Crossings and patterns in signed permutations

Matthieu Josuat-Vergès Université Paris-Sud 11

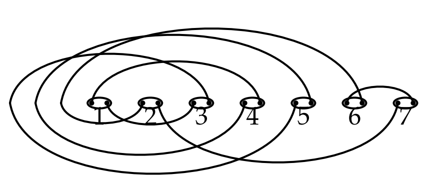
We use the permutation tableaux of type B introduced by Lam and Williams [3] to define notions of crossings [2] and 31-2 patterns for signed permutations. We define some q-analogues of Eulerian polynomials of type B (se for example [1] for a combinatorial definition of these). These polynomials are such that $E^B_{n,k}(q) = E^B_{n,n-k}(q)$ and $E^B_{n,k}(0) = \binom{n}{k}^2$.

A *type B permutation tableau* of length n is a 0,1-filling of a shifted Ferrers diagram of length n satisfying the following conditions: (1) each column has at least one 1, (2) there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row, and (3) if a 0 is in a diagonal cell, then it does not have a 1 to the left of it in the same row. A signed permutation on $[n]$ is a permutation of $[n]$ where each integer may be negated We denote by B_n the set of signed permutations on $[n]$. For $\pi = \pi_1 \cdots \pi_n \in B_n$, we define $\mathrm{wex}(\pi)$ to be the number of *weak excedances* ($i \in [n]$ with $\pi_i \geq i$), $\mathrm{des}(()\pi)$ to be the number of *descents* ($i \in [n-1]$ with $\pi_i > \pi_{i+1}$), $\mathrm{des}()B(\pi)$ to be the number of *type B descents* ($i \in [0, n-1]$ with $\pi_i > \pi_{i+1}$ where $\pi_0 = 0$), and $\mathrm{neg}(\pi)$ to be the number of negative integers in π. Let $\mathrm{twex}(\pi) = 2\,\mathrm{wex}(\pi) + \mathrm{neg}(\pi)$.

For $\pi \in B_n$, a *crossing* is a pair (i, j) of integers $i, j \in [n]$ with $i < j \leq \pi(i) < \pi(j)$, $i > j > \pi(i) > \pi(j)$, or $-i < j \leq -\pi(i) < \pi(j)$. For $\pi \in S_n$ or $\pi \in B_n$, let $\mathrm{cr}(\pi)$ denote the number of crossings of π.

The *type B Eulerian number* $E^B_{n,k}$ is the number of $\pi \in B_n$ with $\mathrm{des}()B(\pi) = k$. Equivalently, $E^B_{n,k}$ is the number of $\pi \in B_n$ with $\lfloor \mathrm{twex}(\pi) \rfloor = k$. We define the *type B q-Eulerian number* $E^B_{n,k}(q)$ as follows: $E^B_{n,k}(q) = \sum_{\substack{\pi \in B_n \\ \lfloor \mathrm{twex}(\pi) \rfloor = k}} q^{\mathrm{cr}(\pi)}$. Let $B_{n,k}(q) = \sum_{\substack{\pi \in B_n \\ \mathrm{twex}(\pi) = k}} q^{\mathrm{cr}(\pi)}$. Then we have $E^B_{n,k}(q) = B_{n,2k}(q) + B_{n,2k+1}(q)$.

We can prove that $B_{n,k}(q) = B_{n,2n+1-k}(q)$, using the *pig-nose diagram* of $\pi = \pi_1 \cdots \pi_n \in B_n$ as follows. For example, the following is the pig-nose diagram of $\pi = 4, -6, 1, -5, -3, 7, 2$. A nice feature of this diagram is that the crossings of $\pi \in B_n$ exactly correspond to the two crossing arcs.



We have defined $B_{n,k}(q)$ in terms of excedances and crossings, but there is an alternative description in terms of ascents of patterns, that generalize the 31-2 pattern that appears in the case of (non-signed) permutations. This is done by using some weighted Motzkin paths, that were defined in [2] using a matrix formulation for the enumeration of type B permutations tableaux. What is nice about these paths is that can adapt some known bijections such as the Françon-Viennot bijection, and obtain that $B_{n,k} = \sum_{\substack{\pi \in B_n \\ \mathrm{pasc}(ß) = k}} q^{31-2(\pi)}$, where we use the following statistics: pasc(ß) is the twice number of i with $|\sigma(i)| < \sigma_{i+1}$ plus $\mathrm{neg}(\pi)$, and $31-2(\pi)$ is the number of pairs (i, j) such that either $|\pi(i)| > |\pi(j)| > \pi(i+1)$ and $i < j$, or $|\pi(i)| > -\pi(j) \geq |\pi(i+1)|$. A nice property of this definition is that we immediately recover the notion of 31-2 pattern when all the entries of π are positive.

This is joint work with Sylvie Corteel and Jang Soo Kim.

References

[1] F. Brenti, *q*-Eulerian polynomials arising from Coxeter groups, European J. Combin. 15 (1994), 417—441.

[2] S. Corteel, M. Josuat-Vergès and L.K. Williams, The Matrix Ansatz, orthogonal polynomials, and permutations, to appear in Advances in Applied Mathematics, 2010.

[3] T. Lam and L.K. Williams, Total positivity for cominuscule Grassmannians, New York Journal of Mathematics, Volume 14, 2008, 53–99.