Pattern match avoidance in cycle structure

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The reduction $\mathrm{red}(n_1, \ldots, n_k)$ is a function that replaces the i-th largest number with i

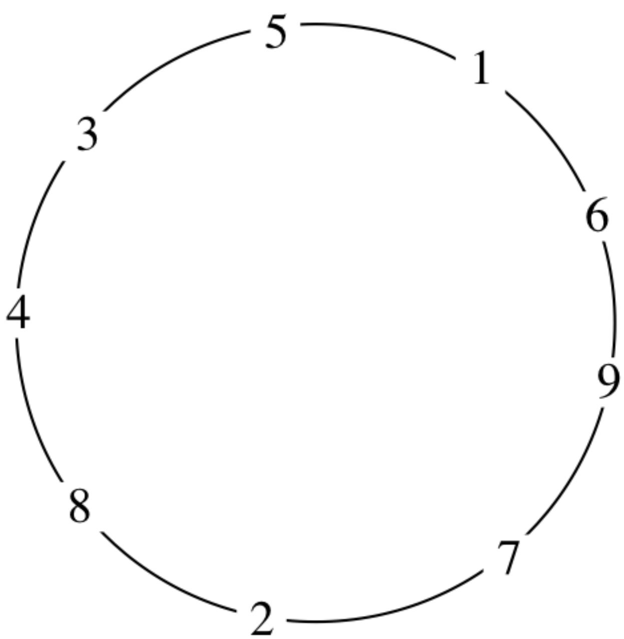
Example:

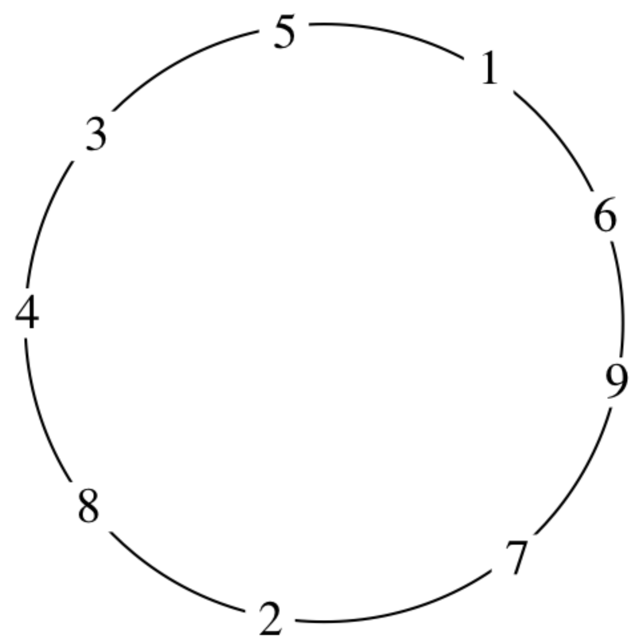
$\mathrm{red}(21\ 5\ 10\ 9\ 7) = 5\ 1\ 4\ 3\ 2$

For $\tau \in S_m$ there is a τ-cycle-occurrence in a cycle $C = (c_0, \ldots, c_{p-1})$ if there is an r and indices $0 < i_1 < \cdots < i_{m-1} \leq p-1$ such that $\mathrm{red}(c_r, c_{r+i_1}, \ldots, c_{r+i_{m-1}}) = \tau$ (Where the subscripts are taken mod p)

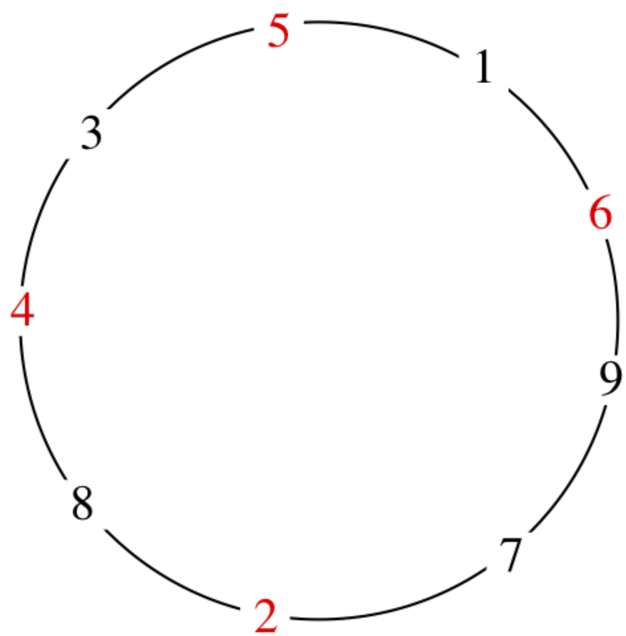
If the occurrence involves consecutive entries it is called a τ-cycle-match

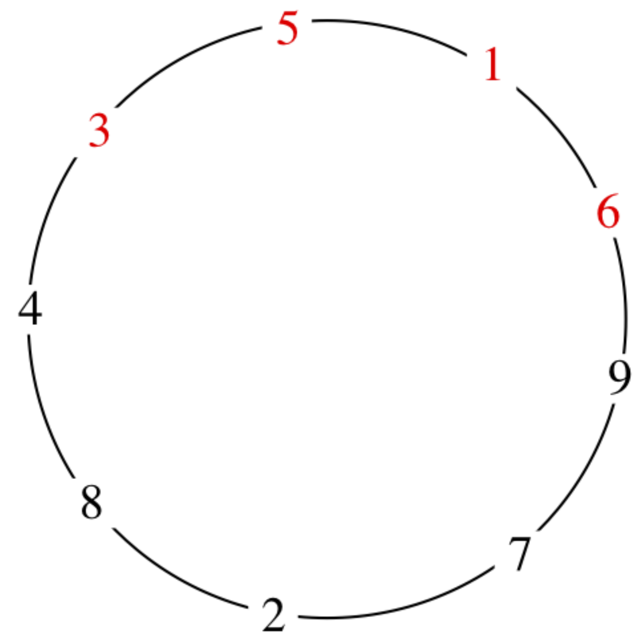
$\tau = 1\ 2\ 3\ 4$ $\beta = 2\ 3\ 1\ 4$





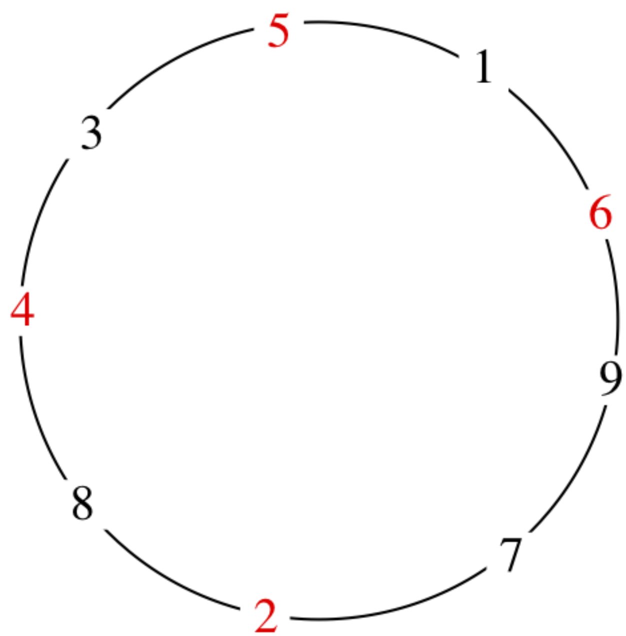
$\tau =$ 1 2 3 4 $\quad\quad\quad$ $\beta =$ 2 3 1 4

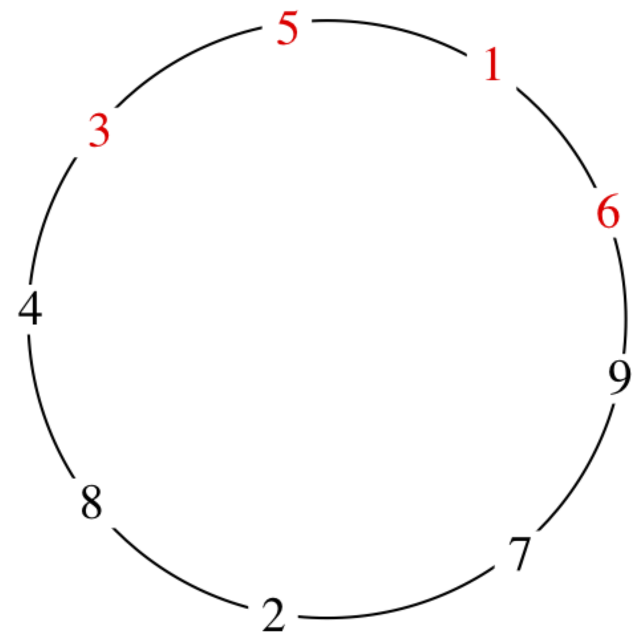




$\tau = 1\ 2\ 3\ 4 \qquad \tau_2 = 2\ 3\ 4\ 1 \qquad \beta = 2\ 3\ 1\ 4$

$\tau_2 = 3\ 4\ 1\ 2 \qquad \tau_3 = 4\ 1\ 2\ 3$





The focus of this talk will be to count

1. the number of n-permutations that have no τ-cycle-occurrences in any cycle denoted
$$ncoS_n(\tau)$$

2. the number of n-permutations that have no τ-cycle-matches in any cycle denoted
$$ncmS_n(\tau)$$

$$NCO_\tau(t) = 1 + \sum_{n\geq 1} ncoS_n(\tau)\frac{t^n}{n!}, \tag{1}$$

$$NCM_\tau(t) = 1 + \sum_{n\geq 1} ncmS_n(\tau)\frac{t^n}{n!}, \tag{2}$$

Two patterns τ and β are cycle-occurrence-Wilf-equivalent (co-Wilf-equivalent) if $NCO_\tau(t) = NCO_\beta(t)$

Two patterns τ and β are cycle-match-Wilf-equivalent (cm-Wilf-equivalent) if $NCM_\tau(t) = NCM_\beta(t)$

Any cyclic permutation of τ is co-Wilf-equivalent to τ

Reversing a Cycle:

$(c_0, c_1, \ldots c_{m-1})^{cr} = (c_{m-1}, c_{m-2}, \ldots c_0)$

Complementing a Cycle:

$(c_0, c_1, \ldots c_{m-1})^{cc} = (n+1-c_0, n+1-c_1, \ldots n+1-c_{m-1})$

σ^{cr} is the result when you reverse all cycles of σ

σ^{cc} is the result when you complement all cycles of σ

Note: Reversing (complementing) the cycle structure of σ is different from reversing (complementing) σ

$$\sigma = 1324 = (1)(23)(4)$$

$$\sigma^r = \sigma^c = 4231 = (14)(2)(3)$$

$$\sigma^{cr} = \sigma^{cc} = (4)(23)(1) = \sigma$$

τ is co-Wilf-equivalent and cm-Wilf-equivalent to τ^r and τ^c

The basic theorem of exponential structures allows us to simplify the problem of enumerating $ncoS_n(\tau)$ and $ncmS_n(\tau)$ to just enumerating single cycles.

Let $W_m : \mathcal{C}_m \to R$ be a weight function that sends cycles of length m to a ring R. Then let $\mathcal{C}_{n,k}$ be the set of all permutations of length n with k cycles.

For $\sigma \in \mathcal{C}_{n,k}$, let

$$W(\sigma) = \sum_{i=1}^{k} W_{p_i}(C_i)$$

where p_i is the length of C_i.

Theorem of Exponential Structures:

Theorem 0.1.

$$1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{k=1}^{n} x^k \sum_{\sigma \in \mathcal{C}_{n,k}} W(\sigma) = e^{x \sum_{m \geq 1} \frac{W_m(C_m) t^m}{m!}}. \qquad (3)$$

For some pattern τ. if we let the weight function be

$W_p(C) = 1$ if C has no τ-matches and $W_p(C) = 0$ if C does have a τ-match then we get the generating functions:

$$NCO_\tau(t,x) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\substack{\sigma\in\mathcal{S}_n \\ c \text{ has no} \\ \tau\text{-cycle-occurrences}}} x^{cyc(\sigma)} = e^{x\sum_{m\geq 1} ncoC_m(\tau)\frac{t^m}{m!}}. \tag{4}$$

and

$$NCM_\tau(t,x) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\substack{\sigma\in\mathcal{S}_n \\ c \text{ has no} \\ \tau\text{-cycle-matches}}} x^{cyc(\sigma)} = e^{x\sum_{m\geq 1} ncmC_m(\tau)\frac{t^m}{m!}}. \tag{5}$$

Example 1: Let $\tau = 1\ 2$. It is clear that the only cycles that have no 1 2-cycle-occurrences or 1 2-cycle-matches are cycles of length 1.

$ncoC_1(1\ 2) = ncmC_1(1\ 2) = 1$

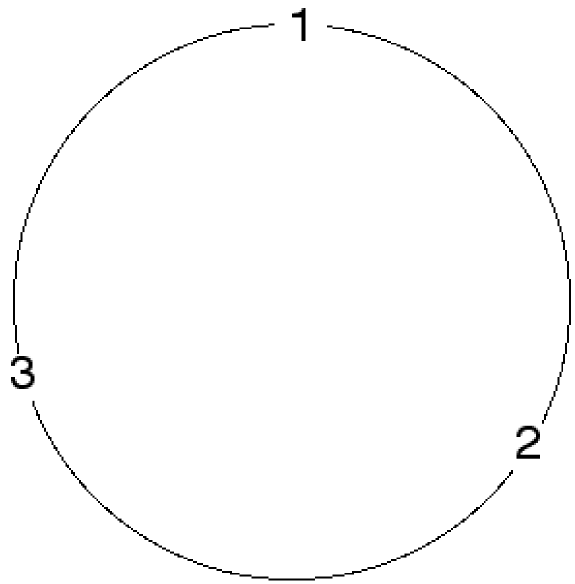
$ncoC_m(1\ 2) = ncmC_m(1\ 2) = 0$ for $m \geq 2$

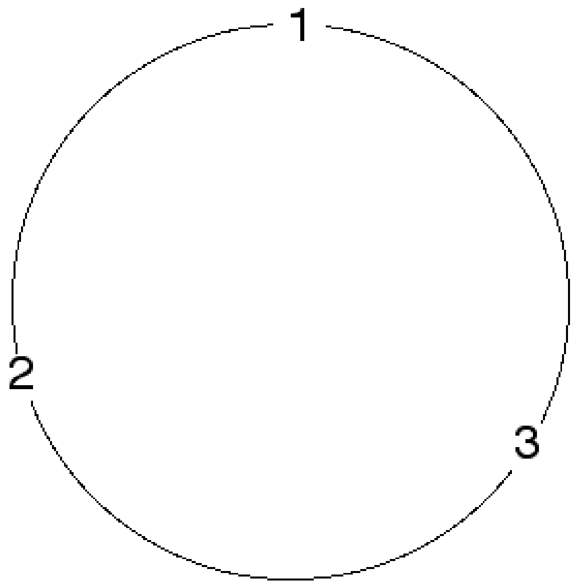
$$\sum_{m\geq 1} ncoC_m(\tau)\frac{t^m}{m!} = \sum_{m\geq 1} ncmC_m(\tau)\frac{t^m}{m!} = t$$

$$NCO_\tau(t,x) = NCM_\tau(t,x) = e^{xt}$$

Consider $\Upsilon = \{1\ 2\ 3, 3\ 2\ 1\}$.

It is clear that $ncoC_1(\Upsilon) = ncoC_2(\Upsilon) = 1$ and $ncoC_n(\Upsilon) = 0$ for $n \geq 3$





Consider $\Upsilon = \{1\ 2\ 3, 3\ 2\ 1\}$.

It is clear that $ncoC_1(\Upsilon) = ncoC_2(\Upsilon) = 1$ and $ncoC_n(\Upsilon) = 0$ for $n \geq 3$

$$\sum_{m \geq 1} ncoC_m(\Upsilon) \frac{t^m}{m!} = t + \frac{t^2}{2}$$

$$NCO_\Upsilon(t, x) = e^{x(t+t^2/2)}$$

$\Upsilon = \{1\ 2\ 3, 3\ 2\ 1\}$

Let $C = (c_0, c_1, \dots, c_{n-1})$ have no Υ-cycle-matches and $c_0 = 1$. If $n \geq 3$

1. $c_0 < c_1 > c_2$
2. $c_0 < c_1 > c_2 < c_3$

   $c_0 < c_1 > c_2 < c_3 > c_4 < c_5 > c_6 < c_7 \dots$

But if $n = 2k+1$ is odd then we will be forced to have $c_{2k-1} > c_{2k} > c_0$ which will give a 3 2 1-match. Therefore n must be even and $c_1, \dots, c_{n-1}$ must be an alternating permutation.

$$\sum_{n \geq 0} Alt_{2n+1} \frac{t^{2n+1}}{(2n+1)!} = \tan(t)$$

$$\sum_{m \geq 1} ncmC_m(\Upsilon) \frac{t^m}{m!} = t - \log|\cos(t)|$$

$\Upsilon = \{1\ 2\ 3, 3\ 2\ 1\}$

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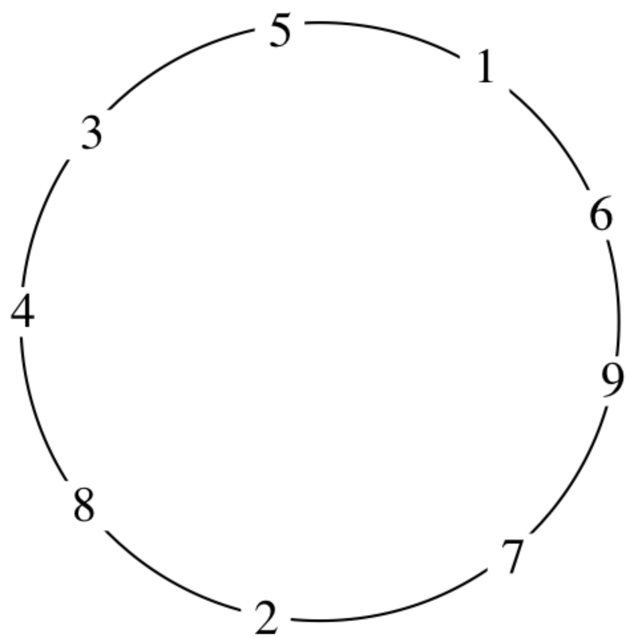
But if $n = 2k + 1$ is odd then we will be forced to have $c_{2k-1} > c_{2k} > c_0$ which will give a 3 2 1-match. Therefore n must be even and $c_1, \ldots, c_{n-1}$ must be an alternating permutation.

$$\sum_{n \geq 0} Alt_{2n+1} \frac{t^{2n+1}}{(2n+1)!} = \tan(t)$$

$$NCM_{\Upsilon}(t, x) = e^{x(t - \log|\cos(t)|)} = e^{xt} \sec^x(t)$$

Let $C = (c_0, \ldots, c_{p-1})$ be a cycle with its smallest element c_0 written first. Then $\mathrm{cdes}(C) = \mathrm{des}(c_0, \ldots, c_{p-1}) + 1$

$$(1\ 6\ 9\ 7\ 2\ 8\ 4\ 3\ 5)$$



And if $\sigma = C_1, \ldots C_k$ is a permutation in cycle structure then $\mathrm{cdes}(\sigma) = \sum_i \mathrm{cdes}(C_i)$

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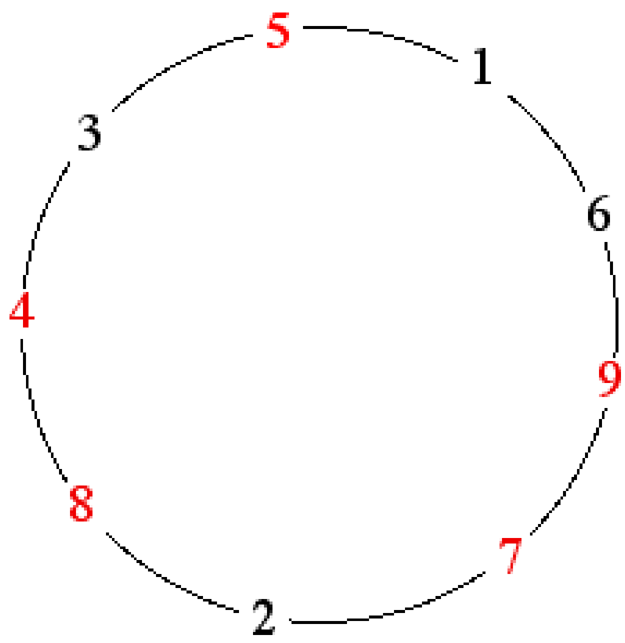

And if $\sigma = C_1, \ldots C_k$ is a permutation in cycle structure then $\mathrm{cdes}(\sigma) = \sum_i \mathrm{cdes}(C_i)$

Let $nmS_n(\tau)$ be the number of n-permutations that have no τ-matches.

If τ starts with 1 then $ncmS_n(\tau) = nmS_n(\tau)$ for all n.

$$1 + \sum_{n \geq 1} ncmS_n(\tau) \frac{t^n}{n!} = 1 + \sum_{n \geq 1} nmS_n(\tau) \frac{t^n}{n!}$$

If τ starts with 1 then $ncmS_n(\tau) = nmS_n(\tau)$ for all n.

Example:

$$\sigma = (5\ 10\ 11)(3\ 6\ 4\ 8)(1\ 9\ 7\ 2)$$
$$\overline{\sigma} = 5\ 10\ 11 \mid 3\ 6\ 4\ 8 \mid 1\ 9\ 7\ 2$$

Lemma 0.2. *If $\tau \in S_j$ and τ starts with 1, then for any $\sigma \in S_n$,*

1. *σ has k cycles if and only if $\bar{\sigma}$ has k left-to-right minima,*
2. $\mathrm{cdes}(\sigma) = 1 + \mathrm{des}(\bar{\sigma})$, *and*
3. *σ has no cycle-τ-matches if and only if $\bar{\sigma}$ has no τ-matches.*

If τ starts with 1 then $ncmS_n(\tau) = nmS_n(\tau)$ for all n.

Example:

$$\sigma = (5\ 10\ 11)(3\ 6\ 4\ 8)(1\ 9\ 7\ 2)$$

$$\overline{\sigma} = 5\ 10\ 11 \mid 3\ 6\ 4\ 8 \mid 1\ 9\ 7\ 2$$

If τ starts with 1,

$$NCM_\tau(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\substack{\sigma \text{ has no} \\ \tau\text{-cycle-matches}}} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{cdes}(\sigma)}$$

$$NM_\tau(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\substack{\sigma \text{ has no} \\ \tau\text{-matches}}} x^{\mathrm{LtoRmin}(\sigma)} y^{1+\mathrm{des}(\sigma)}$$

$$= \exp\left(x \sum_{m\geq 1} \frac{t^m}{m!} \sum_{\substack{c \text{ has no} \\ \tau\text{-cycle-matches}}} y^{\mathrm{cdes}(c)} \right)$$

Suppose τ starts with 1.

$$NM_\tau(t) = NCM_\tau(t) = \exp\left(\sum_{m\geq 1} ncmC_m(\tau)\frac{t^m}{m!}\right)$$

$$\sum_{m\geq 1} ncmC_m(\tau)\frac{t^m}{m!} = \log\left(NM_\tau(t)\right)$$

$$NM_\tau(t,x) = NCM_\tau(t,x) = \exp\left(x\sum_{m\geq 1} ncmC_m(\tau)\frac{t^m}{m!}\right)$$

$$NM_\tau(t)^x = NM_\tau(t,x) = \sum_{n\geq 0}\frac{t^n}{n!}\sum_{\substack{\sigma \text{ has no}\\ \tau\text{-matches}}} x^{\mathrm{LtoRmin}(\sigma)}$$

Example: Let $\tau = 132$ and let

$$A_n(y) = \sum_{\substack{c \in \mathcal{C}_n \\ c \text{ has no} \\ \tau\text{-cycle-matches}}} y^{\text{cdes}(c)} = \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma \text{ starts with 1} \\ \sigma \text{ has no} \\ \tau\text{-cycle-matches}}} y^{1+\text{des}(\sigma)}$$

$A_1 = y \quad (1)$

$A_2 = y \quad (12)$

$A_3 = y \quad (123), \sout{(132)}$

$\tau = 132$

A_n with $n \geq 4$

$k = 2$ $\quad 1\ \underbrace{2\ \_\ \_\ \ldots\ \_}_{n-1}$ $\quad$ we get $A_{n-1}(y)$

$k = 3$ $\quad 1\ \_\ 2\ \_\ \ldots\ \_$ $\quad$ can't avoid 132

$k \geq 4$ $\quad \underbrace{1\ \_\ \_\ \ldots\ \_}_{k-1}\ \underbrace{2\ \_\ \ldots\ \_}_{n-k+1}$ $\quad \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y)$

So now we get the recursion

$$A_n = A_{n-1}(y) + \sum_{k=4}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y)$$

Let $A(t,y) = \sum_{n\geq 0} A_n(y)\frac{t^n}{n!}$

Then the recursion

$$A_n(y) = A_{n-1}(y) + \sum_{k=4}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y)$$

leads to the differential equation:

$$A''(t,y) = (A'(t,y))^2 + (1 - y - yt)(A'(t,y))$$

and solution

$$A(t,y) = -\log(1 - y\int_0^t e^{s-ys-y\frac{s^2}{2}} ds)$$

$$A(t,y) = -\log(1 - y\int_0^t e^{s-ys-y\frac{s^2}{2}}\,ds)$$

$$NCM_\tau(t,x,y) =$$

$$\begin{aligned} NM_\tau(t,x,y) &= \exp\left[x(-\log(1 - y\int_0^t e^{s-ys-y\frac{s^2}{2}}\,ds))\right] \\ &= \left(\frac{1}{1 - y\int_0^t e^{s-ys-y\frac{s^2}{2}}\,ds}\right)^x \end{aligned}$$

The method we used for 132 will give us a differential equation for all patterns $\tau \in S_m$ that have the form

$$\tau = 1\ 2\ \ldots\ (j-1)\ \alpha\ j$$

Where α is any permutation of the last $m - j$ numbers.

Example: 1 2 3 4 9 6 8 10 7 5

Suppose $\tau = 1\ 2\ \ldots\ (j-1)\ \alpha\ j$

$NM_\tau(t,x,y) = \frac{1}{1+\sum_{n\geq 1} U_n^\tau(x,y)\frac{t^n}{n!}}$

$$U_n^\tau = (1-y)U_{n-1}^\tau + y^{\mathrm{des}(\alpha)+1}\binom{n-j}{m}U_{n-m-j+1}^\tau$$

Remmel and Mendes proved the following theorem:

Theorem 0.3. *For $j \geq 2$ and $\tau = j \ \ldots \ 2 \ 1$,*

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\substack{\sigma \in S_n \\ \tau-match(\sigma)=0}} y^{\mathrm{des}(\sigma)} = \left(\sum_{n\geq 0} \frac{t^n}{n!} \sum_{i\geq 0} (-1)^i \mathcal{R}_{n-1,i,j-1} y^i \right)^{-1}$$

where $\mathcal{R}_{n,i,j}$ is the number of rearrangements of i zeroes and $n-1$ ones such that j zeroes never appear consecutively.

This theorem gives the following result:

If $\tau = 1\ 2 \ldots j$ then

$$NM_\tau(t,x,y) = NCM_\tau(t,x,y) = \left(\frac{1}{\sum_{n \geq 0} \frac{t^n}{n!} \sum_{i \geq 0} (-1)^i \mathcal{R}_{n-1,i,j-1} y^i} \right)^x$$

$$\tau = 1\ 2\ \ldots\ (j-1)\ \alpha\ j$$

$\boxed{1\ 2\ 3\ 4}$......Remmel and Mendes

$\boxed{1\ 2\ 4\ 3}$......$j = 3, \alpha = 1$

$\boxed{1\ 3\ 2\ 4}$......$U_n = (1-y)U_{n-1} + \sum_{k=2}^{\frac{n}{2}} (-y)^{k-1} C_{k-1} U_{n-2k+1}$

$\boxed{1\ 3\ 4\ 2}$......$j = 2, \alpha = 1\ 2$

$\boxed{1\ 4\ 2\ 3}$......$U_n = (1-y)U_{n-1} + \sum_{k=2}^{\frac{n}{2}} (-y)^{k-1} \binom{n-k-1}{k-1} U_{n-2k+1}$

$\boxed{1\ 4\ 3\ 2}$......$j = 2, \alpha = 2\ 1$