The Number of Distinct Minors of a Permutation

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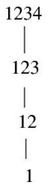
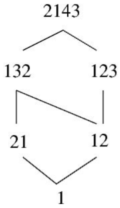
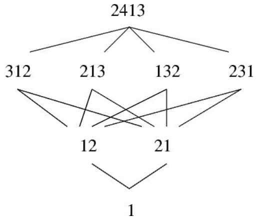
$p = 1234$



$p = 2143$



$p = 2413$



Definition

A $(n-k)$-*minor* of an n-permutation is a pattern of size $(n-k)$ contained in p. Define $M_k(p)$ to be the *set* of $(n-k)$-minors of p. We call an $(n-1)$-minor a *maximal minor*.

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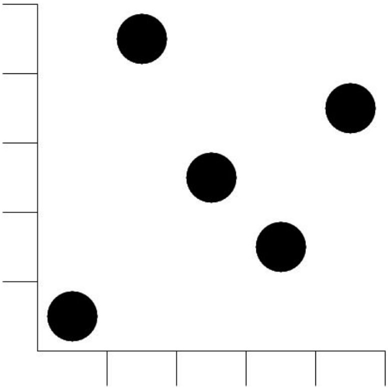
Fact

For any n-permutation p, $|M_k(p)| \leq \binom{n}{k}$. *In particular, p has at most n maximal minors.*

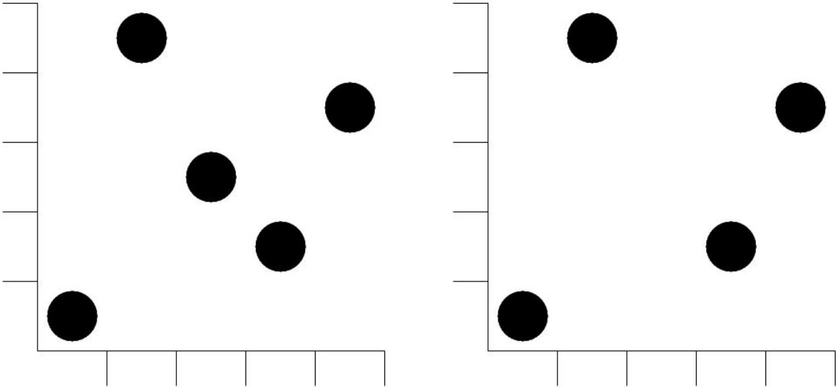
Definition

Let $p \in S_n$, and $i \in [n]$.
Define $M(p, i) \in S_{n-1}$ to be the $(n-1)$-permutation obtained by deleting the i'th entry of p, and renumbering the remaining elements with respect to order.

$p = 15324$



$p = 15324$ $\quad$ $M(p, 3) = 1423$



Definition

Let $p = p_1 p_2 \ldots p_n$ be an n-permutation. Define a *consecutive pair* to be a pair of entries (p_i, p_{i+1}) such that $|p_i - p_{i+1}| = 1$.

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Definition

We say that the sequence $(p_j, p_{j+1}, \dots p_{j+k-1})$ is a *consecutive run of length k* when the pair (p_i, p_{i+1}) is consecutive for each $j \leq i \leq j + k - 2$.

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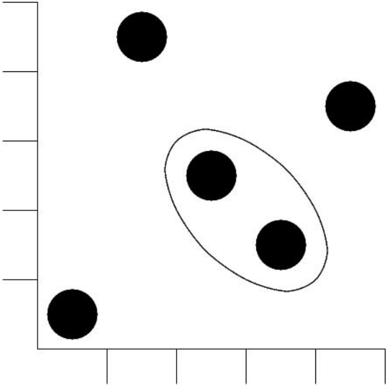

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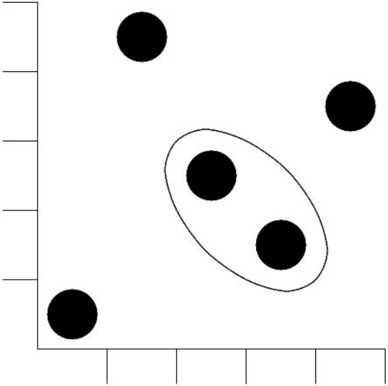
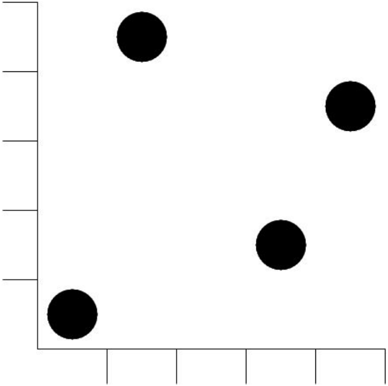


$M(p, 3) = 1423 = M(p, 4)$



Lemma

Let $p = p_1 p_2 \ldots p_n$ be any n permutation, and $i, j \in [n]$ with $i \neq j$. It follows that $M(p, i) = M(p, j)$ if and only if p_i and p_j are a part of the same consecutive run.

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Theorem

Define $C(p)$ *to be the number of consecutive pairs of entries of* p*. Then* $|M_1(p)| = n - C(p)$*.*

Corollary

Let q be any n-permutation. Then q is contained as a pattern in exactly $n^2 + 1$ distinct $(n + 1)$-permutations.

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Proof.

We can insert an entry into q in exactly $(n+1)^2$ different ways. By the lemma, inserting an entry in two different locations will result in the same permutation only when we create the same consecutive run in two different ways.

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Therefore, q is contained in exactly $(n+1)^2 - 2n = n^2 + 1$ $(n+1)$-permutations. □

Corollary

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Proof.

$$(n-1)!\Big((n-1)^2+1\Big) = n!\left(n - 2\frac{n-1}{n}\right). \qquad \square$$

Lemma

Let $b_{n,k}$ be the number of n-permutations with exactly k distinguished consecutive pairs, and let $B(z,u) = \sum_{n,k \geq 0} b_{n,k} z^n u^k$. Set $b_{0,0} = 1$. Then

$$B(z,u) = \sum_{m \geq 0} m! \left(z + \frac{2z^2 u}{1 - zu} \right)^m .$$

Theorem

Let $a_{n,k}$ be the number of n-permutations with exactly k consecutive pairs (and hence $n-k$ distinct minors). Set $a_{0,0}=1$, and $A(z,u)=\sum_{n,k\geq 0} a_{n,k}z^n u^k$. Then

$$A(z,u)=\sum_{m\geq 0} m!\left(z+\frac{2z^2(u-1)}{1-z(u-1)}\right)^m.$$

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Proof.

$A(z,u+1)=B(z,u)$, and so $A(z,u)=B(z,u-1)$. □

Corollary

The generating function for the number of n-permutations with all distinct maximal minors is given by

$$A(z,0) = 1 + z + 2z^4 + 14z^5 + 90z^6 + 646z^7 + 5242z^8 \ldots.$$

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Theorem (Tauraso 2006)

$$a_{n,0} \sim \frac{n!}{e^2}.$$

Theorem

Fix $n \geq 1$, and let $\chi : S_n \to [n]$ be the variable indicating the number of distinct maximal minors. Then

$$\mathbb{E}(\chi) = n - 2\frac{n-1}{n}$$

and

$$\mathbb{V}(\chi) = 4\frac{(n-2)^2}{n(n-1)} + 2\frac{n-1}{n} - 4\frac{(n-1)^2}{n^2}.$$

Definition

Let $p = p_1 p_2 \ldots p_n$ be a permutation. Define the *gap* between entries p_i and p_j to be $gap(p_i, p_j) = |i - j| + |p_i - p_j|$.

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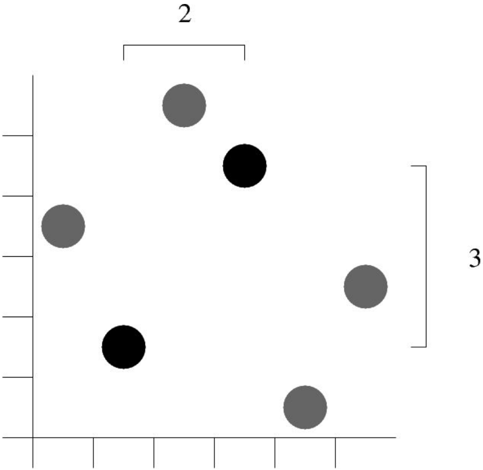
Define the *minimum gap* of p by

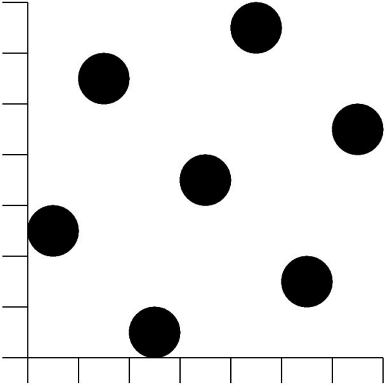
$$mingap(p) = min\{gap(p_i, p_j) : i, j \in [n]\}.$$

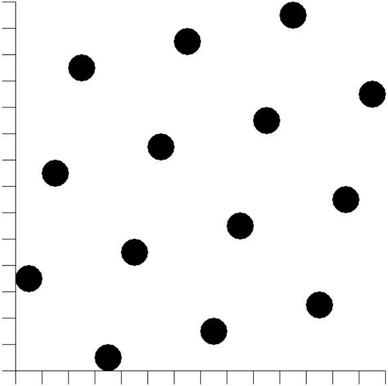


For, $p = 426513$, $gap(2,5) = 5$.

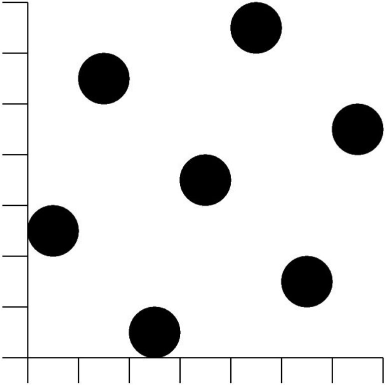
Theorem

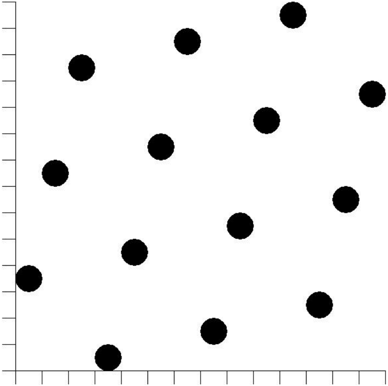
A permutation p has exactly $\binom{n}{k}$ $(n-k)$*-minors if and only if* $mingap(p) \geq k+2$.





$p^4 = 3\ 6\ 1\ 4\ 7\ 2\ 5$ and $p^5 = 4\ 8\ 12\ 1\ 5\ 9\ 13\ 2\ 6\ 10\ 14\ 3\ 7\ 11$





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$\left(\text{In general, } |p^m| = (m-1)^2 - 2\right)$

Further Questions