

Counting Paths and Walks with Several Step Vectors

Shanzhen Gao

Florida Atlantic University

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Dartmouth College

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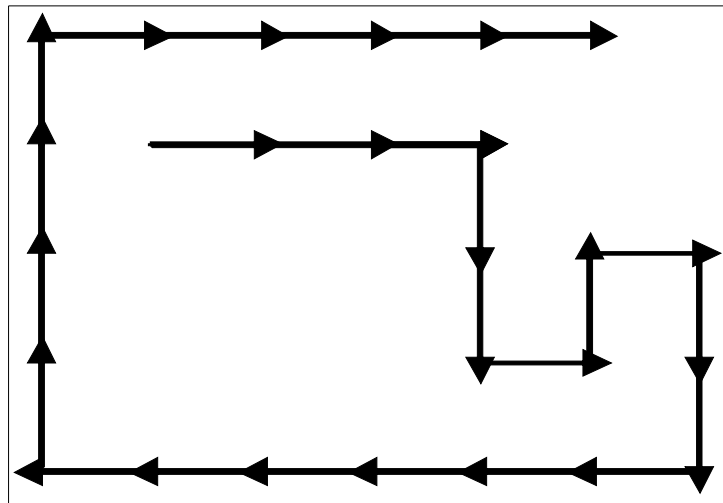
Prudent self-avoiding walks:

A PSAW is a proper subset of SAW on the square lattice. The walk starts at $(0,0)$, and the empty walk is a PSAW. A PSAW grows by adding a step to the end point of a PSAW such that the extension of this step - by any distance - never intersects the walk. Hence the name prudent. The walk is so careful to be self-avoiding that it refuses to take a single step in any direction where it can see - no matter how far away - an occupied vertex.

A.J. Guttmann, E. Duchi, P. Pr ea, T.M. Garoni, I. Jensen and J.C. Dethridge.

Prudent Self-Avoiding Walks

A *PSAW* is a proper subset of *SAWs* on the square lattice. The walk starts at $(0,0)$, and the empty walk is a *PSAW*. A *PSAW* grows by adding a step to the end point of a *PSAW* such that the extension of this step - by any distance - never intersects the walk. The walk is so careful to be self-avoiding that it refuses to take a single step in any direction where it can see - no matter how far away - an occupied vertex.



One-sided PSAWs

What is the number (say $f(n)$) of one-sided n -step prudent walks, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$?

The generating function equals

$$\sum_{n \geq 0} f(n)t^n = \frac{1+t}{1-2t-t^2} = 1 + 3t + 7t^2 + 17t^3 + 41t^4 + 99t^5 + 239t^6 + \dots$$

Also,

$$\begin{aligned} f(n) &= 2f(n-1) + f(n-2) \\ &= \frac{(1-\sqrt{2})^n + (1+\sqrt{2})^n}{2} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\sum_{k=0}^{n-1} \binom{n-1}{k} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}^k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

We obtain sequence A001333 of the On-Line Encyclopedia of Integer Sequences.

What is the number of the one-sided n -step prudent walks, avoiding k or more consecutive east steps, $\rightarrow^{\geq k}$?

The generating function equals

$$\frac{1 + t - t^k}{1 - 2t - t^2 + t^{k+1}}$$

If $k = 2$, we obtain sequence A006356, counting the number of paths for a ray of light that enters two layers of glass and then is reflected exactly n times before leaving the layers of glass.

If $k = 3$, we obtain sequence A033303.

What is the number of one-sided n -step prudent walks in the first quadrant, starting from $(0,0)$ and ending on the y -axis, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$?

The generating function is

$$\frac{1}{2t^3} \left((1+t)(1-t)^2 - \sqrt{(1-t^4)(1-2t-t^2)} \right).$$

What is the number of one-sided n -step prudent walks *exactly* avoiding \leftarrow^k , taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$?

The generating function equals

$$\frac{1 + t - t^k + t^{k+1}}{1 - 2t - t^2 + t^{k+1} - t^{k+2}}.$$

If $k = 1$, we obtain sequence A078061 .

What is the number of one-sided n -step prudent walks exactly avoiding $\leftarrow^{=k}$ and $\uparrow^{=k}$ (both at the same time)?

The generating function is

$$\frac{1 + t - 2t^k + 2t^{k+1}}{1 - 2t - t^2 + 2t^{k+1} - 2t^{k+2}}.$$

For $k = 1$,

$$f(n) = (2^{n+2} - (-1)^{\lfloor n/2 \rfloor} + 2(-1)^{\lfloor (n+1)/2 \rfloor})/5,$$

also,

$$f(n) = 2f(n-1) - f(n-2) + 2f(n-3)$$

$$\text{with } f(1) = 1, f(2) = 3, f(3) = 7.$$

This is sequence A007909.

Two-sided PSAWs

What is the number of two-sided, n -step prudent walks ending on the top side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$ (both at the same time), taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

Two-sided PSAWs

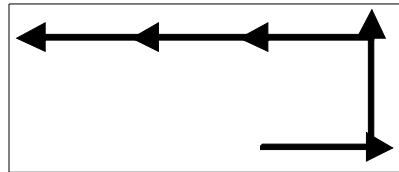
What is the number of two-sided, n -step prudent walks ending on the top side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$ (both at the same time), taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

Theorem *The generating function (say $T(t, u)$) of the above two-sided prudent walks ending on the top side of their box satisfies*

$$\left(1 - t^2u - \frac{tu}{u-t}\right)T(t, u) = 1 + tu + T(t, t)t\frac{u-2t}{u-t},$$

where u counts the distance between the endpoint and the north-east (NE) corner of the box.

A walk takes 5 steps, and the distance between the endpoint and the north-east corner is 3. So we can use $t^5 u^3$ to count this walk.



Solve this generating function for $T(t, u)$ using the Kernel Method:

From

$$\left(1 - t^2u - \frac{tu}{u-t}\right)T(t, u) = 1 + tu + T(t, t)\left(t - \frac{t^2}{u-t}\right),$$

we can get

$$\begin{aligned}(1 - tu)(u - tu - t - t^2u^2 + t^3u)T(t, u) \\ = (u - t)(1 - tu)(1 + tu) - T(t, t)(1 - tu)t(2t - u)\end{aligned}$$

Set $(1 - tu)(u - tu - t - t^2u^2 + t^3u) = 0$, then there is only one *power series* solution for u

$$u = \frac{1}{2t^2} \left(1 - t + t^3 - \sqrt{(1 - t - t^3)^2 - 4t^4}\right).$$

Set

$$(1 + tu)(u - t)(1 - tu) + T(t, t)(1 - tu)t(u - 2t) = 0,$$

and replace u by U :

$$T(t, t) = (1 + tU) \frac{t - U}{t(U - 2t)}.$$

From

$$\begin{aligned} & (1 - tu)(u - t - tu - t^2u^2 + t^3u)T(t, u) \\ & = (u - t)(1 - tu)(1 + tu) - T(t, t)(1 - tu)t(2t - u) \end{aligned}$$

get

$$T(t, u) = \frac{(t - u)(1 - tu)(1 + tu) + T(t, t)(1 - tu)t(2t - u)}{(1 - tu)(u - t - tu - t^2u^2 + t^3u)}.$$

Replace $T(t, t)$ by $T(t, t) = (1 + tU) \frac{t-U}{t(U-2t)}$.

$$T(t, u) = \frac{(1 + tu)(u - t)}{u - t - tu - t^2u^2 + t^3u} - \frac{(1 + tU)(U - t)(1 - tu)(u - 2t)}{(U - 2t)(1 - tu)(u - t - tu - t^2u^2 + t^3u)}$$

Notice that $T(t, 1)$ is the generating function of the number of two-sided n -step prudent walks ending on the top side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$, thus $T(t, 1) =$

$$\frac{(1 - 2t)(1 - t) \sqrt{(1 - t - t^3)^2 - 4t^4} - (1 + t)(1 - 7t + 14t^2 - 11t^3 + 10t^4 - 4t^5)}{2t(1 - 2t - t^2 + t^3)(1 - 2t - 2t^3)}$$

$$= 1 + 3t + 6t^2 + 15t^3 + 35t^4 + 83t^5 + 195t^6 + 460t^7 + 1085t^8 + \dots$$

Note that $T(t, 0)$ is the generating function of the number of two-sided n -step prudent walks ending at the north-east corner of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$, so $T(t, 0) =$

$$\frac{(1-t)\sqrt{(1-t-t^3)^2-4t^4}-1+3t-t^2+t^3+t^4}{(1-2t-2t^3)t}$$

$$= 1 + 2t + 4t^2 + 10t^3 + 24t^4 + 56t^5 + 130t^6 + 304t^7 + 714t^8 + 1678t^9 + \dots$$

Furthermore, $2T(t, 1) - T(t, 0)$ is the generating function of the number of two-sided n -step prudent walks ending on the top side or right side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$, thus

$$2T(t, 1) - T(t, 0) =$$

$$\frac{t(1-t)^2 \sqrt{(1-t-t^3)^2 - 4t^4} + 1 - t - 2t^2 - 2t^3 - 2t^4 + 4t^5 - t^6}{(1-2t-t^2+t^3)(1-2t-2t^3)}$$

$$= 1 + 4t + 8t^2 + 20t^3 + 46t^4 + 110t^5 + 260t^6 + 616t^7 + 1456t^8 + 3442t^9 + \dots$$

Open Problem 1

What is the number of two-sided n -step prudent walks, ending on the top side of their box, avoiding both $\leftarrow^{\geq k}$, and $\downarrow^{\geq t}$ ($k \neq t$) taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

Open Problem 2

What is the number of two-sided n -step prudent walks, ending on the top side of their box, exactly avoiding both $\leftarrow^=k$, $\downarrow^=t$ ($k, t \geq 3$), taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

Let $f(m, n)$ be the number of paths from $(0, 0)$ to (n, n) avoiding $\uparrow^{\geq k}$ and $\rightarrow^{\geq k}$, taking steps from $\{\uparrow, \rightarrow\}$. Then

$$f(m, n) = f(m - 1, n) + f(m, n - 1) - f(m - k, n - 1) - f(m - 1, n - k) + f(m - k, n - k).$$

For $k = 3$, using finite operator calculus:

$$\frac{(1 - t)^2 \sqrt{(1 + t + t^2)(1 - 3t + t^2)} - (1 - 3t + t^2)(1 + t^2)}{t^2(1 - 3t + t^2)} \\ = 2t + 6t^2 + 14t^3 + 34t^4 + 84t^5 + 208t^6 + 518t^7 + \dots$$

This solved a conjecture of R. Stephan

Part Two

Consider lattice walks in the plane with *East*, *North* steps. We count the number of paths from $(0, 1)$ to $((s - r)n + 1, rn + 1)$ that stay weakly above the boundary $(E^{s-r}N^r)^n$.

Let $t_n(i)$ represent the number of paths to (n, i) for points above the boundary. The number of walks to (n, i) follows the recursion $t_n(i) = t_{n-1}(i) + t_n(i - 1)$. Because $t_0(i) = 1$ for all $i \geq 1$, and $t_n(i) = 0$ at all points (n, i) directly below the boundary ($n > 0$), we can uniquely extend the values of $t_n(i)$ to polynomials of degree n on points below the boundary. We call the polynomials again $t_n(i)$. Note that $t_0(x) = 1$ for all x, r, s .

i	1	7	28	84	195	381	662	662
6	1	6	21	56	111	186	281	0
5	1	5	15	35	55	75	95	-281
4	1	4	10	20	20	20	20	-376
3	1	3	6	10	0	0	0	-396
2	1	2	3	4	-10	0	0	-396
1	1	1	1	1	-14	-10	0	-396
0	1	0	0	0	-15	4	-10	-396
-1	1	-1	0	0	-15	19	-14	-386
$n :$	0	1	2	3	4	5	6	7

The polynomials $t_n(i)$ for the case $s = 6, r = 3$

Anna de Mier and Marc Noir showed that the generating function

$$f(z) = \sum_{n \geq 0} t_{(s-r)n+1}(rn + 1)z^n$$

satisfies $f(z) = -(1 - w_1) \cdots (1 - w_{s-r})/z$, where w_1, \dots, w_{s-r} are the unique fractional power series solutions of $(w - 1)^{s-r} - zw^s = 0$. Explicit solutions are hard to get from this relationship when $s - r \nmid r$. The case $r = 2, s = 4$ has been solved by D. Merlini, R. Sprugnoli, M. C. Verri, by different methods.

For $s = 2r$, we proved:

$$\frac{1}{r} \sum_{i=0}^{r-1} t_{rn}(rn - i) = C_{rn},$$

where C_n is the n^{th} Catalan number, $C_n = \frac{1}{n+1} \binom{2n}{n}$.

Example If $r = 2$, then $t_{2n}(2n) + t_{2n}(2n - 1) = 2C_{2n}$.

For $r = 2$ and $s = 4$:

"The Finite Operator Calculus" by Rota, Kahaner, and Odlyzko:

Definition *A B -Sheffer sequence is a sequence of polynomials $(p_i)_{i \in \mathbb{N}_0}$ such that $\deg p_i = i$, $Bp_i = p_{i-1}$, and $p_0 \neq 0$, associated with an operator B that can be written as a power series of order 1 in the derivative operator, \mathcal{D} . If a B -Sheffer sequence has the initial value $p_n(0) = \delta_{0,n}$, then it is a B -basic sequence.*

Lattice paths with steps N and E describe a Sheffer sequence $(t_i(x))$ for the backwards difference operator ∇ , because $\nabla t_n(x) = t_{n-1}(x)$. Since we want to know the values $t_{2n+1}(2n+1)$, we need an operator for the sequence $(t_n(n+x))$, and so we compose the operators E^{-1} and ∇ , where $E^{-1}p_i(x) = p_i(x-1)$, so that $E^{-1}\nabla t_n(n+x) = t_{n-1}(n-1+x)$. The operator $E^{-1}\nabla$ has basic polynomials $b_n(x) = \frac{x}{x+n} \binom{2n+x-1}{n}$.

i	1	7	28	80	185	343	554	554
6	1	6	21	52	105	158	211	0
5	1	5	15	31	53	53	53	-211
4	1	4	10	16	22	0	0	-264
3	1	3	6	6	6	-22	0	-264
2	1	2	3	0	0	-28	-22	-264
1	1	1	1	-3	0	-28	6	-244
0	1	0	0	-4	-3	-28	34	-250
-1	1	-1	0	-4	1	-25	62	-284
$n :$	0	1	2	3	4	5	6	7

The polynomials $t_n(i)$ for the case $s = 4, r = 2$

Because of the boundary given, we get $t_{2n}(2n - 3) = 0$ for all $n > 0$, and $t_{2n+1}(2n - 2) = -t_{2n}(2n) - t_{2n}(2n - 1) = -2C_{2n}$ for all $n \geq 0$. By the Binomial Theorem for Sheffer sequences,

$$\begin{aligned}
 t_n(n) &= \sum_{i=0}^n t_i(i-3) \frac{3}{n+3-i} \binom{2n-2i+2}{n-i} \\
 &= \frac{3}{n+3} \binom{2n+2}{n} + \sum_{i=0}^{(n-1)/2} \frac{3t_{2i+1}(2i-2)}{n+2-2i} \binom{2n-4i}{n-2i-1} \\
 &= \frac{3}{n+3} \binom{2n+2}{n} - 2 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \frac{3C_{2i}}{n+2-2i} \binom{2n-4i}{n-2i-1}
 \end{aligned}$$

For $r = 3$ and $s = 6$:

$$t_n(n) = \sum_{i=0}^n t_i(i-3) \frac{3}{3+n-i} \binom{2(n-i+1)}{n-i}.$$

The zeroes below the given boundary enforce the condition $t_i(i-3) = 0$ unless $i \equiv 1 \pmod{3}$. Therefore this sum simplifies to

$$t_n(n) = \frac{3}{3+n} \binom{2n+2}{n} - \frac{6}{n+2} \binom{2n}{n-1} + \sum_{i=1}^{\frac{n-1}{3}} \frac{t_{3i+1}(3i-2)}{n-3i+2} \binom{2n-6i}{n-3i+1}.$$

Part Three

Paths from $(0, 0)$ to (n, m) , taking steps from $\{\uparrow, \rightarrow, \nearrow\}$, weakly above $y = x$, avoiding $\uparrow^{\geq 2}, \rightarrow^{\geq 2}, \nearrow^{\geq 2}$.

m								
5			1	13	45	47		
4			4	18	20			
3		1	7	9				
2		3	4					
1	1	2						
0	*							
	0	1	2	3	4	5	n	

On the diagonal : 2,4,9,20,47, 273, 676, 1694, 4296,

A035084 $\text{BIK}(b)-b$ where b is A035082 .

A035082 Number of rooted polygonal cacti (Husimi graphs) with n nodes.

FORMULA: Shifts left under transform T where $Ta = \text{EULER}(\text{BIK}(a)-a)$.

$(Y_{n,m}, C_{n,m}, X_{n,m})$: the number of paths from $(0, 0)$ to (n, m) , with a last step $\uparrow, \nearrow, \rightarrow$ respectively

m							
5			(1, 0, 0)	(9, 3, 1)	(20, 13, 12)	(0, 14, 33)	
4			(3, 1, 0)	(9, 5, 4)	(0, 6, 14)		
3		(1, 0, 0)	(4, 2, 1)	(0, 3, 6)			
2		(2, 1, 0)	(0, 1, 3)				
1	(1, 0, 0)	(0, 1, 1)					
0	*						
	0	1	2	3	4	5	n

$$Y_{n,m} = C_{n,m-1} + X_{n,m-1}$$

$$C_{n,m} = Y_{n-1,m-1} + X_{n-1,m-1}$$

$$X_{n,m} = Y_{n-1,m} + C_{n-1,m}$$

$$Y_{n,m} = 0 \text{ for } n \geq m$$

$$C_{n,m} = C_{n-1,m-2} + 2C_{n-2,m-2} + C_{n-2,m-1} + C_{n-1,m-1} \text{ for } m > n$$

$$C_{n,n} = C_{n-1,n-2} + C_{n-2,n-1} + C_{n-1,n-1}$$

$$X_{n,m} = X_{n-1,m-2} + 2X_{n-2,m-2} + X_{n-2,m-1} + X_{n-1,m-1} \text{ for } m > n$$

$$X_{n,n} = X_{n-1,n-1} + 2X_{n-2,n-2} + X_{n-2,n-1} + X_{n-3,n-3}$$

$$Y_{n,m} = Y_{n-1,m-1} + Y_{n-1,m-2} + 2Y_{n-2,m-2} + Y_{n-2,m-1} \text{ for } m > n$$

Using Rota's Finite Operator Calculus to solve $Y_{n,m}$:

m									
9				1	22	151			
8				5	49	179			
7			1	15	67	112			
6			4	25	47	0			
5		1	9	20	0				
4		3	9	0				,	
3	1	4	0						
2	2	0							
1	1	0							
0	*								
	0	1	2	3	4	5	6	n	

<i>m</i>									
6	1	7	30	82	151	179	112		
5	1	6	22	49	67	47	0		
4	1	5	15	25	20	0			
3	1	4	9	9	0				
2	1	3	4	0					
1	1	2	0						
0	1	1							
		0	1	2	3	4	5	6	<i>n</i>

Let $S_n(m)$ be the number.

Let $T_n(m) = S_n(m - n + 1)$

$T_n(m)$:

6	1	7								
5	1	6	30							
4	1	5	22	82						
3	1	5	15	49	151					
2	1	3	9	25	67	179				
1	1	2	4	9	20	47	112	273	676	
0	1	1	0	0	0	0	0	0	0	
		0	1	2	3	4	5	6	7	9

$$\sum_{n \geq 0} T_n(x)t^n = (1 + t) \left(\frac{1 - t - 2t^2 - \sqrt{4t^2(t^2 - 1) + (t - 1)^2}}{2t^3} \right)^x$$

Part Four

Number of “lattice paths” in the first octant satisfying the recursion

$$K(n, m) = K(n, m + 1) + K(n - 1, m - 1) + \eta K(n - 2, m + 1)$$

If $\eta = 1$ then $K(n, 0)$ equals the number of 132-segmented permutations of length n .

1, 2, 6, 18, 57, 190, 654, ...

A125305 in OEIS, **Anders Claesson**:

A permutation is *not* 132-segmented if it has more occurrences of the pattern (1-3-2) than of the pattern (132). For example, $K(4, 0) = 18$, because of the 24 permutations of $\{1, 2, 3, 4\}$ only 1243, 1342, 1423, 1432, 2143, 2413 have more occurrences of the pattern (1-3-2) than of the pattern (132).

Facts about $K(n, m)$

				1	6	27	
			1	5	20	80	
		1	4	14	52	193	
	1	3	9	31	108	381	
1	2	5	16	52	174	602	
1	1	2	6	18	57	190	654

The sequence $K(n, m)$ has the generating function

$$\sum_{n \geq 0} K(n, j) t^n = t^j \mu(t)^{j+1},$$

where $\mu(t) = C(t + \eta t^3) = \frac{1 - \sqrt{1 - 4t(1 + \eta t^2)}}{2t(1 + \eta t^2)}$, and $C(t)$ is the generating function of the Catalan numbers. The compositional inverse $t/\phi(t)$ of $t\mu(t)$ is $\frac{2t(1-t)}{1 + \sqrt{1 + 4\eta t^3(1-t)}}$, hence

$$\phi(t) = \frac{1 + \sqrt{1 + 4\eta t^3(1-t)}}{2(1-t)}$$

$$\begin{aligned}
\sum_{j \geq 0} x^j \sum_{n \geq j} K(n, j) t^n &= \frac{\mu(t)}{1 - xt\mu(t)} = \frac{\frac{1 - \sqrt{1 - 4t(1 + \eta t^2)}}{2t(1 + \eta t^2)}}{1 - xt \frac{1 - \sqrt{1 - 4t(1 + \eta t^2)}}{2t(1 + \eta t^2)}} \\
&= \frac{1 - \sqrt{1 - 4t(1 + \eta t^2)}}{2t(1 + \eta t^2) - xt \left(1 - \sqrt{1 - 4t(1 + \eta t^2)}\right)} \\
&= \frac{1 - \sqrt{1 - 4t(1 + \eta t^2)}}{2t(1 + \eta t^2) - xt + xt \sqrt{1 - 4t(1 + \eta t^2)}} \\
&= \frac{\left(1 - \sqrt{1 - 4t(1 + \eta t^2)}\right) \left(2t(1 + \eta t^2) - xt - xt \sqrt{1 - 4t(1 + \eta t^2)}\right)}{(2t(1 + \eta t^2) - xt)^2 - x^2 t^2 (1 - 4t(1 + \eta t^2))} \\
&= \frac{1 - \sqrt{1 - 4t(1 + \eta t^2)} - 2xt}{2t(1 + \eta t^2)(\eta t^2 + tx^2 + 1 - x)/(1 + \eta t^2)} \\
&= \frac{(1 + \eta t^2)}{1 + \eta t^2 + tx^2 - x} \left(\frac{1 - \sqrt{1 - 4t(1 + \eta t^2)}}{2t(1 + \eta t^2)} - \frac{x}{1 + \eta t^2} \right) \\
&= \frac{(1 + \eta t^2)}{1 + \eta t^2 + tx^2 - x} \left(\mu(t) - \frac{x}{1 + \eta t^2} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{1-(x-\eta t^2-tx^2)} &= \sum_{j \geq 0} (x - \eta t^2 - tx^2)^j = \sum_{j \geq 0} \sum_{i,k} \binom{j}{i,k} x^{j-i+k} (-1)^{i+k} \eta^i t^{2i+k} \\
&= \sum_{n \geq 0} t^n \sum_{j \geq 0} \sum_{i=0}^{n/2} \binom{j}{i, n-2i} x^{j-3i+n} (-1)^{n-i} \eta^i \\
&= \sum_{n \geq 0} t^n \sum_{J \geq n} x^J \sum_{i=0}^{n/2} \binom{J+3i-n}{i, n-2i} (-1)^{n-i} \eta^i = \sum_{j \geq 0} x^j \sum_{n=0}^j t^n \sum_{i=0}^{n/2} \binom{j+3i-n}{i, n-2i} (-1)^{n-i} \eta^i \\
S_{j,n} &= \sum_{i=0}^{n/2} \binom{j+3i-n}{i, n-2i} (-1)^{n-i} \eta^i = \sum_{i=0}^{\infty} \binom{j+3i-n}{i} \binom{j+2i-n}{n-2i} (-1)^{n-i} \eta^i
\end{aligned}$$

Hankel determinants: The first seven values of $|K(i+j, 0)|_{0 \leq i+j \leq n}$ are
 $1, 1, -2, -29, -305, -3761, 20827$ for $n = 0, 1, 2, \dots$

$$\begin{vmatrix}
 1 & 1 & 2 & 6 & 18 & 57 & 190 \\
 1 & 2 & 6 & 18 & 57 & 190 & 654 \\
 2 & 6 & 18 & 57 & 190 & 654 & 2306 \\
 6 & 18 & 57 & 190 & 654 & 2306 & 8290 \\
 18 & 57 & 190 & 654 & 2306 & 8290 & 30272 \\
 57 & 190 & 654 & 2306 & 8290 & 30272 & 111973 \\
 190 & 654 & 2306 & 8290 & 30272 & 111973 & 418666
 \end{vmatrix} = 20827$$

Inverse matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 5 & 3 & 1 & 0 & 0 & 0 & 0 \\ 18 & 16 & 9 & 4 & 1 & 0 & 0 & 0 \\ 57 & 52 & 31 & 14 & 5 & 1 & 0 & 0 \\ 190 & 174 & 108 & 52 & 20 & 6 & 1 & 0 \\ 654 & 602 & 381 & 193 & 80 & 27 & 7 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 2 & -2 & 3 & -4 & 1 & 0 & 0 & 0 \\ -1 & 6 & -4 & 6 & -5 & 1 & 0 & 0 \\ 2 & -6 & 12 & -8 & 10 & -6 & 1 & 0 \\ -6 & 7 & -18 & 21 & -15 & 15 & -7 & 1 \end{pmatrix}$$

The entries of the inverse matrix are the coefficients of the following generating function :

$$\begin{aligned}
\sum_{j \geq 0} t^j \sum_{n \geq j} s_{n,j} x^n &= \sum_{j \geq 0} t^j x^j \phi(x)^{-j-1} = \frac{1}{\phi(x)} \frac{1}{1-xt/\phi(x)} \\
&= \frac{1}{\phi(x)-xt} = \frac{2(1-x)}{1-2(1-x)xt + \sqrt{1+4\eta x^3(1-x)}} \\
\frac{2(1-x)}{1-2(1-x)xt + \sqrt{1+4x^3(1-x)}} &= 1 + (-1+t)x + t(-2+t)x^2 + (-1+t-3t^2+t^3) \\
& x^3 + (-1+t)(t^3-3t^2-2)x^4 + (-1+6t-4t^2+6t^3-5t^4+t^5) \\
& x^5 + (2-6t+12t^2-8t^3+10t^4-6t^5+t^6)x^6 \\
& + (7t-18t^2+21t^3-15t^4+15t^5-7t^6-6+t^7)x^7 + \dots
\end{aligned}$$

$$\begin{aligned}
\sum_{j \geq 0} t^j \sum_{n \geq j} s_{n,j} x^n &= \sum_{j \geq 0} t^j x^j \phi(x)^{-j-1} = \frac{1}{\phi(x)} \frac{1}{1-xt/\phi(x)} = \frac{1}{\phi(x)-xt} = \frac{2(1-x)}{1-2(1-x)xt+\sqrt{1+4\eta x^3(1-x)}} \\
\frac{2(1-x)}{1-2(1-x)xt+\sqrt{1+4x^3(1-x)}} &= 1 + (-1+t)x + t(-2+t)x^2 + (-1+t-3t^2+t^3) \\
& x^3 + (-1+t)(t^3-3t^2-2)x^4 + (-1+6t-4t^2+6t^3-5t^4+t^5) \\
& x^5 + (2-6t+12t^2-8t^3+10t^4-6t^5+t^6)x^6 + (7t-18t^2+21t^3-15t^4+15t^5-7t^6-6+t^7) \\
& x^7 + O(x^{10})
\end{aligned}$$

Thank you very much for attending!