Counting Paths and Walks with Several Step Vectors

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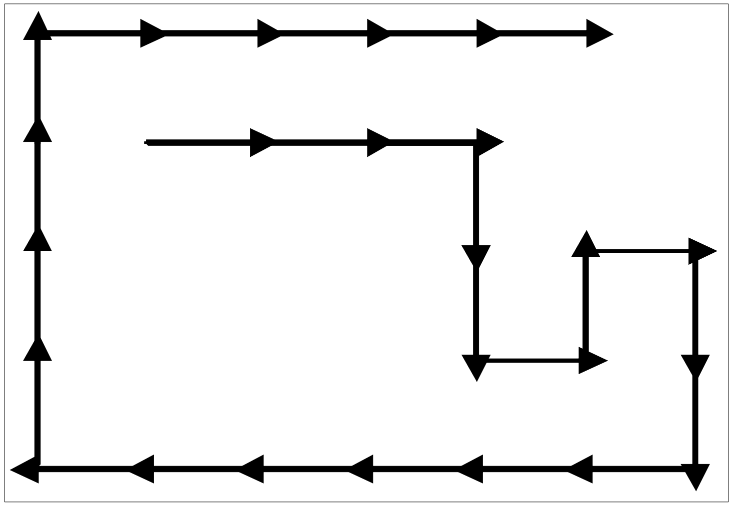
Prudent self-avoiding walks:

A PSAW is a proper subset of SAW on the square lattice. The walk starts at $(0,0)$, and the empty walk is a PSAW. A PSAW grows by adding a step to the end point of a PSAW such that the extension of this step - by any distance - never intersects the walk. Hence the name prudent. The walk is so careful to be self-avoiding that it refuses to take a single step in any direction where it can see - no matter how far away - an occupied vertex.

A.J. Guttmann, E. Duchi, P. Préa, T.M. Garoni, I. Jensen and J.C. Dethridge.

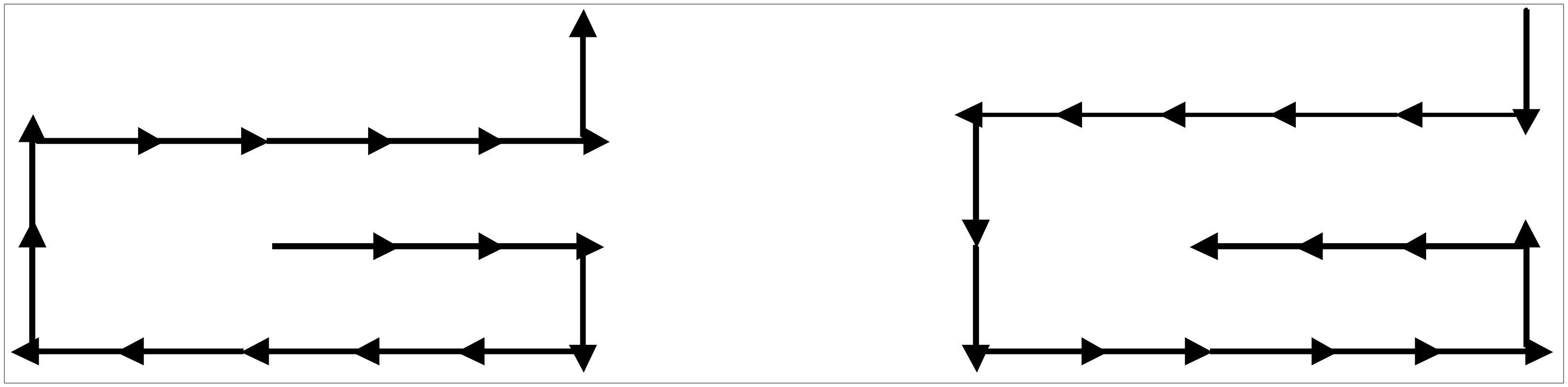
Prudent Self-Avoiding Walks

A PSAW is a proper subset of SAWs on the square lattice. The walk starts at $(0, 0)$, and the empty walk is a PSAW. A PSAW grows by adding a step to the end point of a PSAW such that the extension of this step - by any distance - never intersects the walk. The walk is so careful to be self-avoiding that it refuses to take a single step in any direction where it can see - no matter how far away - an occupied vertex.



Properties of a PSAW

Unlike SAW, PSAW are usually not reversible.



Each PSAW possesses a minimum bounding rectangle, which we call *box*.

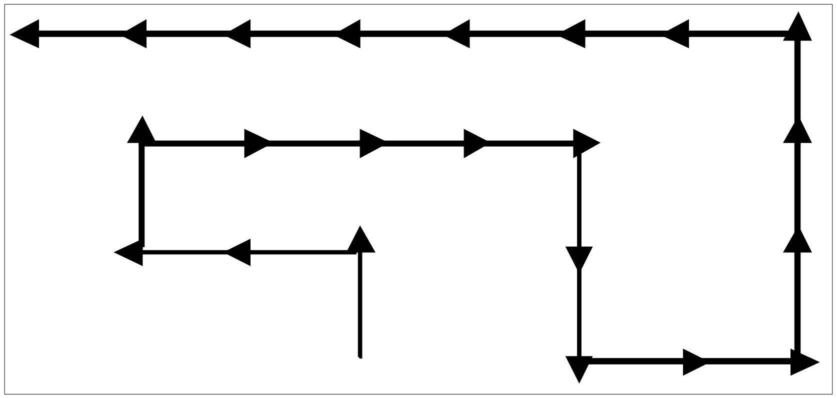
The endpoint of a prudent walk is always a point on the boundary of the box.

Each new step either inflates the box or walks (prudently) along the border.

After an inflating step, there are 3 possibilities for a walk to go on. Otherwise, only 2.

In a *one-sided* PSAW, the endpoint lies always on the top side of the box. The walk is *partially directed*.

A prudent walk is *two-sided* if its endpoint lies always on the top side, or on the right side of the box.



One-sided PSAWs

What is the number (say $f(n)$) of one-sided n-step prudent walks, taking steps from $\{\uparrow,\leftarrow,\rightarrow\}$?

The generating function equals

$$\sum_{n\geq 0} f(n)t^n = \frac{1+t}{1-2t-t^2} = 1+3t+7t^2+17t^3+41t^4+99t^5+239t^6+\dots$$

Also,

$$\begin{aligned} f(n) &= 2f(n-1)+f(n-2) \\ &= \frac{(1-\sqrt{2})^n+(1+\sqrt{2})^n}{2} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\sum_{k=0}^{n+1} \binom{n+1}{k} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}^k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

We obtain sequence $A001333$ of the On-Line Encyclopedia of Integer Sequences.

What is the number of the one-sided n-step prudent walks, avoiding k or more consecutive east steps, $\rightarrow^{\geq k}$?

The generating function equals

$$\frac{1 + t - t^k}{1 - 2t - t^2 + t^{k+1}}$$

If $k = 2$, we obtain sequence $A006356$, counting the number of paths for a ray of light that enters two layers of glass and then is reflected exactly n times before leaving the layers of glass.

If $k = 3$, we obtain sequence $A033303$.

What is the number of one-sided n-step prudent walks in the first quadrant, starting from $(0,0)$ and ending on the y-axis, taking steps from $\{\uparrow,\leftarrow,\rightarrow\}$?

The generating function is

$$\frac{1}{2t^3}\left((1+t)(1-t)^2-\sqrt{(1-t^4)(1-2t-t^2)}\,\right).$$

What is the number of one-sided n-step prudent walks *exactly* avoiding $\leftarrow^{=k}$, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$?

The generating function equals

$$\frac{1 + t - t^k + t^{k+1}}{1 - 2t - t^2 + t^{k+1} - t^{k+2}}.$$

If $k = 1,$ we obtain sequence $A078061$.

What is the number of one-sided n-step prudent walks exactly avoiding $\leftarrow^{=k}$ and $\uparrow^{=k}$ (both at the same time)?

The generating function is

$$\frac{1 + t - 2t^k + 2t^{k+1}}{1 - 2t - t^2 + 2t^{k+1} - 2t^{k+2}}.$$

For $k = 1$,

$$f(n) = (2^{n+2} - (-1)^{\lfloor n/2 \rfloor} + 2(-1)^{\lfloor (n+1)/2 \rfloor})/5,$$

also,

$$f(n) = 2f(n-1) - f(n-2) + 2f(n-3)$$

with $f(1) = 1, f(2) = 3, f(3) = 7.$

This is sequence $A007909$.

Two-sided PSAWs

What is the number of two-sided, n-step prudent walks ending on the top side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$ (both at the same time), taking steps from $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$?

Two-sided PSAWs

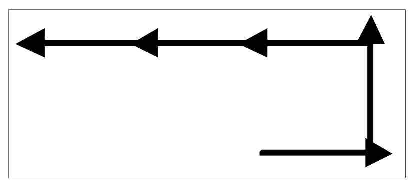
What is the number of two-sided, n-step prudent walks ending on the top side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$ (both at the same time), taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

Theorem *The generating function (say $T(t,u)$) of the above two-sided prudent walks ending on the top side of their box satisfies*

$$\left(1 - t^2 u - \frac{tu}{u-t}\right) T(t,u) = 1 + tu + T(t,t)t\frac{u-2t}{u-t},$$

where u counts the distance between the endpoint and the north-east (NE) corner of the box.

A walk takes 5 steps, and the distance between the endpoint and the north-east corner is 3. So we can use t^5u^3 to count this walk.



Solve this generating function for $T(t,u)$ using the Kernel Method:

From

$$\left(1 - t^2u - \frac{tu}{u-t}\right)T(t,u) = 1 + tu + T(t,t)\left(t - \frac{t^2}{u-t}\right),$$

we can get

$$(1-tu)(u - tu - t - t^2u^2 + t^3u)T(t,u)$$
$$= (u-t)(1-tu)(1+tu) - T(t,t)(1-tu)t(2t-u)$$

Set $(1-tu)(u - tu - t - t^2u^2 + t^3u) = 0$, then there is only one *power series* solution for u

$$u = \frac{1}{2t^2}\left(1 - t + t^3 - \sqrt{(1-t-t^3)^2 - 4t^4}\right).$$

Set

$$(1+tu)(u-t)(1-tu)+T(t,t)(1-tu)t(u-2t) = 0,$$

and replace u by U:

$$T(t,t) = (1+tU)\frac{t-U}{t(U-2t)}.$$

From

$$\begin{aligned}&(1-tu)(u-t-tu-t^2u^2+t^3u)T(t,u)\\ &= (u-t)(1-tu)(1+tu)-T(t,t)(1-tu)t(2t-u)\end{aligned}$$

get

$$T(t,u) = \frac{(t-u)(1-tu)(1+tu)+T(t,t)(1-tu)t(2t-u)}{(1-tu)(u-t-tu-t^2u^2+t^3u)}.$$

Replace $T(t,t)$ by $T(t,t) = (1+tU)\frac{t-U}{t(U-2t)}$.

$$T(t,u) = \frac{(1+tu)(u-t)}{u-t-tu-t^2u^2+t^3u} - \frac{(1+tU)(U-t)(1-tu)(u-2t)}{(U-2t)(1-tu)(u-t-tu-t^2u^2+t^3u)}$$

Notice that $T(t,1)$ is the generating function of the number of two-sided n-step prudent walks ending on the top side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, taking steps from $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$, thus $T(t,1) =$

$$\frac{(1-2t)(1-t)\sqrt{(1-t-t^3)^2-4t^4}-(1+t)(1-7t+14t^2-11t^3+10t^4-4t^5)}{2t(1-2t-t^2+t^3)(1-2t-2t^3)}$$

$$= 1+3t+6t^2+15t^3+35t^4+83t^5+195t^6+460t^7+1085t^8+\ldots$$

Note that $T(t,0)$ is the generating function of the number of two-sided n-step prudent walks ending at the north-east corner of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, taking steps from $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$, so $T(t,0) =$

$$\frac{(1-t)\sqrt{(1-t-t^3)^2-4t^4}-1+3t-t^2+t^3+t^4}{(1-2t-2t^3)t}$$

$$= 1+2t+4t^2+10t^3+24t^4+56t^5+130t^6+304t^7+714t^8+1678t^9+\ldots$$

Furthermore, $2T(t,1) - T(t,0)$ is the generating function of the number of two-sided n-step prudent walks ending on the top side or right side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, taking steps from $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$, thus

$2T(t,1) - T(t,0) =$

$$\frac{t(1-t)^2\sqrt{(1-t-t^3)^2-4t^4}+1-t-2t^2-2t^3-2t^4+4t^5-t^6}{(1-2t-t^2+t^3)(1-2t-2t^3)}$$

$$= 1+4t+8t^2+20t^3+46t^4+110t^5+260t^6+616t^7+1456t^8+3442t^9+\ldots$$

Open Problem 1

What is the number of two-sided n-step prudent walks, ending on the top side of their box, avoiding both $\leftarrow^{\geq k}$, and $\downarrow^{\geq t}$ ($k \neq t$) taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

Open Problem 2

What is the number of two-sided n-step prudent walks, ending on the top side of their box, exactly avoiding both $\leftarrow^{=k}$, $\downarrow^{=t}$ ($k, t \geq 3$), taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

Let $f(m,n)$ be the number of paths from $(0,0)$ to (n,n) avoiding $\uparrow^{\geq k}$ and $\rightarrow^{\geq k}$,taking steps from $\{\uparrow,\rightarrow\}$. Then

$$f(m,n) = f(m-1,n) + f(m,n-1) - f(m-k,n-1) - f(m-1,n-k) + f(m-k,n-k).$$

For $k = 3,$ using finite operator calculus:

$$\frac{(1-t)^2\sqrt{(1+t+t^2)(1-3t+t^2)} - (1-3t+t^2)(1+t^2)}{t^2(1-3t+t^2)}$$

$$= 2t + 6t^2 + 14t^3 + 34t^4 + 84t^5 + 208t^6 + 518t^7 + \dots .$$

This solved a conjecture of R. Stephan

Part Two

Consider lattice walks in the plane with East, North steps. We count the number of paths from $(0, 1)$ to $((s - r)n + 1, rn + 1)$ that stay weakly above the boundary $(E^{s-r}N^{r})^{n}$.

Let $t_n(i)$ represent the number of paths to (n, i) for points above the boundary. The number of walks to (n, i) follows the recursion $t_n(i) = t_{n-1}(i) + t_n(i - 1)$. Because $t_0(i) = 1$ for all $i \geq 1$, and $t_n(i) = 0$ at all points (n, i) directly below the boundary ($n > 0$), we can uniquely extend the values of $t_n(i)$ to polynomials of degree n on points below the boundary. We call the polynomials again $t_n(i)$. Note that $t_0(x) = 1$ for all x, r, s.

i								
	1	7	28	84	195	381	662	*662*
6	1	6	21	56	111	186	281	0
5	1	5	15	35	55	75	95	-281
4	1	4	10	20	*20*	20	20	-376
3	1	3	6	10	0	0	0	-396
2	1	2	3	4	-10	0	0	-396
1	1	*1*	1	1	-14	-10	0	-396
0	1	0	0	0	-15	4	-10	-396
-1	1	-1	0	0	-15	19	-14	-386
n :	0	1	2	3	4	5	6	7

The polynomials $t_n(i)$ for the case $s = 6$, $r = 3$

Anna de Mier and Marc Noir showed that the generating function

$$f(z) = \sum_{n\geq 0} t_{(s-r)n+1}(rn+1)z^n$$

satisfies $f(z) = -(1-w_1)\cdots(1-w_{s-r})/z$, where $w_1, \ldots, w_{s-r}$ are the unique fractional power series solutions of $(w-1)^{s-r} - zw^s = 0$. Explicit solutions are hard to get from this relationship when $s-r \nmid r$. The case $r=2$, $s=4$ has been solved by D. Merlini, R. Sprugnoli, M. C. Verri, by different methods.

For $s = 2r$, we proved:

$$\frac{1}{r}\sum_{i=0}^{r-1} t_{rn}(rn - i) = C_{rn},$$

where C_n is the n^{th} Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Example If $r = 2$, then $t_{2n}(2n) + t_{2n}(2n-1) = 2C_{2n}$.

For $r = 2$ and $s = 4$:

"The Finite Operator Calculus” by Rota, Kahaner, and Odlyzko:

Definition *A B-Sheffer sequence is a sequence of polynomials $(p_i)_{i\in\mathbb{N}_0}$ such that $\deg p_i = i$, $Bp_i = p_{i-1}$, and $p_0 \neq 0$, associated with an operator B that can be written as a power series of order 1 in the derivative operator, $\mathcal{D}$. If a B-Sheffer sequence has the initial value $p_n(0) = \delta_{0,n}$, then it is a B-basic sequence.*

Lattice paths with steps N and E describe a Sheffer sequence $(t_i(x))$ for the backwards difference operator ∇, because $\nabla t_n(x) = t_{n-1}(x)$. Since we want to know the values $t_{2n+1}(2n+1)$, we need an operator for the sequence $(t_n(n+x))$, and so we compose the operators E^{-1} and ∇, where $E^{-1}p_i(x) = p_i(x-1)$, so that $E^{-1}\nabla t_n(n+x) = t_{n-1}(n-1+x)$. The operator $E^{-1}\nabla$ has basic polynomials $b_n(x) = \frac{x}{x+n}\binom{2n+x-1}{n}$.

i	1	7	28	80	185	343	554	*554*
6	1	6	21	52	105	158	211	0
5	1	5	15	31	53	*53*	53	-211
4	1	4	10	16	22	0	0	-264
3	1	3	6	*6*	6	-22	0	-264
2	1	2	3	0	0	-28	-22	-264
1	1	*1*	1	-3	0	-28	6	-244
0	1	0	0	-4	-3	-28	34	-250
-1	1	-1	0	-4	1	-25	62	-284
n :	0	1	2	3	4	5	6	7

The polynomials $t_n(i)$ for the case $s = 4$, $r = 2$

Because of the boundary given, we get $t_{2n}(2n-3) = 0$ for all $n > 0$, and $t_{2n+1}(2n-2) = -t_{2n}(2n) - t_{2n}(2n-1) = -2C_{2n}$ for all $n \geq 0$. By the Binomial Theorem for Sheffer sequences,

$$
\begin{aligned}
t_n(n) &= \sum_{i=0}^{n} t_i(i-3)\frac{3}{n+3-i}\binom{2n-2i+2}{n-i} \\
&= \frac{3}{n+3}\binom{2n+2}{n} + \sum_{i=0}^{(n-1)/2} \frac{3t_{2i+1}(2i-2)}{n+2-2i}\binom{2n-4i}{n-2i-1} \\
&= \frac{3}{n+3}\binom{2n+2}{n} - 2\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \frac{3C_{2i}}{n+2-2i}\binom{2n-4i}{n-2i-1}
\end{aligned}
$$

For $r = 3$ and $s = 6$:

$$t_n(n) = \sum_{i=0}^{n} t_i(i-3)\frac{3}{3+n-i}\binom{2(n-i+1)}{n-i}.$$

The zeroes below the given boundary enforce the condition $t_i(i-3) = 0$ unless $i \equiv 1 \bmod 3$. Therefore this sum simplifies to

$$t_n(n) = \frac{3}{3+n}\binom{2n+2}{n} - \frac{6}{n+2}\binom{2n}{n-1} + \sum_{i=1}^{\frac{n-1}{3}} \frac{t_{3i+1}(3i-2)}{n-3i+2}\binom{2n-6i}{n-3i+1}.$$

Part Three

Paths from $(0,0)$ to (n,m), taking steps from $\{\uparrow, \rightarrow, \nearrow\}$, weakly above $y = x$, avoiding $\uparrow^{\geq 2}$, $\rightarrow^{\geq 2}$, $\nearrow^{\geq 2}$.

m							
5			1	13	45	47	
4			4	18	20		
3		1	7	9			
2		3	4				
1	1	2					
0	$*$						
	0	1	2	3	4	5	n

On the diagonal : 2,4,9,20,47, 273, 676, 1694, 4296,

*A*035084 BIK(b)-b where b is *A*035082 .

*A*035082 Number of rooted polygonal cacti (Husimi graphs) with n nodes.

FORMULA: Shifts left under transform T where Ta = EULER(BIK(a)-a).

$(Y_{n,m}, C_{n,m}, X_{n,m})$: the number of paths from $(0,0)$ to (n,m), with a last step $\uparrow$, $\nearrow$, $\rightarrow$ respectively

m							
5			$(1,0,0)$	$(9,3,1)$	$(20,13,12)$	$(0,14,33)$	
4			$(3,1,0)$	$(9,5,4)$	$(0,6,14)$		
3		$(1,0,0)$	$(4,2,1)$	$(0,3,6)$			
2		$(2,1,0)$	$(0,1,3)$				
1	$(1,0,0)$	$(0,1,1)$					
0	$*$						
	0	1	2	3	4	5	n

$$Y_{n,m} = C_{n,m-1} + X_{n,m-1}$$
$$C_{n,m} = Y_{n-1,m-1} + X_{n-1,m-1}$$
$$X_{n,m} = Y_{n-1,m} + C_{n-1,m}$$
$$Y_{n,m} = 0 \text{ for } n \geq m$$

$$C_{n,m} = C_{n-1,m-2} + 2C_{n-2,m-2} + C_{n-2,m-1} + C_{n-1,m-1} \text{ for } m > n$$
$$C_{n,n} = C_{n-1,n-2} + C_{n-2,n-1} + C_{n-1,n-1}$$
$$X_{n,m} = X_{n-1,m-2} + 2X_{n-2,m-2} + X_{n-2,m-1} + X_{n-1,m-1} \text{ for } m > n$$
$$X_{n,n} = X_{n-1,n-1} + 2X_{n-2,n-2} + X_{n-2,n-1} + X_{n-3,n-3}$$
$$Y_{n,m} = Y_{n-1,m-1} + Y_{n-1,m-2} + 2Y_{n-2,m-2} + Y_{n-2,m-1} \text{ for } m > n$$

Using Rota's Finite Operator Calculus to solve $Y_{n,m}$:

m							
9					1	22	151
8					5	49	179
7				1	15	67	112
6				4	25	47	0
5			1	9	20	0	
4			3	9	0		
3		1	4	0			
2		2	0				
1	1	0					
0	$*$						
	0	1	2	3	4	5	6 n

,

m							
6	1	7	30	82	151	179	112
5	1	6	22	49	67	47	**0**
4	1	5	15	25	20	**0**	
3	1	4	9	9	**0**		
2	1	3	4	**0**			
1	1	2	**0**				
0	1	**1**					
	0	1	2	3	4	5	6 n

Let $S_n(m)$ be the number.

Let $T_n(m) = S_n(m-n+1)$

$T_n(m)$:

6	1	7								
5	1	6	30							
4	1	5	22	82						
3	1	5	15	49	151					
2	1	3	9	25	67	179				
1	1	2	4	9	20	47	112	273	676
0	1	1	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	9

$$\sum_{n\geq 0} T_n(x)t^n = (1+t)\left(\frac{1-t-2t^2-\sqrt{4t^2(t^2-1)+(t-1)^2}}{2t^3}\right)^x$$

Part Four

Number of “lattice paths” in the first octant satisfying the recursion

$$K(n,m) = K(n,m+1) + K(n-1,m-1) + \eta K(n-2,m+1)$$

If $\eta = 1$ then $K(n,0)$ equals the number of 132-segmented permutations of length n.

1, 2, 6, 18, 57, 190, 654, ...

A125305 in OEIS, **Anders Claesson**:

A permutation is *not* 132-segmented if it has more occurrences of the pattern (1-3-2) than of the pattern (132). For example, $K(4,0) = 18$, because of the 24 permutations of $\{1,2,3,4\}$ only 1243, 1342, 1423, 1432, 2143, 2413 have more occurrences of the pattern (1-3-2) than of the pattern (132).

Facts about $K(n,m)$

					1	6	27
				1	5	20	80
			1	4	14	52	193
		1	3	9	31	108	381
	1	2	5	16	52	174	602
1	1	2	6	18	57	190	654

The sequence $K(n,m)$ has the generating function

$$\sum_{n\geq 0} K(n,j)t^n = t^j \mu(t)^{j+1},$$

where $\mu(t) = C(t + \eta t^3) = \frac{1-\sqrt{1-4t\left(1+\eta t^2\right)}}{2t\left(1+\eta t^2\right)}$, and $C(t)$ is the generating function of the Catalan numbers. The compositional inverse $t/\phi(t)$ of $t\mu(t)$ is $\frac{2t(1-t)}{1+\sqrt{1+4\eta t^3(1-t)}}$, hence

$$\phi(t) = \frac{1 + \sqrt{1 + 4\eta t^3(1-t)}}{2(1-t)}$$

$$
\begin{aligned}
\sum_{j\ge 0} x^j \sum_{n\ge j} K(n,j)t^n &= \frac{\mu(t)}{1-xt\mu(t)} = \frac{\frac{1-\sqrt{1-4t\left(1+\eta t^2\right)}}{2t\left(1+\eta t^2\right)}}{1-xt\frac{1-\sqrt{1-4t\left(1+\eta t^2\right)}}{2t\left(1+\eta t^2\right)}} \\
&= \frac{1-\sqrt{1-4t(1+\eta t^2)}}{2t(1+\eta t^2)-xt\left(1-\sqrt{1-4t(1+\eta t^2)}\right)} \\
&= \frac{1-\sqrt{1-4t(1+\eta t^2)}}{2t(1+\eta t^2)-xt+xt\sqrt{1-4t(1+\eta t^2)}} \\
&= \frac{\left(1-\sqrt{1-4t(1+\eta t^2)}\right)\left(2t(1+\eta t^2)-xt-xt\sqrt{1-4t(1+\eta t^2)}\right)}{(2t(1+\eta t^2)-xt)^2-x^2t^2(1-4t(1+\eta t^2))} \\
&= \frac{1-\sqrt{1-4t(1+\eta t^2)}-2xt}{2t(1+\eta t^2)(\eta t^2+tx^2+1-x)/(1+\eta t^2)} \\
&= \frac{(1+\eta t^2)}{1+\eta t^2+tx^2-x}\left(\frac{1-\sqrt{1-4t(1+\eta t^2)}}{2t(1+\eta t^2)}-\frac{x}{1+\eta t^2}\right) \\
&= \frac{(1+\eta t^2)}{1+\eta t^2+tx^2-x}\left(\mu(t)-\frac{x}{1+\eta t^2}\right)
\end{aligned}
$$

$$\frac{1}{1-\left(x-\eta t^2-tx^2\right)} = \sum_{j\geq 0}(x - \eta t^2 - tx^2)^j = \sum_{j\geq 0}\sum_{i,k}\binom{j}{i,k}x^{j-i+k}(-1)^{i+k}\eta^i t^{2i+k}$$

$$= \sum_{n\geq 0} t^n \sum_{j\geq 0}\sum_{i=0}^{n/2}\binom{j}{i,n-2i}x^{j-3i+n}(-1)^{n-i}\eta^i$$

$$= \sum_{n\geq 0} t^n \sum_{J\geq n} x^J \sum_{i=0}^{n/2}\binom{J+3i-n}{i,n-2i}(-1)^{n-i}\eta^i = \sum_{j\geq 0} x^j \sum_{n=0}^{j} t^n \sum_{i=0}^{n/2}\binom{j+3i-n}{i,n-2i}(-1)^{n-i}\eta^i$$

$$s_{j,n} = \sum_{i=0}^{n/2}\binom{j+3i-n}{i,n-2i}(-1)^{n-i}\eta^i = \sum_{i=0}^{\infty}\binom{j+3i-n}{i}\binom{j+2i-n}{n-2i}(-1)^{n-i}\eta^i$$

Hankel determinants: The first seven values of $|K(i+j,0)|_{0\leq i+j\leq n}$ are $1, 1, -2, -29, -305, -3761, 20\,827$ for $n = 0, 1, 2, \ldots$

$$\begin{vmatrix} 1 & 1 & 2 & 6 & 18 & 57 & 190 \\ 1 & 2 & 6 & 18 & 57 & 190 & 654 \\ 2 & 6 & 18 & 57 & 190 & 654 & 2306 \\ 6 & 18 & 57 & 190 & 654 & 2306 & 8290 \\ 18 & 57 & 190 & 654 & 2306 & 8290 & 30\,272 \\ 57 & 190 & 654 & 2306 & 8290 & 30\,272 & 111\,973 \\ 190 & 654 & 2306 & 8290 & 30\,272 & 111\,973 & 418\,666 \end{vmatrix} = 20\,827$$

Inverse matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 5 & 3 & 1 & 0 & 0 & 0 & 0 \\ 18 & 16 & 9 & 4 & 1 & 0 & 0 & 0 \\ 57 & 52 & 31 & 14 & 5 & 1 & 0 & 0 \\ 190 & 174 & 108 & 52 & 20 & 6 & 1 & 0 \\ 654 & 602 & 381 & 193 & 80 & 27 & 7 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 2 & -2 & 3 & -4 & 1 & 0 & 0 & 0 \\ -1 & 6 & -4 & 6 & -5 & 1 & 0 & 0 \\ 2 & -6 & 12 & -8 & 10 & -6 & 1 & 0 \\ -6 & 7 & -18 & 21 & -15 & 15 & -7 & 1 \end{pmatrix}$$

The entries of the inverse matrix are the coefficients of the following generating function :

$$\sum_{j\geq 0} t^j \sum_{n\geq j} s_{n,j} x^n = \sum_{j\geq 0} t^j x^j \phi(x)^{-j-1} = \frac{1}{\phi(x)} \frac{1}{1-xt/\phi(x)}$$

$$= \frac{1}{\phi(x)-xt} = \frac{2(1-x)}{1-2(1-x)xt+\sqrt{1+4\eta x^3(1-x)}}$$

$$\frac{2(1-x)}{1-2(1-x)xt+\sqrt{1+4x^3(1-x)}} = 1 + (-1+t)x + t(-2+t)x^2 + (-1+t-3t^2+t^3)$$

$$x^3 + (-1+t)(t^3-3t^2-2)x^4 + (-1+6t-4t^2+6t^3-5t^4+t^5)$$

$$x^5 + (2-6t+12t^2-8t^3+10t^4-6t^5+t^6)x^6$$

$$+ (7t-18t^2+21t^3-15t^4+15t^5-7t^6-6+t^7)x^7 + \ldots$$

$$\sum_{j\geq 0} t^j \sum_{n\geq j} s_{n,j} x^n = \sum_{j\geq 0} t^j x^j \phi(x)^{-j-1} = \frac{1}{\phi(x)} \frac{1}{1-xt/\phi(x)} = \frac{1}{\phi(x)-xt} = \frac{2(1-x)}{1-2(1-x)xt+\sqrt{1+4\eta x^3(1-x)}}$$

$$\frac{2(1-x)}{1-2(1-x)xt+\sqrt{1+4x^3(1-x)}} = 1 + (-1+t)x + t(-2+t)x^2 + (-1+t-3t^2+t^3)$$

$$x^3 + (-1+t)(t^3-3t^2-2)x^4 + (-1+6t-4t^2+6t^3-5t^4+t^5)$$

$$x^5 + (2-6t+12t^2-8t^3+10t^4-6t^5+t^6)x^6 + (7t-18t^2+21t^3-15t^4+15t^5-7t^6-6+t^7)$$

$$x^7 + O(x^{10})$$

Thank you very much for attending!