Separation Anxiety

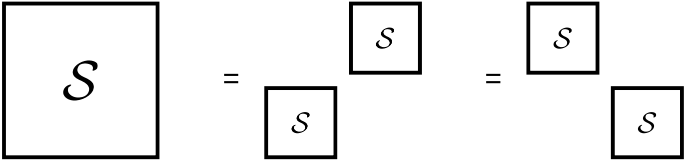
Michael Albert, Mike Atkinson, Vince Vatter

PP2010, Dartmouth

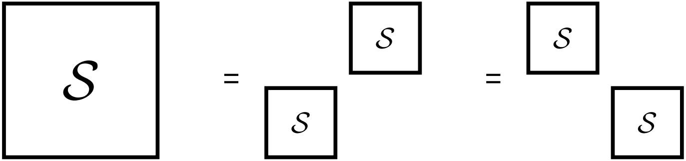
http://arxiv.org/abs/1007.1014

UNIVERSITY of OTAGO
SAPERE AUDE
Te Whare Wānanga o Otāgo
NEW ZEALAND

Starring: Separable Permutations



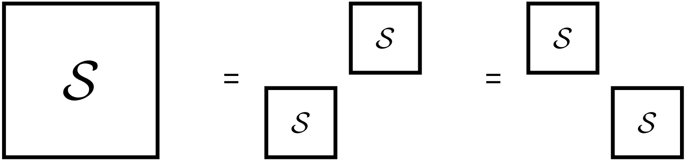
Starring: Separable Permutations



$$\mathcal{S} = \text{Av}(2413, 3142)$$

$$\mathcal{S} = \mathcal{S} \oplus \mathcal{S} = \mathcal{S} \ominus \mathcal{S}$$

Starring: Separable Permutations

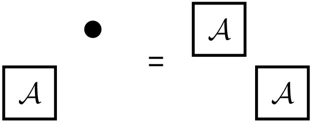


$$\mathcal{S} = \text{Av}(2413, 3142)$$

$$\mathcal{S} = \mathcal{S} \oplus \mathcal{S} = \mathcal{S} \ominus \mathcal{S}$$

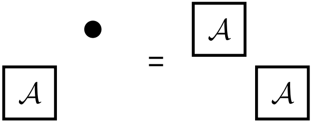
$$S(t) = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2t}$$

Also featuring:

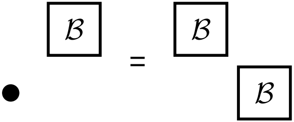


$\mathcal{A} = \text{Av}(132)$

Also featuring:

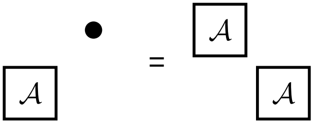


$\mathcal{A} = \text{Av}(132)$

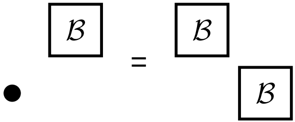


$\mathcal{B} = \text{Av}(213)$

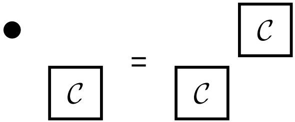
Also featuring:



$\mathcal{A} = \text{Av}(132)$

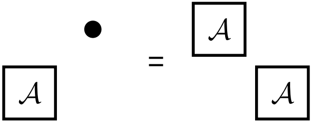


$\mathcal{B} = \text{Av}(213)$

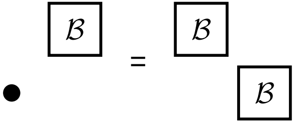


$\mathcal{C} = \text{Av}(231)$

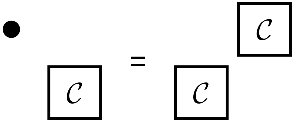
Also featuring:



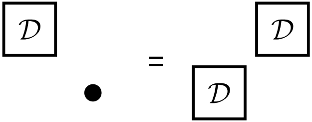
$\mathcal{A} = \text{Av}(132)$



$\mathcal{B} = \text{Av}(213)$

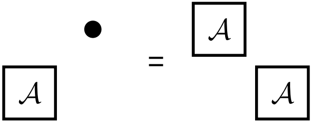


$\mathcal{C} = \text{Av}(231)$

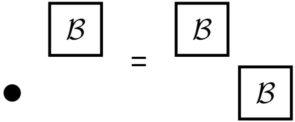


$\mathcal{D} = \text{Av}(312)$

Also featuring:

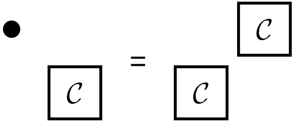


$\mathcal{A} = \text{Av}(132)$

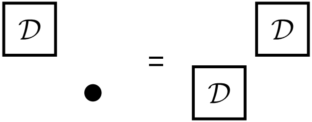


$\mathcal{B} = \text{Av}(213)$

$$\frac{1 - 2t - \sqrt{1 - 4t}}{2t}$$

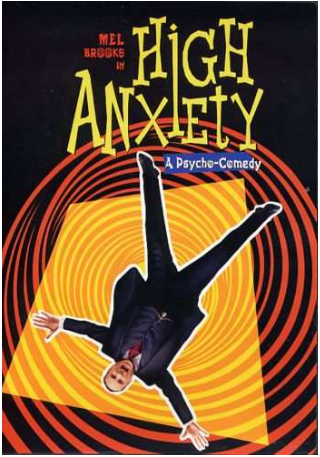


$\mathcal{C} = \text{Av}(231)$

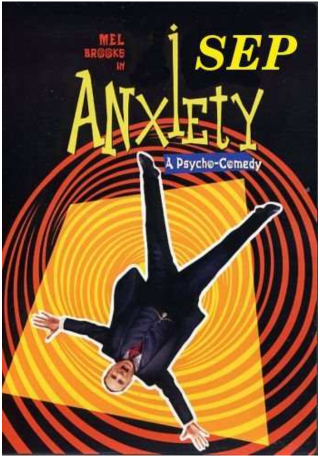


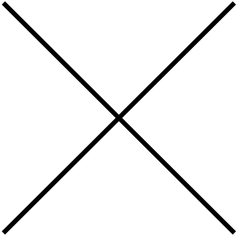
$\mathcal{D} = \text{Av}(312)$

And Introducing: Mr X

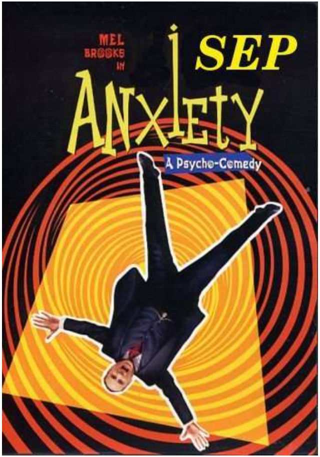


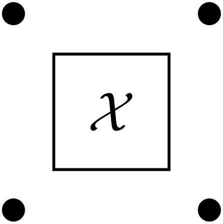
And Introducing: Mr X





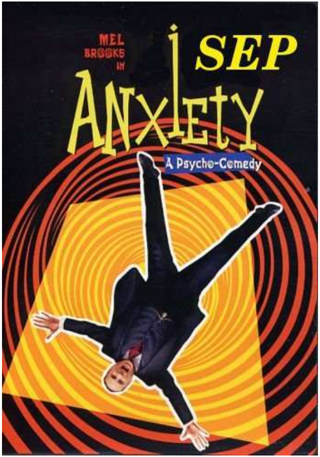
And Introducing: Mr X

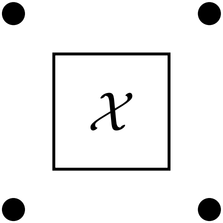




$$\begin{aligned} \mathcal{X} &= 1 \left\{ \begin{matrix} \oplus \\ \ominus \end{matrix} \right\} \mathcal{X} \left\{ \begin{matrix} \oplus \\ \ominus \end{matrix} \right\} 1 \\ &= \text{Av}(2143, 2413, 3142, 3412). \end{aligned}$$

And Introducing: Mr X





$$\begin{aligned}\mathcal{X} &= 1\left\{\begin{matrix}\oplus\\ \ominus\end{matrix}\right\}\mathcal{X}\left\{\begin{matrix}\oplus\\ \ominus\end{matrix}\right\}1\\ &= \mathrm{Av}(2143, 2413, 3142, 3412).\end{aligned}$$

$$X(t) = \frac{t - 2t^2}{1 - 4t + 2t^2}$$

The spoiler

Theorem

Every subclass of $\mathcal{S}$ either:

- *contains (at least) one of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$, or,*
- *has a rational generating function.*

The spoiler

Theorem

Every subclass of $\mathcal{S}$ either:

- *contains (at least) one of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$, or,*
- *has a rational generating function.*

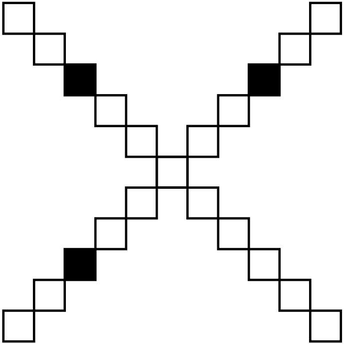
But where's the rôle for Mr X?

The real spoiler

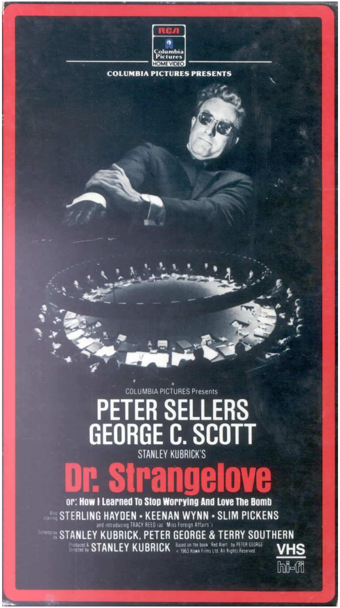
Theorem

Every atomic *subclass of $\mathcal{S}$ either:*

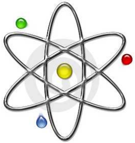
- *contains (at least) one of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$, or,*
- *(condition X) is contained in the inflation, $\mathcal{X}[\mathcal{U}]$ of one of its proper subclasses.*



Well order in the atomic age

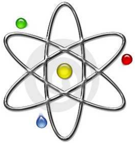


Well order in the atomic age



A class is atomic if every two permutations in the class have a common extension in the class.

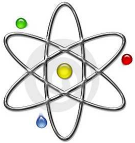
Well order in the atomic age



A class is atomic if every two permutations in the class have a common extension in the class.

- An atomic class is not the finite union of proper subclasses.

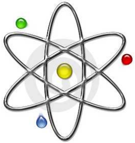
Well order in the atomic age



A class is atomic if every two permutations in the class have a common extension in the class.

- An atomic class is not the finite union of proper subclasses.
- A partially well ordered class is a finite union of atomic classes.

Well order in the atomic age



A class is atomic if every two permutations in the class have a common extension in the class.

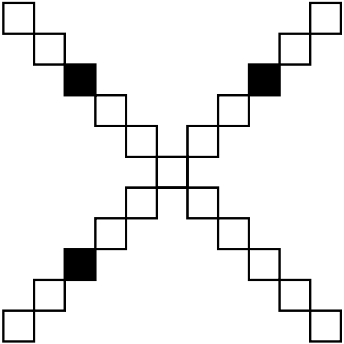
- An atomic class is not the finite union of proper subclasses.
- A partially well ordered class is a finite union of atomic classes.
- If P and Q are hereditary properties and every element of an atomic class satisfies P or Q, then every member satisfies P or every member satisfies Q.

The real spoiler (again)

Theorem

Every atomic *subclass of $\mathcal{S}$ either:*

- *contains (at least) one of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$, or,*
- *(condition X) is contained in the inflation, $\mathcal{X}[\mathcal{U}]$ of one of its proper subclasses.*



The highlight reel (I)

Let an atomic subclass $\mathcal{Y}$ of $\mathcal{S}$ be given.

The highlight reel (I)

Let an atomic subclass $\mathcal{Y}$ of $\mathcal{S}$ be given.

- If the subclass consisting of all subpermutations of plus (minus) indecomposable elements of $\mathcal{Y}$ is proper, then condition X holds.

The highlight reel (I)

Let an atomic subclass $\mathcal{Y}$ of $\mathcal{S}$ be given.

- If the subclass consisting of all subpermutations of plus (minus) indecomposable elements of $\mathcal{Y}$ is proper, then condition X holds.
- If not, and $\mathcal{Y}$ is $\oplus$-closed, then for $\pi \in \mathcal{Y}$, $1 \oplus \pi \preceq \theta$ for some $\oplus$-indecomposable θ. So, either $1 \ominus (1 \oplus \pi)$, or $(1 \oplus \pi) \ominus 1$ belongs to $\mathcal{Y}$. By atomicity, $1 \ominus \mathcal{Y} \subseteq \mathcal{Y}$ or $\mathcal{Y} \ominus 1 \subseteq \mathcal{Y}$. So $\mathcal{C} \subseteq \mathcal{Y}$ or $\mathcal{D} \subseteq Y$.

The highlight reel (I)

Let an atomic subclass $\mathcal{Y}$ of $\mathcal{S}$ be given.

- If the subclass consisting of all subpermutations of plus (minus) indecomposable elements of $\mathcal{Y}$ is proper, then condition X holds.
- If not, and $\mathcal{Y}$ is $\oplus$-closed, then for $\pi \in \mathcal{Y}$, $1 \oplus \pi \preceq \theta$ for some $\oplus$-indecomposable θ. So, either $1 \ominus (1 \oplus \pi)$, or $(1 \oplus \pi) \ominus 1$ belongs to $\mathcal{Y}$. By atomicity, $1 \ominus \mathcal{Y} \subseteq \mathcal{Y}$ or $\mathcal{Y} \ominus 1 \subseteq \mathcal{Y}$. So $\mathcal{C} \subseteq \mathcal{Y}$ or $\mathcal{D} \subseteq Y$.
- Similarly if $\mathcal{Y}$ is $\ominus$ closed.

The highlight reel (II)

So, $\mathcal{Y}$ is atomic, neither $\oplus$ nor $\ominus$ closed, and we may assume is neither the sum or skew sum of two proper subclasses (lest condition X hold).

Define:

$$
\begin{aligned}
\mathcal{Y}_{SW} &= \{\sigma \in \mathcal{Y} : \sigma \oplus \mathcal{Y} \subseteq \mathcal{Y}\} \\
\mathcal{Y}_{SE} &= \{\sigma \in \mathcal{Y} : \mathcal{Y} \ominus \sigma \subseteq \mathcal{Y}\} \\
\mathcal{Y}_{NW} &= \{\sigma \in \mathcal{Y} : \sigma \ominus \mathcal{Y} \subseteq \mathcal{Y}\} \\
\mathcal{Y}_{NE} &= \{\sigma \in \mathcal{Y} : \mathcal{Y} \oplus \sigma \subseteq \mathcal{Y}\}
\end{aligned}
$$

Each is a proper subclass of $\mathcal{Y}$ (possibly empty). Put their union in $\mathcal{U}$.

The highlight reel (III)

Consider $\pi = \gamma \oplus \tau \in \mathcal{Y}$. If $\gamma \in \mathcal{Y}_{SE}$, or $\tau \in \mathcal{Y}_{NW}$ then “locally” condition X holds. If not, define:

$$\begin{aligned} \mathcal{F}_\gamma &= \{\sigma \in \mathcal{Y} : \gamma \oplus \sigma \in \mathcal{Y}\} \\ \mathcal{E}_\gamma &= \{\sigma \in \mathcal{Y} : \sigma \oplus \mathcal{F}_\gamma \subseteq \mathcal{Y}\} \end{aligned}$$

Both are proper. Moreover (magic), there are only finitely many different possibilities. So, put them all in $\mathcal{U}$ (and do the same for minus decomposable permutations).

The highlight reel (III)

Consider $\pi = \gamma \oplus \tau \in \mathcal{Y}$. If $\gamma \in \mathcal{Y}_{SE}$, or $\tau \in \mathcal{Y}_{NW}$ then "locally" condition X holds. If not, define:

$$\begin{aligned} \mathcal{F}_\gamma &= \{\sigma \in \mathcal{Y} : \gamma \oplus \sigma \in \mathcal{Y}\} \\ \mathcal{E}_\gamma &= \{\sigma \in \mathcal{Y} : \sigma \oplus \mathcal{F}_\gamma \subseteq \mathcal{Y}\} \end{aligned}$$

Both are proper. Moreover (magic), there are only finitely many different possibilities. So, put them all in $\mathcal{U}$ (and do the same for minus decomposable permutations).

Finis.

The credits