Minimal overlapping patterns

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Given a sequence $\sigma = \sigma_1 \dots \sigma_n$ of distinct integers, let $\text{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\text{red}(\sigma) = 1\ 4\ 3\ 2$. Given a permutation $\tau = \tau_1 \dots \tau_j$ in the symmetric group S_j, we say a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$ to have a *τ-match starting at position i* provided $\text{red}(\sigma_i \dots \sigma_{i+j-1}) = \tau$. Let $\tau\text{-mch}(\sigma)$ be the number of τ-matches in the permutation σ. We say that a permutation $\tau \in S_j$ is a **minimal overlapping permutation** if the smallest n such that there is $\sigma \in S_n$ where $\tau\text{-mch}(\sigma) = 2$ is $2j-1$. For example, $\tau_3 = 132$, $\tau_4 = 1243$, and, in general, $\tau_n = 12 \cdots (n-2)n(n-1)$ are minimal overlapping permutations while $\alpha = 1234$, $\beta = 1324$ are not minimal overlapping permutations.

In this paper, we shall study the following generating functions for minimal overlapping permutations.

$$A_\tau(t) = \sum_{n \geq 0} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}| \text{ and} \tag{8}$$

$$P_\tau(x,t) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)}. \tag{9}$$

Given permutations $\alpha, \beta \in S_n$, we say that α is **c-Wilf equivalent** (**strongly c-Wilf equivalent**) to β if $A_\alpha(t) = A_\beta(t)$ ($P_\alpha(u,t) = P_\beta(u,t)$). It is a conjecture of Elizalde [2] that if $\alpha = \alpha_1 \dots \alpha_n$ and $\beta = \beta_1 \dots \beta_n$ are minimal overlapping permutations in S_n such that $\alpha_1 = \beta_1$ and $\alpha_n = \beta_n$, then α is strongly c-Wilf equivalent to β. Our results will prove this conjecture. This conjecture has been proved by a different methods by Dotsenko and Khoroshkin [1].

To state our results, we need one more definition. Given a minimal overlapping permutation $\tau \in S_j$, we say that $\sigma \in S_{j+s(j-1)}$ is a **maximum packing** for τ if and only if σ has τ-matchings starting at positions $1, j, 2j-1, 3j-2, \dots, sj-(j-1)$. We let $\mathcal{MP}_{\tau,j+s(j-1)}$ be the such that of $\sigma \in S_{j+s(j-1)}$ such that σ is a maximum packing for τ and

$$MP_{\tau,j+s(j-1)}(q) = \sum_{\sigma \in \mathcal{MP}_{\tau,j+s(j-1)}} q^{inv(\sigma)}.$$

Then we prove that

Theorem 1.

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} q^{inv(\sigma)} \frac{1}{1 - (t + \sum_{s \geq 0} \frac{t^{j+s(j-1)}}{[j+s(j-1)]_q!}(x-1)^{s+1} MP_{\tau,j+s(j-1)}(q))}.$$

In particular, when $q = 1$, we get

Theorem 2. *Suppose that τ is a minimal overlapping permutation. Then*

$$\begin{aligned} P_\tau(x,t) &= \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} \\ &= \frac{1}{1 - (t + \sum_{s \geq 0} \frac{t^{j+s(j-1)}}{(j+s(j-1)!)}(x-1)^{s+1} MP_{\tau,j+s(j-1)})}. \end{aligned} \tag{10}$$

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Then using an idea of Kitaev, we prove that if $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are minimal overlapping permutations in S_j and $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$, then for all $s \geq 0$, $MP_{\alpha,j+s(j-1)} = MP_{\beta,j+s(j-1)}$. Hence it follows from Theorem 2, that $P_\alpha(x,t) = P_\beta(x,t)$.

In many cases where τ is a minimal overlapping permutation, we can explicitly compute $MP_{\tau,j+s(j-1)}$. For example, if $\tau = \tau_1 \dots \tau_j$ where $\tau_1 = 1$ and $\tau_j = k$, then we can show that for $s \geq 1$,

$$MP_{\tau,j+s(j-1)} = \prod_{i=1}^{s} \binom{j+i(j-1)-k}{j-k}.$$

Hence we can get an explicit expression for $P_\tau(x,t)$.

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References

[1] V. Dotsenko and A. Khoroshkin, Anick-type resolutions and consecutive pattern avoidance, preprint.

[2] S. Elizalde, Consecutive Patterns and statistics on restricted permutations, Ph.D. thesis, Universitat Politécnica de Catalunya, 2004.