})(k-2)^{d+p} k^e.$$

$$(k-1)^{n-a-b-d-e-m-p-2}.$$

Back

Avoiding 112 – Sketch of Proof

The Proof of this Theorem can be broken into 3 cases.

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Avoiding 112 – Sketch of Proof

The Proof of this Theorem can be broken into 3 cases.

Case 1: No elements are colored blue.

Case 2: Exactly one block contains elements colored blue.

Case 3: At least two blocks contain elements colored blue.

No elements are colored blue.

In this case there can't possibly be a copy of 112.

- $B(n)$ (n^{th} Bell number) ways to partition the elements in $[n]$
- $(k - 1)^n$ ways to color each element any color but blue.

Thus there are $B(n)(k - 1)^n$ such 112 avoiding partitions in $\Pi_n \setminus C_k$.

Exactly one block contains elements colored blue.

In this case there can't possibly be a copy of 112.

- Select m elements to be in the block with the elements that are colored blue.
- Select j of these elements to be colored blue.
- Partition the remaining $n - m$ elements in $B(n - m)$ ways.
- Color the non-blue elements in $(k - 1)^{n-j}$ ways.

Thus there are

$$\sum_{m=1}^n \sum_{j=1}^m \binom{n}{m} \binom{m}{j} B(n - m) (k - 1)^{n-j}$$

such partitions avoiding 112.

The Final Case

At least two blocks contain elements colored blue and there is no copy of 112 .

SEE BOARD!

[Jump to theorem](#)

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- Set Partition Statistics

Thank You

THANK YOU

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