Grid pattern classes

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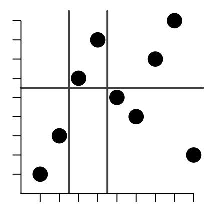
Matrix griddings are structural means of representing permutations as built of finitely many increasing and decreasing permutations. More specifically, let $\mathcal{M}$ be an $m \times n$ matrix with entries $m_{ij} \in \{0, 1, -1\}$. We say that a permutation π *admits* an $\mathcal{M}$-*gridding* if the xy-plane, in which the graph Γ of π has been plotted, can be partitioned into an xy-parallel, $m \times n$ rectangular grid with cells C_{ij}, such that the following hold:

- if $m_{ij} = 1$ then $\Gamma \cap C_{ij}$ is an increasing sequence of points;
- if $m_{ij} = -1$ then $\Gamma \cap C_{ij}$ is decreasing;
- if $m_{ij} = 0$ then $\Gamma \cap C_{ij} = \emptyset$.

For example, the permutation 136854792 admits the following $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$-gridding:



Let us denote by $\mathrm{Grid}(\mathcal{M})$ the set (pattern class) of all permutations which admit $\mathcal{M}$-griddings. Grid classes have been present in the pattern classes literature from very early on. For example, Atkinson (1999) observed that the class of permutations avoiding 321 and 2143 is equal to $\mathrm{Grid}\begin{pmatrix} 1 & 1 \end{pmatrix} \cup \mathrm{Grid}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and used this to enumerate the class. Much more recently, grid classes have played a crucial role in Vatter's (to appear) classification of small growth rates of pattern classes.

These past uses hint strongly at the natural importance of grid classes in the general theory of pattern classes. If this is to be so, the next step is to establish 'nice' general properties of grid classes themselves. A number of researchers, including M.H. Albert, M.D. Atkinson, M. Bouvel, R. Brignall, V. Vatter and myself, have been engaged on such a project over the past year, and I will report on their findings.

To our grid class $\mathrm{Grid}(\mathcal{M})$ we can associate a bipartite graph with vertices $r_1, \dots, r_m, c_1, \dots, c_n$ with $r_i \sim c_j$ iff $m_{ij} \neq 0$. If this graph is a forest then the following hold:

- $\mathrm{Grid}(\mathcal{M})$ is finitely based.
- $\mathrm{Grid}(\mathcal{M})$ is partially well ordered (Murphy & Vatter (2003)).
- Every subclass of $\mathrm{Grid}(\mathcal{M})$ is a finite union of (slightly generalised) forest grid classes.
- Every subclass of $\mathrm{Grid}(\mathcal{M})$ has a rational generating function.
- The basis and the generating function for $\mathrm{Grid}(\mathcal{M})$ can be effectively computed from $\mathcal{M}$.

The above results are proved by an intriguing interplay of language-theoretic and combinatorial-geometric methods, the flavour of which I will try to convey. The talk will conclude with a discussion of some open problems concerning general grid classes (i.e. removing the forest assumption) which ought to point the way for the next stage in this project.

This is joint work with Michael Albert, Mike Atkinson, Mathilde Bouvel, Robert Brignall and Vincent Vatter.