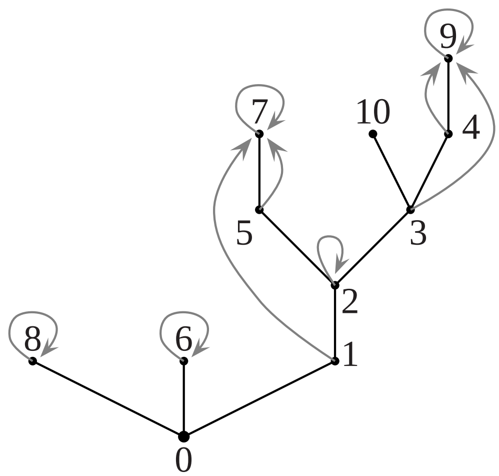
A Bijection to Count (1-23-4)-Avoiding Permutations

n-edge labeled trees = $(n+1)^{n-1}$

n-edge increasing trees = $n!$

n-edge increasing trees in which each non-root vertex has a designated favorite descendant = ?



decorated tree

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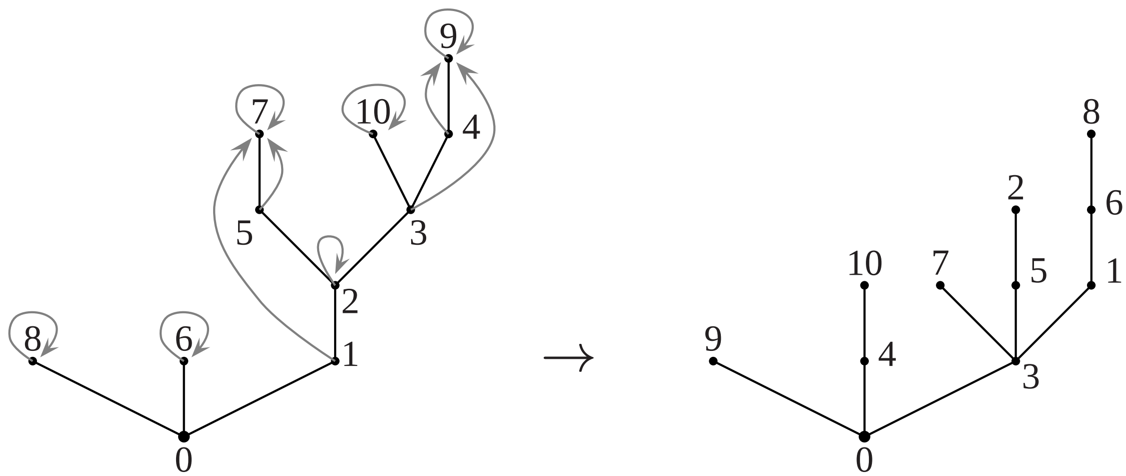


decorated tree labeled tree

List the vertices of the decorated tree in preorder:

$$0\ \ 8\ \ 6\ \ 1\ \ 2\ \ 5\ \ 7\ \ 3\ \ 10\ \ 4\ \ 9.$$

To obtain the parent of non-root vertex i in the labeled tree, follow arc from i in decorated tree and take the successor (in the preorder list) of its terminal vertex. Thus the parent of 1 is 3 (successor of 7).

(1-23-4)-avoiding permutations

Terminology: *ascent, initiator, terminator, record-low segment*. Also *initial* ascent and *interior* ascent (relative to record-low segments).

$$* * * 2 * * 5\,7 * * 8 * * *$$

A permutation is (1-23-4)-avoiding

$\Leftrightarrow$ it contains no four entries, increasing left to right, with the middle two adjacent

$\Leftrightarrow$ for each ascent, either its initiator is a left-to-right minimum or its terminator is a right-to-left maximum.

Another viewpoint using the notion of interior ascent. The left-to-right minima split a permutation into segments:

$$5\ 12\ 8\ 6\ 11\ 9 \qquad 4 \qquad 3\ 7\ 10 \qquad 1\ 2$$

A permutation is

- (1-23)-avoiding $\Leftrightarrow$ it has no interior ascents
- (1-23-4)-avoiding $\Leftrightarrow$ each interior ascent terminates at a right-to-left maximum

Direct counting of (1-23-4)-avoiding permutations

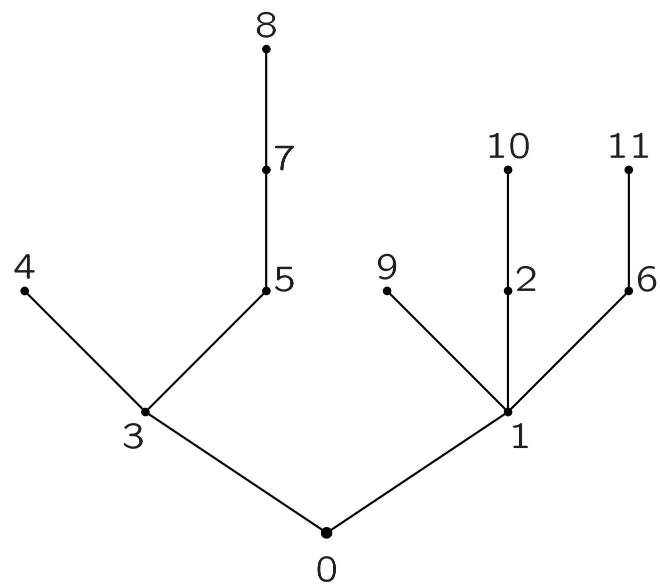
Let $v[n, a]$ = # permutations of $[n]$ that avoid the 3-letter pattern 1-23 and start with a.

Let $u[n, a, m, k]$ = # (1-23-4)-avoiding permutations of $[n]$ that start with a, have n in position k and for which m is the minimum of the first $k - 1$ entries.

There is a complicated recurrence for $u[n, a, m, k]$ in OEIS. Summing over a, m, k counts (1-23-4)-avoiding permutations of $[n]$. The counting sequence begins

$$1,\ 2,\ 6,\ 23,\ 105,\ 549,\ 3207\ldots$$

Increasing Ordered Trees with Increasing Leaves



An increasing ordered tree with increasing leaves

Let $u(n)$ denote the number of size-n increasing ordered trees with increasing leaves and let $u(n,k)$ denote the number of these trees in which the root has k children.

We have $u(n) = \sum_{k=1}^{n} u(n,k)$ and the recurrence

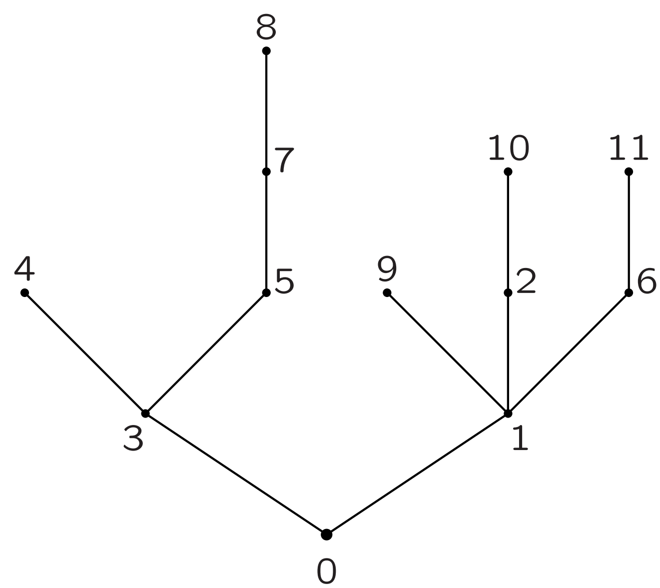
$$u(n,k) = \underbrace{u(n-1,k-1)}_{\text{1 is a leaf}} + \underbrace{k \sum_{j=k}^{n-1} u(n-1,j)}_{\text{1 is not a leaf}}$$

for $1 \le k \le n$.

Jansen: IOT $\rightarrow$ Stirling Permutation

A Stirling permutation of size n is a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that for all $i \in [n]$, all entries between the two occurrences of i exceed i (the Stirling property for i).

Fact: n-edge increasing ordered trees and size-n Stirling permutations are both counted by the double factorial $(2n - 1)!! = 1 \cdot 3 \cdot 5 \cdots 2n - 1$.



Transfer labels to both sides of parent edges. Walk around tree recording labels going up and going down.

3 4 4 5 7 8 8 7 5 3 1 9 9 2 10 10 2 6 11 11 6 1

leaves in tree $\leftrightarrow$ plateaus in permutation

Jansen: IOT $\to$ Stirling Permutation

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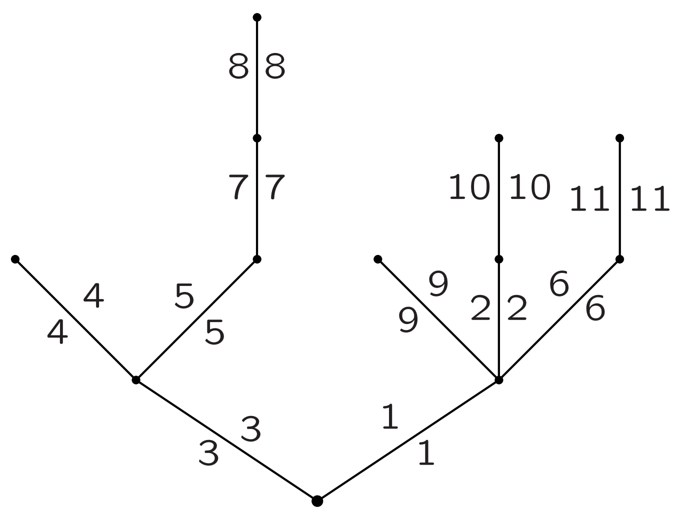

Transfer labels to both sides of parent edges. Walk around tree recording labels going up and going down.

$$3\ 4\ 4\ 5\ 7\ 8\ 8\ 7\ 5\ 3\ 1\ 9\ 9\ 2\ 10\ 10\ 2\ 6\ 11\ 11\ 6\ 1$$

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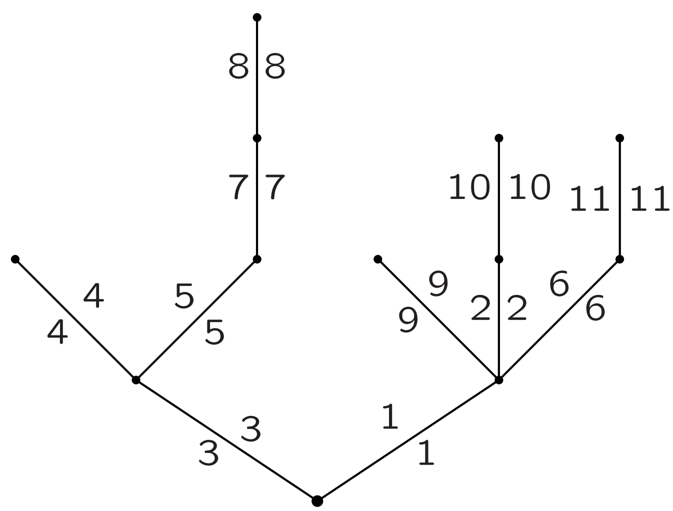

Transfer labels to both sides of parent edges. Walk around tree recording labels going up and going down.

$$3\ \underline{4\ 4}\ 5\ 7\ \underline{8\ 8}\ 7\ 5\ 3\ 1\ \underline{9\ 9}\ 2\ \underline{10\ 10}\ 2\ 6\ \underline{11\ 11}\ 6\ 1$$

leaves in tree $\leftrightarrow$ plateaus in permutation

Bijection: Gessel perms $\to$ Stirling perms

Ira Gessel noted that permutations of the multiset $\{1,1,2,2,...,n,n\}$ in which the second occurrences of $1,2,3,...,n$ occur in that order (the Gessel property) are also counted by $(2n-1)!!$.

$$2\;3\;1\;5\;4\;1\;2\;3\;4\;6\;5\;6$$

A Gessel permutation

Terminology. *Stirling violator* and its *delinquents*.

Given a Gessel permutation, find the largest Stirling violator. Ensquare its *second* occurrence and all its delinquents. Then cyclically shift right the contents of the squares. Repeat until there is no Stirling violator.

$$2\;3\;1\;5\;4\;1\;2\;3\;4\;6\;\boxed{5}\;\boxed{6} \quad\to\quad 2\;3\;1\;5\;4\;1\;2\;3\;4\;6\;\boxed{6}\;\boxed{5} \quad =$$

$$2\;3\;1\;5\;\boxed{4}\;\boxed{1}\;\boxed{2}\;\boxed{3}\;\boxed{4}\;6\;6\;\boxed{5} \quad\to\quad 2\;3\;1\;5\;\boxed{5}\;\boxed{4}\;\boxed{1}\;\boxed{2}\;\boxed{3}\;6\;6\;\boxed{4} \quad =$$

$$\ldots \quad =$$

$$2\;3\;3\;5\;5\;4\;4\;2\;1\;6\;6\;1$$

A Stirling permutation

Bijection: Gessel perms $\to$ Stirling perms

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2 3 1 5 4 1 2 3 4 6 5 6

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$$2\;3\;1\;5\;4\;1\;2\;3\;4\;6\;\boxed{5}\;\boxed{6} \;\to\; 2\;3\;1\;5\;4\;1\;2\;3\;4\;6\;\boxed{6}\;\boxed{5} =$$

$$2\;3\;1\;5\;\boxed{4}\;\boxed{1}\;\boxed{2}\;\boxed{3}\;\boxed{4}\;6\;6\;\boxed{5} \;\to\; 2\;3\;1\;5\;\boxed{5}\;\boxed{4}\;\boxed{1}\;\boxed{2}\;\boxed{3}\;6\;6\;\boxed{4} =$$

$$\ldots =$$

2 3 3 5 5 4 4 2 1 6 6 1

A Stirling permutation

Bijection: Gessel perms $\to$ Stirling perms

Ira Gessel noted that permutations of the multiset $\{1, 1, 2, 2, ..., n, n\}$ in which the second occurrences of $1, 2, 3, ..., n$ occur in that order (the Gessel property) are also counted by $(2n-1)!!$.

$$2\ 3\ 1\ 5\ 4\ 1\ 2\ 3\ 4\ 6\ 5\ 6$$

A Gessel permutation

Terminology. *Stirling violator* and its *delinquents*.

Given a Gessel permutation, find the largest Stirling violator. Ensquare its *second* occurrence and all its delinquents. Then cyclically shift right the contents of the squares. Repeat until there is no Stirling violator.

$$2\ 3\ 1\ 5\ 4\ 1\ 2\ 3\ 4\ 6\ \boxed{5}\ \boxed{6} \quad \to \quad 2\ 3\ 1\ 5\ 4\ 1\ 2\ 3\ 4\ 6\ \boxed{6}\ \boxed{5} \quad =$$

$$2\ 3\ 1\ 5\ \boxed{4}\ \boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ 6\ 6\ \boxed{5} \quad \to \quad 2\ 3\ 1\ 5\ \boxed{5}\ \boxed{4}\ \boxed{1}\ \boxed{2}\ \boxed{3}\ 6\ 6\ \boxed{4} \quad =$$

$$\ldots \quad =$$

$$2\ 3\ 3\ 5\ 5\ 4\ 4\ 2\ 1\ 6\ 6\ 1$$

A Stirling permutation

Bijection: IOT w/ $\uparrow$ leaves $\to$ Stirling Configuration

Apply Jansen to IOT. Then, for each plateau, overline the longest decreasing run starting at its second entry and extract these runs to form a list of decreasing blocks with increasing first entries:

$$3\ 4\ 4\ 5\ 7\ 8\ 8\ 7\ 5\ 3\ 1\ 9\ 9\ 2\ 10\ 10\ 2\ 6\ 11\ 11\ 6\ 1$$

$$\to \quad 4 \quad 8\ 7\ 5\ 3\ 1 \quad 9\ 2 \quad 10\ 2 \quad 11\ 6\ 1$$

A Stirling configuration

A size-n *Stirling configuration* is a list of decreasing blocks characterized by

1. Each $i \in [n]$ occurs at least once and at most twice (a *repeater*)
2. The first entries of the blocks are increasing and are never repeaters
3. The first occurrences of repeaters are all in different blocks
4. The Stirling property for all repeaters

Bijection: IOT w/ $\uparrow$ leaves $\rightarrow$ Stirling Configuration

Apply Jansen to IOT. Then, for each plateau, overline the longest decreasing run starting at its second entry and extract these runs to form a list of decreasing blocks with increasing first entries:

$$3\ 4\ \overline{4}\ 5\ 7\ 8\ \overline{8\ 7\ 5\ 3\ 1}\ 9\ \overline{9\ 2}\ 10\ \overline{10\ 2}\ 6\ 11\ \overline{11\ 6\ 1}$$

$$\rightarrow \quad 4 \quad 8\,7\,5\,3\,1 \quad 9\,2 \quad 10\,2 \quad 11\,6\,1$$

A Stirling configuration

A size-n *Stirling configuration* is a list of decreasing blocks characterized by

1. Each $i \in [n]$ occurs at least once and at most twice (a *repeater*)
2. The first entries of the blocks are increasing and are never repeaters
3. At most one repeater per block
4. The Stirling property

Bijection: (1-23-4)-avoider $\rightarrow$ Avoider Configuration

Step 1. Split into segments and draw overlines starting at ascent and segment initiators.

$$\overline{4\ \ 21\ \ \overline{6\ \ 24\ \ 23\ \ \overline{14\ \ 22\ \ \overline{18\ \ 20}\ \ 16}\ \ 13\ \ \overline{11\ \ 19}\ \ 7}\ \ 5}\ \ /$$

$$\overline{2\ \ \overline{8\ \ 17\ \ 12\ \ 10}}\ \ \ /\ \ \ \overline{1\ \ 15\ \ \overline{3\ \ 9}}$$

Step 2. Extract blocks covered by overlines.

4 21 5   6 24 23 13 7   14 22 16   18 20   11 19 /

2   8 17 12 10   /   1 15   3 9

Step 3. Transform singleton blocks.

4 21 5   6 24 23 13 7   14 22 16   18 20   11 19 /

17   8 2 12 10   /   1 15   3 9

Step 4. Sort each block into decreasing order.

21 5 4   24 23 13 7 6   22 16 14   20 18   19 11 /

17   12 10 8 2   /   15 1   9 3

Bijection: (1-23-4)-avoider → Avoider Configuration

Step 1. Split into segments and draw overlines starting at ascent and segment initiators.

4 21 6 24 23 14 22 18 20 16 13 11 19 7 5 /

2 8 17 12 10 / 1 15 3 9

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21 5 4   24 23 13 7 6   22 16 14   20 18   19 11 /

17   12 10 8 2 / 15 1   9 3

Bijection: (1-23-4)-avoider $\rightarrow$ Avoider Configuration

Step 5. Arrange blocks so first entries increase.

19 11   20 18   21 5 4   22 16 14   24 23 13 7 6 /
12 10 8 2   17 / 9 3   15 1

Step 6. Insert a second copy of the last entry of a block into the next block, maintaining monotonicity.

19 11   20 18 11   21 18 5 4   22 16 14 4   24 23 14 13 7 6 /
12 10 8 2   17 2 / 9 3   15 3 1

Step 7. Finally, erase the dividers and arrange all segments in order of increasing first entry.

9 3   12 10 8 2   15 3 1   17 2   19 11   20 18 11
21 18 5 4   22 16 14 4   24 23 14 13 7 6

The resulting list of blocks is an *avoider configuration* and lives in the same universe as Stirling configurations: Conditions 1 - 3 hold, but the Stirling condition is replaced by other, less transparent, conditions.

Bijection: Avoider Config $\rightarrow$ Stirling Config

So the problem is reduced to finding a bijection from avoider configurations to Stirling configurations.

Recall a Stirling violator is a repeater whose two occurrences enclose a smaller entry.

We must turn Stirling violators in an avoider configuration into Stirling compliers. A Stirling violator is

- of Type 1 if its first occurrence is not the last entry in its block
- of Type 2 if its second occurrence is not the last entry in its block
- of Type 3 if it is neither of Type 1 or 2.

7 5   10 9 5   11 9 8   13 2   15 4   16 8 1   17 14 12 6 1   18 6   19 3 2   20 3

Bijection: Avoider Config $\to$ Stirling Config

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- of Type 2 if its second occurrence is not the last entry in its block
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$7\,5 \quad 10\,9\,5 \quad 11\,9\,8 \quad 13\,2 \quad 15\,4 \quad 16\,8\,1 \quad 17\,14\,12\,6\,1 \quad 18\,6 \quad 19\,3\,2 \quad 20\,3$

Bijection: Avoider Config $\rightarrow$ Stirling Config

Phase 1. As long as there is a Stirling violator of Type 1, take the largest such and transfer the last entry of its first block to its second block.

$$\cdots \qquad *\,4 \qquad \cdots \qquad *\,6\,4 \qquad \cdots \qquad *\,6\,3\,2 \qquad \cdots$$

Phase 2. As long as there is a Stirling violator of Type 2, take the largest such and transfer its second occurrence to the block of its leftmost delinquent.

$$\cdots \qquad *\,6 \qquad *\,7 \qquad *\,8\,5 \qquad \cdots \qquad *\,6\,2 \qquad \cdots$$

Bijection: Avoider Config $\to$ Stirling Config

Phase 3. As long as there is a Stirling violator of Type 3, take the largest such. Let $A_1, A_2, \ldots, A_k$ ($k \geq 1$) denote the blocks containing its delinquents and let A_{k+1} denote the second block of the Stirling violator. Transfer the Stirling violator from A_{k+1} to A_1 and transfer all delinquents in A_i to A_{i+1}, $1 \leq i \leq k$. (This is the "cyclically shift right" operation of the Gessel $\to$ Stirling bijection described earlier).

$$\cdots \quad * \, 6 \quad \cdots \quad * \, \underbrace{5\;3} \quad \cdots \quad * \, \underbrace{4} \quad \cdots \quad * \, 6 \quad \cdots$$

The result is the desired Stirling configuration.

Quotes from two books

Each is the opening sentence in the Preface:

> There are people who feel that a combinatorial result should be given a “purely combinatorial” proof, but I am not one of them.

> First order partial differential equations are encountered in various fields of science and numerous applications (differential geometry, analytical mechanics, solid mechanics, gas dynamics, geometric optics, wave theory, heat and mass transfer, multiphase flows, control theory, differential games, calculus of variations, dynamic programming, chemical engineering sciences, etc...)

A pretty generating function

The mapping

> "Express as a canonical product of cycles,
> then erase parentheses"

is a bijection on permutations—the "First Fundamental Transformation" of Foata. The cycles can be recovered as the record-low segments. Thus cycles $\to$ segments, fixed points $\to$ singleton (short) segments, cycles of length $\geq 2 \to$ long segments, excedances $\to$ ascents.

The GF for permutations by length (x), number of short segments (y), number of long segments = number of initial ascents (z), and number of interior ascents (w) is

$$e^{x(y-z)} \left(\frac{1-w}{1-we^{x(1-w)}} \right)^{\frac{z}{w}} .$$

This GF is obtained by routinely solving a first-order linear PDE with 4 independent variables, x, y, z, w. Lurking in the GF are GFs for Stirling cycle numbers, Eulerian numbers and Bell numbers.