

# An Asymptotic Version of a Theorem of Knuth

Jonathan Novak

MSRI & Waterloo

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# Symmetry



## Schensted pairs

$s(d, N)$  = no. of Schensted pairs on partitions  $\lambda \vdash N$ ,  $\ell(\lambda) \leq d$

$$s(3, 9) = 94\ 359$$

1	2	3	4	5
6	7	8		
9				

1	3	5	7	9
2	4	6		
8				

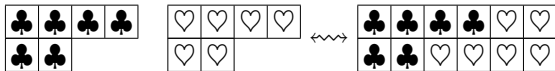
$s(d, N) = ?$
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# Knuth's formula

## Theorem

$$s(2, N) = \dim R(2, N)$$

## Proof.



## Corollary

$$s(2, N) = \frac{1}{N+1} \binom{2N}{N}.$$

## A general formula

$$s(d, N) = \sum_{\substack{\lambda \vdash N \\ \ell(\lambda) \leq d}} (\dim \lambda)^2,$$

where

$$\dim \lambda = \frac{N!}{\prod_{i=1}^d (\lambda_i - i + d)!} \prod_{1 \leq i < j \leq d} (\lambda_i - \lambda_j + j - i).$$

**Challenge:** use this formula to estimate

$$s(3, 10^{10}).$$

## Regev's formula

### Theorem

For any fixed  $d \geq 1$ ,

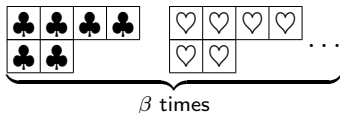
$$\begin{aligned} s(d, N) &\sim (2\pi)^{\frac{1-d}{2}} \left( \prod_{i=0}^{d-1} i! \right) d^{2N + \frac{d^2}{2}} (2N)^{\frac{1-d^2}{2}} \\ &= \underbrace{\left( (2\pi)^{\frac{1-d}{2}} \left( \prod_{i=0}^{d-1} i! \right) d^{\frac{d^2}{2}} 2^{\frac{1-d^2}{2}} \right)}_{\text{GUE partition function}} \underbrace{\left( d^{2N} N^{\frac{1-d^2}{2}} \right)}_{\text{S}(d, N) \text{ growth rate}} \end{aligned}$$

as  $N \rightarrow \infty$ .

# Regev's formula

Proof.

“Continuous” Schensted pairs:

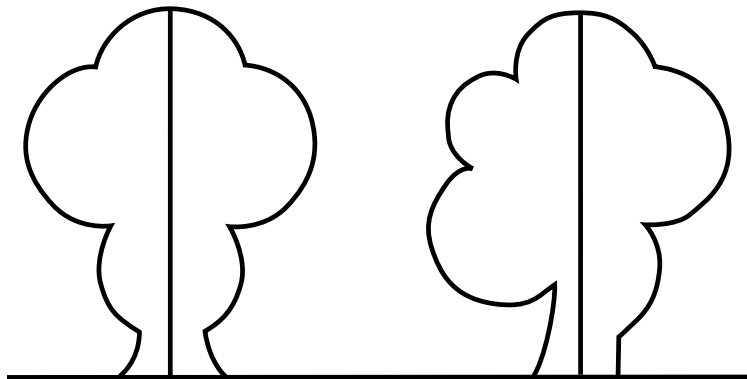


$$s(d, N; \beta) := \sum_{\substack{\lambda \vdash N \\ \ell(\lambda) \leq d}} (\dim \lambda)^\beta$$

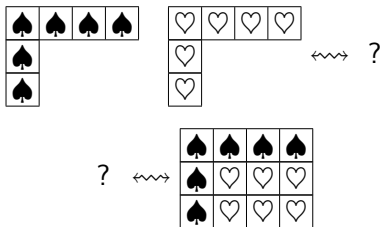
$$C_{d,N}^\beta s(d, N; \beta) \rightarrow \underbrace{\int_{\Omega_{d-1}} e^{-\beta W(y_1, \dots, y_{d-1})} dy}_{\text{Mehta-Dyson-Selberg}}$$



## Asymmetry



# Asymmetry



# Symmetry





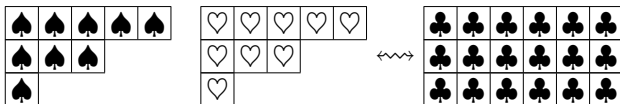
# Symmetry



# Symmetry



# Asymptotic symmetry



Conjecture:

$$s(d, N) \sim \dim R(d, 2N/d)$$

## Verification of asymptotic symmetry

Exact formula:

$$\dim R(d, q) = \frac{(dq)!}{\prod_{i=0}^{d-1} \frac{(q+i)!}{i!}}$$

Dimension of a  $d \times \infty$  strip:

$$\dim R(d, q) \sim (2\pi)^{\frac{1-d}{2}} \left( \prod_{i=0}^{d-1} i! \right) d^{dq + \frac{1}{2}} q^{\frac{1-d^2}{2}}$$

Scaling dictated by symmetry:

$$q \rightsquigarrow 2N/d$$

Reproduces Regge's formula:

$$\dim R(d, 2N/d) \sim (2\pi)^{\frac{1-d}{2}} \left( \prod_{i=0}^{d-1} i! \right) d^{2N + \frac{d^2}{2}} (2N)^{\frac{1-d^2}{2}}$$

# Asymptotic Knuth theorem

## Theorem

For any fixed  $d \geq 1$ ,

$$s(d, dn) \sim \dim R(d, 2n)$$

as  $n \rightarrow \infty$ .

## Corollary

The number of *permutations* in  $\mathbf{S}(dn)$  with no decreasing subsequence of length  $d + 1$  is asymptotically equal to the number of *involutions* in  $\mathbf{S}(2dn)$  with longest decreasing subsequence of length exactly  $d$  and longest increasing subsequence of length exactly  $2n$ .

## Error term

Complements:  $\mu \subset R(d, q)$ ,

$$\mu^{*R(d,q)} = (q - \mu_d, q - \mu_{d-1}, \dots, q - \mu_1)$$

### Theorem

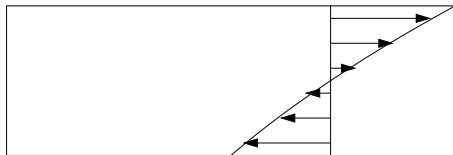
$$s(d, dn) = \dim R(d, 2n) + E(d, dn),$$

where

$$E(d, dn) = \frac{1}{2} \underbrace{\sum_{\substack{\mu \vdash dn \\ \mu \subset R(d, 2n)}} (\dim \mu - \dim \mu^*)^2}_{\text{asymmetry}} + \underbrace{\sum_{\substack{\nu \vdash dn \\ \nu_1 > 2n}} (\dim \nu)^2}_{\text{large deviation}}.$$

## Laplace method

*If you want to understand a sum/integral where the integrand contains a large parameter, the maximum of the integrand is the centre of the universe.*



$$\dim(n + y_1\sqrt{n}, \dots, n + y_d\sqrt{n}) \sim ?$$

## Laplace method

### Theorem

For any fixed  $y_1 > \dots > y_d$ ,  $y_1 + \dots + y_d = 0$ ,

$$\lim_{n \rightarrow \infty} C_{d,dn} \dim(n + y_1 \sqrt{n}, \dots, n + y_d \sqrt{n}) = e^{-W(y_1, \dots, y_d)},$$

where

$$W(y_1, \dots, y_d) = \frac{1}{2} \sum_{i=1}^d y_i^2 - \sum_{1 \leq i < j \leq d} \log(y_i - y_j).$$

Proof.

$$\begin{aligned} & \dim(n + y_1 \sqrt{n}, \dots, n + y_d \sqrt{n}) \\ &= \frac{\Gamma(dn + 1)}{\prod_{i=1}^d \Gamma(n + y_i \sqrt{n} + i + d + 1)} \prod_{1 \leq i < j \leq d} ((y_i - y_j) \sqrt{n} + j - i). \end{aligned}$$

## Laplace method

$$s(d, N; \beta) = \sum_{\substack{\lambda \vdash N \\ \ell(\lambda) \leq d}} (\dim \lambda)^\beta$$

$$\lim_{n \rightarrow \infty} C_{d, dn}^\beta s(d, dn; \beta) = \int_{\Omega_{d-1}} e^{-\beta W(y_1, \dots, y_d)} d\mathbf{y}$$

$$\Omega_{d-1} = \{y_1 > \dots > y_d, y_1 + \dots + y_d = 0\} \subset \mathbb{R}^{d-1}$$

Regev: evaluate this (difficult) integral.

## Laplace method

$$\dim R(d, 2n) = \sum_{\substack{\mu \vdash dn \\ \mu \subset R(d, 2n)}} (\dim \mu)(\dim \mu^*).$$

$$t(d, dn; \gamma, \delta) = \sum_{\substack{\mu \vdash dn \\ \mu \subset R(d, 2n)}} (\dim \mu)^\gamma (\dim \mu^*)^\delta.$$

Exactly the same argument:

$$\lim_{n \rightarrow \infty} C_{d, dn}^{\gamma + \delta} t(d, dn; \gamma, \delta) = \int_{\Omega_{d-1}} e^{-\gamma W(y_1, \dots, y_d)} e^{-\delta W(-y_d, \dots, -y_1)} dy.$$

## Symmetry returns

$$s(d, dn) \sim \dim R(d, 2n)$$

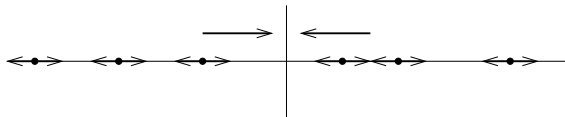


$$\int_{\Omega_{d-1}} e^{-2W(y_1, \dots, y_d)} d\mathbf{y} = \int_{\Omega_{d-1}} e^{-W(y_1, \dots, y_d)} e^{-W(-y_d, \dots, -y_1)} d\mathbf{y}$$



$$W(y_1, \dots, y_d) = W(-y_d, \dots, -y_1)$$

## Symmetry returns



Energy:

$$W(y_1, \dots, y_d) = \frac{1}{2} \sum_{i=1}^d y_i^2 - \sum_{1 \leq i < j \leq d} \log(y_i - y_j).$$

Symmetry:

$$W(y_1, \dots, y_d) = W(-y_d, \dots, -y_1)$$

## Symmetry returns

$$W(y_1, \dots, y_d) = W(-y_d, \dots, -y_1)$$

### Theorem

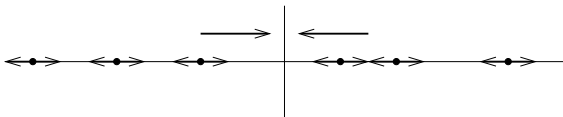
For any  $0 \leq \gamma < \beta$ ,

$$\sum_{\substack{\lambda \vdash dn \\ \ell(\lambda) \leq d}} (\dim \lambda)^\beta \sim \sum_{\substack{\mu \vdash dn \\ \mu \subset R(d, 2n)}} (\dim \mu)^\gamma (\dim \mu^*)^{\beta-\gamma}$$

### Corollary

$$\sum_{\substack{\lambda \vdash dn \\ \ell(\lambda) \leq d}} (\dim \lambda)^2 \sim \sum_{\substack{\mu \vdash dn \\ \mu \subset R(d, 2n)}} (\dim \mu)(\dim \mu^*) = \dim R(d, 2n)$$

## Mehta-Dyson integral



Energy:

$$W(t_1, \dots, t_d) = \frac{1}{2} \sum_{i=1}^d t_i^2 - \sum_{1 \leq i < j \leq d} \log(t_i - t_j).$$

Partition function (Mehta-Dyson integral):

$$\Psi(d; \beta) = \int_{\mathcal{W}_d} e^{-\beta W(t_1, \dots, t_d)} dt$$
$$\mathcal{W}_d = \{t_1 > \dots > t_d\} \subset \mathbb{R}^d$$

Mehta-Dyson conjecture:

$$\Psi(d; \beta) = \frac{1}{d!} (2\pi)^{\frac{d}{2}} \beta^{-\frac{d}{2} - \beta \frac{d(d-1)}{4}} \prod_{i=1}^d \frac{\Gamma(1 + i\frac{\beta}{2})}{\Gamma(1 + \frac{\beta}{2})}.$$

- Bombieri: Selberg  $\implies$  Mehta-Dyson
- Symmetry  $\implies \Psi(d; 2)$
- Dyson:  $\Psi(d; 2k) \implies \Psi(d; \beta)$ .
- Symmetry  $\implies \Psi(d; 2k)???$

## Double-Scaling limit

Baik-Deift-Johansson, Okounkov, Borodin-Okounkov-Olshanski, Johansson:

### Theorem

For  $d, N \rightarrow \infty$  at the rate  $d \sim 2N^{1/2} + tN^{1/6}$ ,

$$s(d, N) \sim F(t)N!,$$

where

$F(t) = \text{Tracy-Widom distribution function.}$

$$s(d, dn) = \dim R(d, 2n) + \frac{1}{2} \sum_{\substack{\mu \vdash dn \\ \mu \subset R(d, 2n)}} (\dim \mu - \dim \mu^*)^2 + \sum_{\substack{\nu \vdash dn \\ \nu_1 > 2n}} (\dim \nu)^2.$$

Asymptotics of  $E(d, dn)$  in double scaling limit???

# Acknowledgements

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