Some General Results for Even-Wilf-Equivalence

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Classical Pattern Avoidance

A permutation $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n$ *contains* pattern $\sigma \in \mathcal{S}_k$ if there is some substring $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}$ which is order-isomorphic to σ. If π does not contain σ, then π *avoids* σ.

Example: 412563 contains 132, but avoids 321.

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Notation

For pattern $\sigma \in \mathcal{S}_k$, let $\mathcal{S}_n(\sigma)$ be the set of permutations of length n which avoid σ, and let $S_n(\sigma) = |\mathcal{S}_n(\sigma)|$ denote the number of such permutations.

Even Permutations

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For permutation π*, let* $\mathrm{inv}(\pi)$ *be the inversion number of* π*, i.e.*

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|.$$

The sign of a permutation is $\mathrm{sgn}(\pi) = (-1)^{\mathrm{inv}(\pi)}$*.*

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Notation

Let $\mathcal{E}_n$ denote the even permutations of length n (i.e., the alternating group).
Let $\mathcal{E}_n(\sigma) = \mathcal{S}_n(\sigma) \cap \mathcal{E}_n$ be the set of even permutations avoiding σ, and $E_n(\sigma) = |\mathcal{E}_n(\sigma)|$ be its size.

Wilf-Equivalence

Two patterns σ, τ are [classically] *Wilf-equivalent* if $S_n(\sigma) = S_n(\tau)$ for all $n \geq 0$. We denote this $\sigma \sim_{\mathcal{S}_n} \tau$.

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Goal

Explore the equivalence relation $\sim_{\mathcal{E}_n}$. In particular, which results regarding classical Wilf-equivalence extend to even-Wilf-equivalence?

Similar Work

Similar questions have already been examined for pattern avoidance by involutions, yielding a concept of involution-Wilf-equivalence. These include:
Guibert (1995), Guibert-Pergola-Pinzani (2001), Jaggard (2003), Bousqet-Mélou-Steingrímsson (2005), Dukes-Jelinek-Mansour-Reifegerste (2007), Jaggard-Marincel (to appear).

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Likewise, explorations of enumeration of classes $\mathcal{E}_n(B)$ for various sets of patterns B have already started.

1. Mansour (2004): Even permutations with k copies of 132
2. Mansour (2006): Even permutations avoiding 132 and another (arbitrary) pattern β
3. Albert-Atkinson-Vatter (2009): Even separable permutations
4. B (PP2009): Enumeration schemes for $E_n(B)$

Outline of Talk

- ✓ Introduction and Statement of Goal
- Elementary Results
- Classification of $\mathcal{S}_2$ and $\mathcal{S}_3$
- Prefix Manipulation
- Classification of $\mathcal{S}_4$
- Partial Classification of $\mathcal{S}_5$ and $\mathcal{S}_6$
- Concluding Remarks

Same Sign Required

We begin with a very simple result.

Lemma

If $\sigma, \tau \in \mathcal{S}_k$ *have different signs, then* $\sigma \not\sim_{\mathcal{E}_n} \tau$.

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Proof.

If σ is even and τ is odd, then $\mathcal{E}_k(\sigma) = \mathcal{E}_k \setminus \{\sigma\}$ while $\mathcal{E}_k(\tau) = \mathcal{E}_k$. □

Symmetries

We define three trivial symmetries, as implied by the dihedral group D_4.

Definition

- The *reverse* of $\pi = \pi_1\pi_2\ldots\pi_n$ is denoted $\pi^r := \pi_n\pi_{n-1}\ldots\pi_1$.
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  Example: $1423^c = 4132$
- The inverse of π is denoted π^{-1}.
  Example: $1423^{-1} = 1342$

Symmetries and Sign

The trivial symmetries affect sign as follows:

Lemma

The sign of a permutation $\pi \in S_n$ in the following ways:

(a.) $\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi^r)$ *if and only if* $n \equiv 0, 1 \pmod 4$.

(b.) $\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi^c)$ *if and only if* $n \equiv 0, 1 \pmod 4$.

(c.) $\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi^{-1})$

Symmetries and Equivalence

For classical Wilf-equivalence, $\sigma \sim_{\mathcal{S}_n} \sigma^r \sim_{\mathcal{S}_n} \sigma^c \sim_{\mathcal{S}_n} \sigma^{-1}$.

This does not transfer to even-Wilf-equivalence, e.g., $1234 \not\sim_{\mathcal{E}_n} 4321$.

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Each orbit over D_4 yields *two* trivial families of even-Wilf-equivlences:

Lemma

For a pattern σ, we have the following trivial equivalences:

- $\sigma \sim_{\mathcal{E}_n} \sigma^{-1} \sim_{\mathcal{E}_n} \sigma^{rc} \sim_{\mathcal{E}_n} \left(\sigma^{-1}\right)^{rc}$
- $\sigma^r \sim_{\mathcal{E}_n} \sigma^c \sim_{\mathcal{E}_n} \left(\sigma^{-1}\right)^r \sim_{\mathcal{E}_n} \left(\sigma^{-1}\right)^c$

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Furthermore, $E_n(21) = 1$ for all $n \geq 1$ while

$$E_n(12) = \begin{cases} 0 & n = 0, 1 \pmod 4, n \geq 2 \\ 1 & \textit{otherwise} \end{cases}$$

Classification of $\mathcal{S}_3$

Simion and Schmidt implicitly classified patterns in $\mathcal{S}_3$ by $\sim_{\mathcal{E}_n}$. They count $E_n(\sigma) - O_n(\sigma)$ for $\sigma \in \mathcal{S}_3$. Their results imply:

Corollary (Simion and Schmidt (1985))

- $123 \sim_{\mathcal{E}_n} 231 \sim_{\mathcal{E}_n} 312$
- $321 \sim_{\mathcal{E}_n} 213 \sim_{\mathcal{E}_n} 132$

Observe the two even-Wilf-equivalence classes are $\mathcal{S}_3 \cap \mathcal{E}_3$ and $\mathcal{S}_3 \setminus \mathcal{E}_3$.
(For pattern-avoidance by involutions, the equivalence class $\mathcal{S}_3$ splits similarly into $\mathcal{S}_3 \cap \mathcal{I}_3$ and $\mathcal{S}_3 \setminus \mathcal{I}_3$)

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This suggests: If $\sigma \sim_{\mathcal{S}_n} \tau$ and $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\tau)$ then $\sigma \sim_{\mathcal{E}_n} \tau$.
This is false: e.g., $1234 \not\sim_{\mathcal{E}_n} 4321$ although $1234, 4321 \in \mathcal{E}_4$

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Direct Sum

The next few results make use of the *direct sum* of two patterns.

The direct sum of two permutations, $\alpha \in \mathcal{S}_k$ and $\beta \in \mathcal{S}_\ell$, is the length-$(k+\ell)$ permutation

$$\alpha \oplus \beta := \alpha_1\alpha_2\cdots\alpha_k(\beta_1+k+1)(\beta_2+k+1)\cdots(\beta_\ell+k+1).$$

This is most easily seen as placing β above and to the right of α.

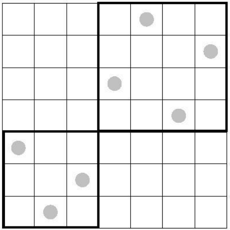

Figure: $312 \oplus 2413 = 3125746$

Prefix Reversal

The following result, nicknamed “prefix reversal,” has been instrumental in the classical case of Wilf-equivalence.

Theorem (Backelin, West, Xin (2007))

$t(t-1)\ldots 21 \oplus \sigma \overset{s}{\sim}_{\mathcal{S}_n} 12\ldots(t-1)t \oplus \sigma$ *for any pattern* σ*.*

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This will not extend directly to even-Wilf-equivalence, as indicated by $123 \not\sim_{\mathcal{E}_n} 321$ and $1234 \not\sim_{\mathcal{E}_n} 4321$. Something weaker *does* extend, however.

Prefix Manipulation

Proposition (Backelin, West, Xin (2007))

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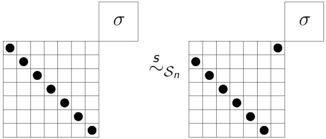

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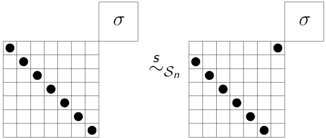


This proposition restricts to even-Wilf-equivalence in certain cases.

Proposition (B. and Jaggard (2010))

If t is odd, then $t(t-1)\ldots 21 \oplus \sigma \overset{s}{\sim}_{\mathcal{E}_n} (t-1)\ldots 21t \oplus \sigma$ for any pattern σ.

Transversals in Young Diagrams

A transversal π in Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a placement of n rooks in boxes of λ such that there is exactly one rook in every row and column. Clearly π can be written as a permutation in $\mathcal{S}_n$.

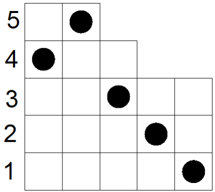


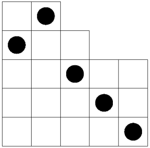
Figure: Transversal $\pi = 45321$ of $\lambda = (5, 5, 5, 3, 2)$.

A transversal π is *even* if π is even as a permutation.

Pattern Avoidance for Transversals

A transversal π of Young diagram λ *contains* σ if

Example:



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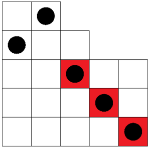

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A transversal π of Young diagram λ *contains* σ if

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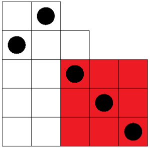

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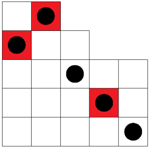

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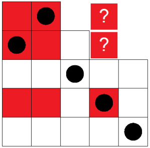

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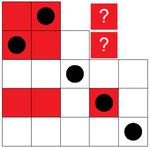

Note: Pattern avoidance is dependent on λ, but sign is not.

Shape-Wilf-Equivalence

Definition

Let $\mathcal{S}_\lambda(\sigma)$ be the set of transversals of λ avoiding σ, and $S_\lambda(\sigma) = |\mathcal{S}_\lambda(\sigma)|$.
If $S_\lambda(\sigma) = S_\lambda(\tau)$ for all λ, then σ and τ are shape-Wilf-equivalent and we write $\sigma \overset{s}{\sim}_{\mathcal{S}_n} \tau$.

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This definition extends to even transversals as well.

Definition

Let $\mathcal{E}_\lambda(\sigma)$ be the set of *even* transversals of λ avoiding σ, and $E_\lambda(\sigma) = |\mathcal{E}_\lambda(\sigma)|$.
If $E_\lambda(\sigma) = E_\lambda(\tau)$ for all λ, then σ and τ are even-shape-Wilf-equivalent and we write $\sigma \overset{s}{\sim}_{\mathcal{E}_n} \tau$.

Shape-Wilf-Equivalence and Direct Sums

Backelin, West, and Xin show that shape-Wilf-equivalence combines well with direct sums.

Lemma (Backelin, West, Xin (2007))

For patterns α and β, $\alpha \overset{s}{\sim}_{\mathcal{S}_n} \beta$ implies $\alpha \oplus \sigma \overset{s}{\sim}_{\mathcal{S}_n} \beta \oplus \sigma$.

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Prefix Manipulation

By the previous lemma, $t(t-1)\cdots 21 \oplus \sigma \overset{s}{\sim}_{\mathcal{S}_n} (t-1)\cdots 21t \oplus \sigma$ follows from a proof that

$$S_\lambda(t(t-1)\cdots 21) = S_\lambda((t-1)\cdots 21t).$$

Backelin, West, and Xin provide a bijection $\phi_t^* : \mathcal{S}_\lambda((t-1)\cdots 21t) \rightarrow \mathcal{S}_\lambda(t(t-1)\cdots 21)$.

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We will demonstrate that ϕ_t^* preserves sign when t is odd.

This implies for odd t,

$$E_\lambda(t(t-1)\cdots 21) = E_\lambda((t-1)\cdots 21t),$$

which implies $t(t-1)\cdots 21 \oplus \sigma \overset{s}{\sim}_{\mathcal{E}_n} (t-1)\cdots 21t \oplus \sigma$.

A bijective proof

Let $J_t = t(t-1)\cdots 21$ and $F_t = (t-1)\cdots 21t$.

We first recall the bijection $\phi_t^* : \mathcal{S}_\lambda(F_t) \to \mathcal{S}_\lambda(J_t)$ as constructed by Backelin et al.

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It works by converting copies of J_t into copies of F_t via an iterated operation ϕ_t.

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

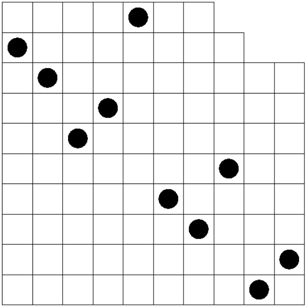

Figure: The transversal π in $\mathcal{S}_\lambda(F_5)$

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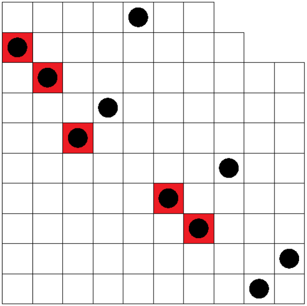

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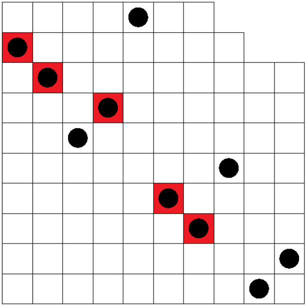

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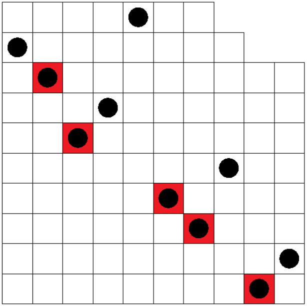

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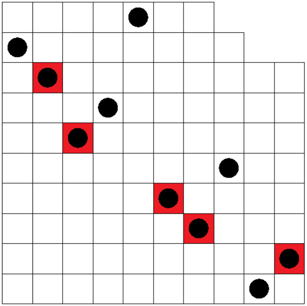

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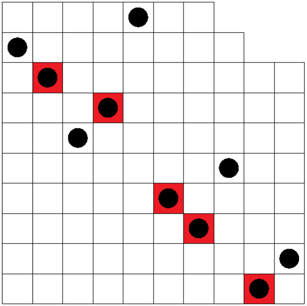

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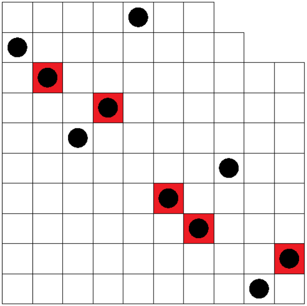

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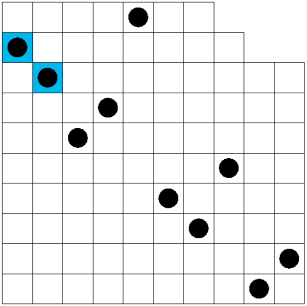

Figure: Candidates for "first letter."

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

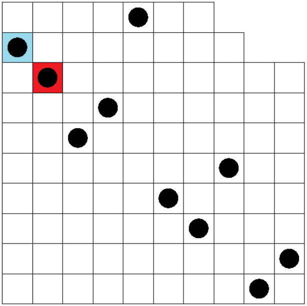

Figure: Select the lowest "first letter," a_1

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

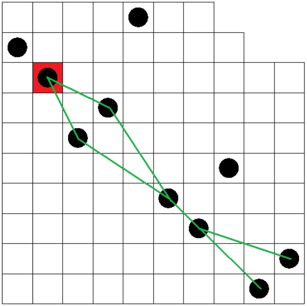

Figure: This a_1 participates in four J_5's

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

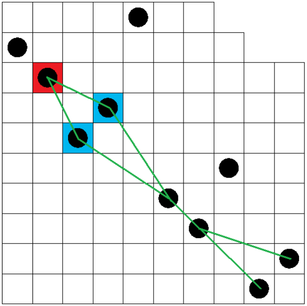

Figure: Two candidates for a_2

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

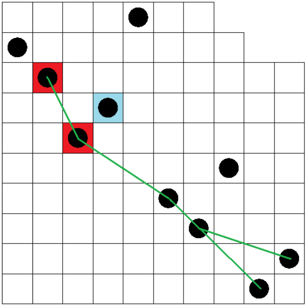

Figure: Select the leftmost candidate for a_2

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

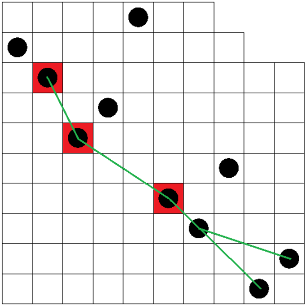

Figure: Choose leftmost candidate for a_3

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

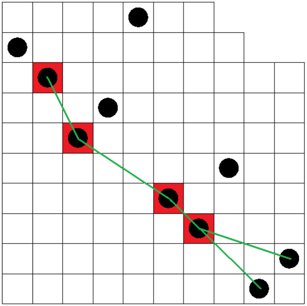

Figure: Choose leftmost candidate for a_4

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

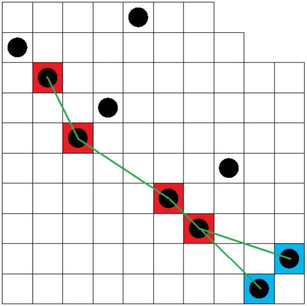

Figure: Two candidates for a_5

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

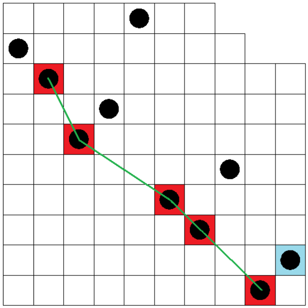

Figure: Choose the leftmost candidate for a_5

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

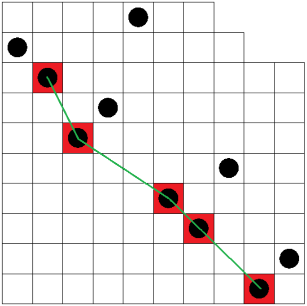

Figure: We have now selected a J_5

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

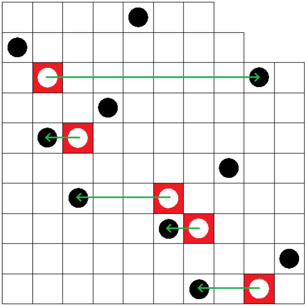

Figure: Rearrange the selected J_5 into an F_5

The map ϕ_t in pictures (for $t = 5$)

Suppose $\pi \in \mathcal{S}_\lambda(F_t)$. Then an application of ϕ_t proceeds as follows:

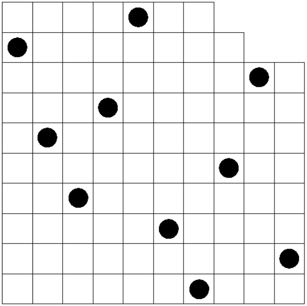

Figure: Left with a new transversal, π'

More about ϕ_t

One application of ϕ_t does not remove all copies of J_t.

The operation ϕ_t is repeated until all occurences of J_t are removed.

The iterated map $\phi_t^* : \mathcal{S}_\lambda(F_t) \to \mathcal{S}_\lambda(J_t)$ is a bijection, with inverse $\left(\phi_t^{-1}\right)^*$

This bijection provides the proof for $F_t \overset{s}{\sim}_{\mathcal{S}_n} J_t$, which implies $F_t \oplus \sigma \overset{s}{\sim}_{\mathcal{S}_n} J_t \oplus \sigma$.

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The map ϕ_t and sign

Claim

The operation ϕ_t preserves sign if and only if t is odd.

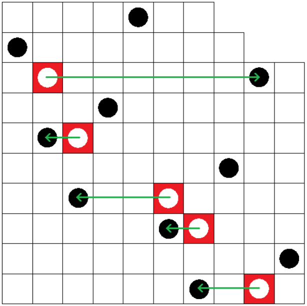

Figure: The change from J_t to F_t

The map ϕ_t and sign

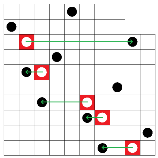

Figure: The change from J_t to F_t

This is multiplication by the cyclic permutation $(a_1\, a_2 \cdots a_t)$, which is an even permutation if and only if t is odd.

Thus ϕ_t is sign-preserving if and only if t is odd.

Therefore, if t is odd then ϕ_t^* preserves sign.

Hence $J_t \overset{s}{\sim}_{\mathcal{E}_n} F_t$ when t is odd, so $J_t \oplus \sigma \overset{s}{\sim}_{\mathcal{E}_n} F_t \oplus \sigma$ when t is odd.

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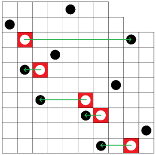

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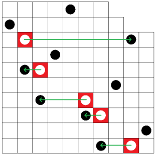

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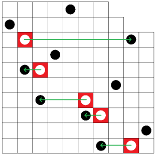

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When t is even

Observe that if t is even, each non-trivial application of ϕ_t reverses sign.

However, ϕ_t may be iterated an even or odd number of times dependent on the given $\pi \in \mathcal{S}_\lambda(F_t)$. Hence ϕ_t^* does not respect sign when t is even.

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Other extensions do not work

Let $I_t = 12 \cdots t$ be the increasing permutation.
Backelin et al. actually prove $J_t \overset{s}{\sim}_{\mathcal{S}_n} J_k \oplus I_{t-k}$ for any $0 \leq k \leq t$.
This does not hold for $\overset{s}{\sim}_{\mathcal{E}_n}$, nor even $\sim_{\mathcal{E}_n}$. Confirmed computationally:

$$E_7(54321) = E_7(4321\,5) < E_7(321\,45) = E_7(21\,345) < E_7(1\,2345)$$

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- ✓ Elementary Results
- ✓ Classification of $\mathcal{S}_2$ and $\mathcal{S}_3$
- ✓ Prefix Manipulation
- Classification of $\mathcal{S}_4$
- Partial Classification of $\mathcal{S}_5$ and $\mathcal{S}_6$
- Concluding Remarks

Classification of $\mathcal{S}_4$

We have thus shown $3214 \sim_{\mathcal{E}_n} 2134$ which, when combined with the equivalences implied by symmetries and computation of $E_n(\sigma)$ for $n \leq 7$, completes the classification of length 4 patterns under even-Wilf-equivalence.

1234	4321
2143	3412
1243	3421
2134	4312
1432	2341
3214	4123

2314	4132
1423	3241
1342	2431
3124	4213
2413	3142

1324	4231

Figure: Equivalence classes under $\sim_{\mathcal{E}_n}$

Classification of $\mathcal{S}_4$

σ	sgn(σ)	$E_4(\sigma)$	$E_5(\sigma)$	$E_6(\sigma)$	$E_7(\sigma)$	$E_8(\sigma)$	$E_9(\sigma)$	$E_{10}(\sigma)$
1243*	−1	12	52	257	1381	7885	47181	293297
2134	−1	12	52	257	1381	7885	47181	293297
3214	−1	12	52	257	1381	7885	47181	293297
1432	−1	12	52	257	1381	7885	47181	293297
3421*	−1	12	52	256	1380	7885	47181	293293
4312	−1	12	52	256	1380	7885	47181	293293
2341	−1	12	52	256	1380	7885	47181	293293
4123	−1	12	52	256	1380	7885	47181	293293
2314	1	11	51	257	1371	7742	45622	277826
1423	1	11	51	257	1371	7742	45622	277826
3124	1	11	51	257	1371	7742	45622	277826
1342	1	11	51	257	1371	7742	45622	277826
4132	1	11	51	255	1369	7742	45622	277836
3241	1	11	51	255	1369	7742	45622	277836
4213	1	11	51	255	1369	7742	45622	277836
2413	1	11	51	255	1369	7742	45622	277836
2413	−1	12	52	256	1370	7743	45623	277831
3142	−1	12	52	256	1370	7743	45623	277831
1234*	1	11	51	258	1382	7879	47175	293311
4321*	1	11	51	255	1379	7879	47175	293279
2143	1	11	51	256	1380	7885	47181	293301
3412	1	11	51	257	1381	7885	47181	293289
1324	−1	12	52	258	1382	7903	47393	296002
4231	−1	12	52	255	1380	7903	47393	295948

Partial Classification of $\mathcal{S}_5$

Computations of $E_n(\sigma)$ for $n \leq 11$ and $\sigma \in \mathcal{S}_5$ suggest there are four even-Wilf-equivalence classes which contain patterns which are not trivially equivalent under symmetries.
Some of the putative equivalences can be proven by prefix manipulation and symmetry:

Corollary

- $54321 \sim_{\mathcal{E}_n} 43215 \sim_{\mathcal{E}_n} 15432$
- $32154 \sim_{\mathcal{E}_n} 21354 \sim_{\mathcal{E}_n} 21543$
- $12345 \sim_{\mathcal{E}_n} 51234 \sim_{\mathcal{E}_n} 23451$
- $45123 \sim_{\mathcal{E}_n} 45312 \sim_{\mathcal{E}_n} 34512$
- $32145 \sim_{\mathcal{E}_n} 21345 \sim_{\mathcal{E}_n} 12354 \sim_{\mathcal{E}_n} 12543$
- $54123 \sim_{\mathcal{E}_n} 54312 \sim_{\mathcal{E}_n} 34521 \sim_{\mathcal{E}_n} 45321$

Conjectures for Length 5 Patterns

There remain a few conjectured equivalences for length 5 patterns. In the classical case, these were proven by symmetries and prefix reversal.

Conjecture

- $12345 \sim_{\mathcal{E}_n} 45123$ *(equivalently,* $54321 \sim_{\mathcal{E}_n} 32154$*)*
- $12354 \sim_{\mathcal{E}_n} 45321$
- $13524 \sim_{\mathcal{E}_n} 42531$

The first conjecture implies
$12345 \sim_{\mathcal{E}_n} 2345\,1 \sim_{\mathcal{E}_n} 345\,12 \sim_{\mathcal{E}_n} 45\,123 \sim_{\mathcal{E}_n} 5\,1234$.

Conjectures for Length 5 Patterns

Two of the previous conjectures have the form $\sigma \sim_{\mathcal{E}_n} \sigma^r$, which suggests:

Question

When is $\sigma \sim_{\mathcal{E}_n} \sigma^r$?

This will only occur for $\sigma \in \mathcal{S}_k$ where $k = 0, 1 \pmod 4$, since otherwise $\operatorname{sgn}(\sigma) \neq \operatorname{sgn}(\sigma^r)$.

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If $\sigma^r = \sigma^{-1}$, then $\sigma \sim_{\mathcal{E}_n} \sigma^r$, but this not necessary.

Length 4 and 5 patterns which are even-Wilf-equivalent to their reverses:

- $2413 \sim_{\mathcal{E}_n} 3142$ $(\sigma^r = \sigma^{-1})$
- $25314 \sim_{\mathcal{E}_n} 41352$ $(\sigma^r = \sigma^{-1})$
- $12354 \sim_{\mathcal{E}_n} 45321$ (conjectured based on $n \leq 11$)
- $12543 \sim_{\mathcal{E}_n} 34521$ (conjectured based on $n \leq 11$)
- $13524 \sim_{\mathcal{E}_n} 42531$ (conjectured based on $n \leq 11$)

Partial Classification of $\mathcal{S}_6$

For patterns of length 6, prefix manipulation and symmetries account for all instances of even-Wilf-equivalence except for one conjectured class (and its reverse)

Conjecture

$231564 \sim_{\mathcal{E}_n} 312564$ *(equivalently,* $465132 \sim_{\mathcal{E}_n} 465213$*)*

We have confirmed $E_n(231564) = E_n(312564)$ for $n \leq 11$.

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It was shown by Stankova and West (2002) that $231564 \overset{s}{\sim}_{\mathcal{S}_n} 312564$ when they showed that $231 \overset{s}{\sim}_{\mathcal{S}_n} 312$. This suggests the following stronger conjecture:

Conjecture

$231 \overset{s}{\sim}_{\mathcal{E}_n} 312$

We have confirmed $E_\lambda(231) = E_\lambda(312)$ for all shapes λ which fit in a 9×9 square.

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Reflections on Even-Wilf-Equivalence

So far, all proven and conjectured even-Wilf-equivalences are between classically Wilf-equivalent patterns. This suggests:

Conjecture

Even-Wilf-equivalence implies classical Wilf-equivalence.

The analogous conjecture is still open for avoidance by involutions.

Number of equivalence classes

The even-Wilf-equivalence relation is a very strong condition. Consider the number of equivalence classes under $\sim_{\mathcal{S}_n}$ versus $\sim_{\mathcal{E}_n}$

n	2	3	4	5	6
Trivial Wilf-classes	1	2	7	23	115
Wilf-equivalence	1	1	3	16	91
Trivial Even-Wilf-classes	2	4	13	45	230
even-Wilf-equivalence	2	2	11	$[35, 39]$	$\{216, 218\}$

It appears that for each n there are at least twice as many equivalence classes under even-Wilf-equivalence as classical Wilf-equivalence.

Relaxing Conditions

There are many pairs (σ, τ) where $E_n(\sigma) = E_n(\tau)$ for infinitely many, but not all, $n \geq 0$:

- $E_n(\sigma) = E_n(\sigma^r) = E_n(\sigma^c)$ for any $n = 0, 1 \pmod 4$
- Data suggest instances of $E_{2n}(\sigma) = E_{2n}(\tau)$, e.g., 12345 and 12354
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Asymptotic equivalence may also be interesting, where σ and τ are asymptotically even-Wilf-equivalent if $E_n(\sigma) \sim E_n(\tau)$ as $n \to \infty$.

Other curious behavior of $E_n(\sigma)$

Using enumeration schemes, it has been determined for $n \leq 15$ that:

$$E_n(1234) - E_n(1243) = 0, 0, 0, -1, -1, 1, 1, -6, -6, 14, 14, -69, -69, 332, 332, \ldots$$

Observe the sign changes, depending on $n \pmod 4$. Perhaps of note is that $1234 \sim_{\mathcal{S}_n} 1243$.

Conclusion

- We have proven that $t(t-1)\cdots 21 \oplus \sigma \sim_{\mathcal{E}_n} (t-1)\cdots 21\, t \oplus \sigma$ when t is odd by refining a result of Backelin, West, and Xin.
- We have classified patterns in $\mathcal{S}_4$ according to $\sim_{\mathcal{E}_n}$, and partially classified $\mathcal{S}_5$ and $\mathcal{S}_6$
- Question: When is $\sigma \sim_{\mathcal{E}_n} \sigma^r$? A full characterization would complete the classification of $\mathcal{S}_5$.
- Conjecture: $231 \overset{s}{\sim}_{\mathcal{E}_n} 312$, which refines a result of Stankova and West. This would complete the classification of $\mathcal{S}_6$.
- Conjecture: If $\sigma \sim_{\mathcal{E}_n} \tau$, then $\sigma \sim_{\mathcal{S}_n} \tau$.

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