Consecutive Pattern Avoidance

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Outline of talk

Notation and definitions

- $\pi \leq_c \alpha$ if α has a consecutive subsequence order isomorphic to π
  - $132 \leq_c 5274316$
  - $132 \not\leq_c 5247316$ "5247316 avoids 132"
- $\leq_c$ is a partial order on permutations.
- $\mathrm{Cav}(\Pi)$ is the set of permutations that avoid every permutation of Π.
- All generating functions are exponential generating functions.

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  - $132 \not\leq_c 5247316$ "5247316 avoids 132"
- $\leq_c$ is a partial order on permutations.
- $\mathrm{Cav}(\Pi)$ is the set of permutations that avoid every permutation of Π. t_n is number of permutations of length n in $\mathrm{Cav}(\Pi)$
- All generating functions are exponential generating functions. Such as

$$\sum_{n=0}^{\infty} \frac{t_n x^n}{n!}$$

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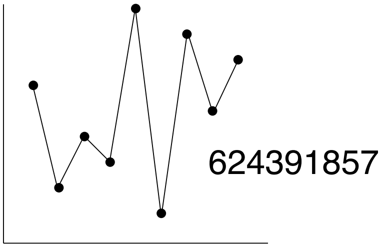

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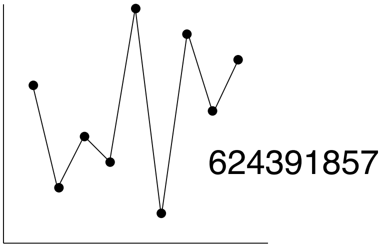
- $2\sec x + 2\tan x - x - 1$ (André 1879)



Some previous work

- [Kitaev, 2003]
  - Results for $\mathrm{Cav}(\Pi)$, Π contains length 3 permutations
  - Incomplete for $\mathrm{Cav}(123, 231, 312)$, $\mathrm{Cav}(123, 231)$, $\mathrm{Cav}(132, 312)$, $\mathrm{Cav}(132, 213)$, $\mathrm{Cav}(123)$, $\mathrm{Cav}(132)$
- [Kitaev and Mansour, 2005]
  - $\mathrm{Cav}(123, 231, 312)$
- [Elizalde and Noy, 2003]
  - $\mathrm{Cav}(12\cdots k)$
  - $\mathrm{Cav}(132)$
  - $\mathrm{Cav}(123, 231)$
- [Elizalde, 2006]
  - For $|\Pi| = 1$, $\lim_{n\to\infty} \sqrt[n]{\frac{t_n}{n!}}$ exists, and
  - lies strictly between 0 and 1
- [Liese and Remmel, 2010]
  - Many results for $|\Pi| = 1$

Revisiting $12\cdots k$ consecutive avoidance

- Elizalde and Noy enumeration of $\mathrm{Cav}(12\cdots k)$ did much more: the entire distribution of permutations according to length and number of occurrences of $12\cdots k$
- Another way of obtaining $\mathrm{Cav}(12\cdots k)$ via a different "more": the entire distribution of permutations in $\mathrm{Cav}(12\cdots k)$ according to length and value of the last term

Major steps 1

- $u_{na}^{(t)}$ defined as the number of permutations π such that
  1. π avoids $12\cdots k$
  2. $|\pi| = n$
  3. π ends in a
  4. π ends with t ascents
- Recurrences

$$u_{na}^{(0)} = \sum_{b:b\geq a}\left(u_{n-1,b}^{(0)} + u_{n-1,b}^{(1)} + \cdots + u_{n-1,b}^{(k-2)}\right)$$

$$u_{na}^{(t)} = \sum_{b:b<a} u_{n-1,b}^{(t-1)}$$

Major steps 2

- Change of variable: $v_{ij}^{(t)} = u_{i+j+1,t+1}^{(t)}$
- Recast as equations for the generating functions $V^{(t)}(x,y)$

$$
\begin{aligned}
\frac{\partial V^{(0)}}{\partial y} &= \frac{\partial V^{(0)}}{\partial x} + V^{(0)} + V^{(1)} + \cdots V^{(k-2)} \\
\frac{\partial V^{(1)}}{\partial x} &= \frac{\partial V^{(1)}}{\partial y} + V^{(0)} \\
\frac{\partial V^{(2)}}{\partial x} &= \frac{\partial V^{(2)}}{\partial y} + V^{(1)} \\
& \cdots \\
\frac{\partial V^{(k-2)}}{\partial x} &= \frac{\partial V^{(k-2)}}{\partial y} + V^{(k-3)}
\end{aligned}
$$

Major steps 3

- Change of variables $w = (x+y)/2$, $z = (x-y)/2$
- This gives

$$\begin{aligned}
\frac{\partial V^{(0)}}{\partial z} &= -\left(V^{(0)} + V^{(1)} + \cdots V^{(k-2)}\right) \\
\frac{\partial V^{(1)}}{\partial z} &= V^{(0)} \\
\frac{\partial V^{(2)}}{\partial z} &= V^{(1)} \\
&\cdots \\
\frac{\partial V^{(k-2)}}{\partial z} &= V^{(k-3)} \text{ and so} \\
\sum_{i=0}^{k-1} \frac{\partial^i V^{(k-2)}}{\partial z^i} &= 0
\end{aligned}$$

Major steps 4

- Solve for all $V^{(t)}$ in terms of $\exp(\lambda_j z)$, $\lambda_j = k^{\mathrm{th}}$ roots of 1
- Matrix of coefficients has van der Monde form
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Example

With $k = 4$

$$V^{(0)}(x, y) = \frac{\cos x - \sin x + \exp(-x)}{\cos(x + y) - \sin(x + y) + \exp(-x - y)}$$

$$V^{(0)}(0, y) = \frac{2}{\cos(y) - \sin(y) + \exp(-y)}$$

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- $\mathrm{Av}(132)$: 1.0 2.0 2.5 2.8 3.0 3.1 3.2 3.3 3.4 3.45 3.5 3.54 3.57 3.6 3.62 3.65 3.67 3.68 3.70 3.71 3.73 3.74 3.75 3.76 3.77 3.78 3.786 3.793 3.800 (30 terms)

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- $\mathrm{Cav}(132)$: 1.0 0.83 0.85 0.82 0.83 0.826 0.828 0.827 0.827 0.827 0.827 0.827 0.827 0.826987 0.826996 0.826992 0.826994 0.826993 0.8269935 0.8269932 0.82699339 0.82699332 0.82699336 0.82699334 0.826993346 0.826993342 0.826993344 0.826993343 (28 terms)

Some empirical Wilf-Stanley limits

- Remaining unsolved cases with Π having two permutations of length 3. Find recurrences and compute numerically:
  1. $\mathrm{Cav}(312, 132)$: $\lim t_n/(nt_{n-1}) = 0.601730727943943$
  2. $\mathrm{Cav}(312, 231)$: $\lim t_n/(nt_{n-1}) = 0.676388228094035$
- When Π has one permutation of length 4. Recurrences again:

$\mathrm{Cav}(\Pi)$	$\lim t_n/(nt_{n-1})$	
$\mathrm{Cav}(1234)$	0.963005	E&N
$\mathrm{Cav}(2413)$	0.957718	
$\mathrm{Cav}(2143)$	0.956174	
$\mathrm{Cav}(1324)$	0.955850	
$\mathrm{Cav}(1423)$	0.954826	
$\mathrm{Cav}(1342)$	0.954611	E&N
$\mathrm{Cav}(1243)$	0.952891	E&N

Wilf-equivalence

- Many equalities of enumeration sequences are explained by symmetries
- "Wilf-equivalence" generally refers to equalities that are not explained by symmetries
- Best example: $\mathrm{Av}(123)$ and $\mathrm{Av}(231)$
- Probably no over-arching theory to explain all Wilf-equivalences

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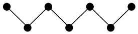

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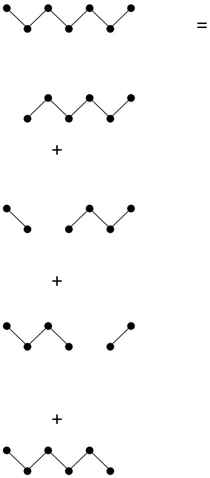

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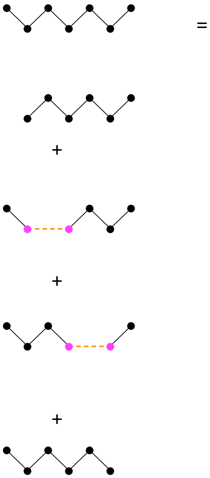

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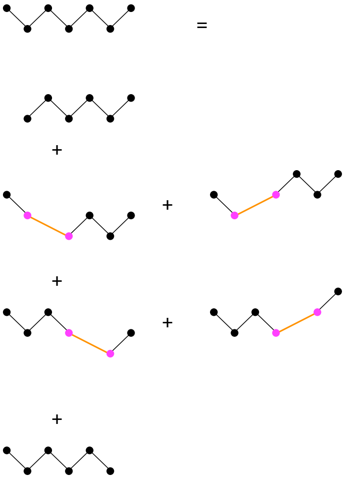

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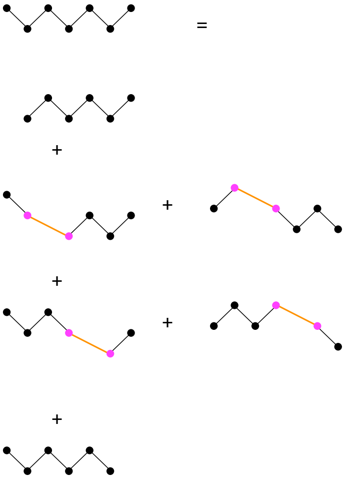

Beyond Wilf-equivalence

- $\alpha\beta$ any permutation such that

$$\alpha \oplus 1 \not\leq_c \beta \text{ and } 1 \ominus \beta \not\leq_c \alpha$$

- $\Pi(\alpha, \beta, k) = \{\alpha\gamma\beta : |\gamma| = k\}$
- Γ any set of t permutations in $\Pi(\alpha, \beta, k)$

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How many permutations of length n have exactly 7 occurrences of 316|978|425 and exactly 9 occurrences of 316|987|425?
This depends only on α, β, k, t, "7", "9".

Theorem

Theorem

The $(t+1)$-variate distribution of permutations according to length and number of occurrences of each permutation of Γ depends on α, β, k, t alone

Some sources

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