Products of cycles

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We will discuss two aspects of multiplying two cycles in the symmetric group S_n of all permutations of $1, 2, \ldots, n$.

(1) It is easy to see that if $n > 1$, then the probability that 1 and 2 are in the same cycle of a random permutation in S_n is $1/2$. Miklos Bóna conjectured that if n is also odd, then the probability that 1 and 2 are in the same cycle of a product of two random n-cycles in S_n is $1/2$. We will explain a proof of this conjecture and of many extensions of it. For instance, if $n > k-1$ and $n-k$ is odd, then the probability that $1, 2, \ldots, k$ are in k different cycles of a product of two random n-cycles is $1/k!$. Many open problems and conjectures remain. Much of this work was done in collaboration with Rosena R. X. Du.

(2) The distribution of the number of cycles in the product of two n-cycles was first obtained by Zagier and has connections with such topics as Riemann surfaces, polynomials with real zeros, graph theory, and Kerov's character polynomials.