Separable permutations, Robinson-Schensted and shortest containing supersequences

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The Robinson-Schensted correspondence associates to any permutation (or word) σ a pair of *Young tableaux*, each of equal partition shape $\lambda = (\lambda_1, \dots, \lambda_k)$. We say that σ has *shape* $\mathrm{sh}(\sigma) = \lambda$. Many properties of σ translate to natural properties of the tableaux and vice versa; the study of one object is made easier through the study of the other. For example, the length of the longest increasing subsequence of σ equals λ_1. In fact, Greene's Theorem [**?**] gives a much more precise correspondence: The sum $\lambda_1 + \cdots + \lambda_k$ equals the maximum number of elements in a disjoint union of k increasing subsequences of σ. However, it is not generally true that one can find k disjoint increasing subsequences $u^1, u^2, \dots, u^k$ with u^i of length λ_i for each i. (An example is afforded by $\sigma = 236145$ whose shape is $(4, 2)$.) Our main result is a sufficient condition for such a collection of subsequences $\{u^i\}$ to exist:

Theorem 1. *Let σ be a* separable *permutation (i.e.,* 2413 *and* 3142*-avoiding) with* $\mathrm{sh}(\sigma) = \lambda = (\lambda_1, \dots, \lambda_k)$. *Then there exist k disjoint, increasing subsequences $u^1, \dots, u^k$, with u^i of maximum length in $\sigma \setminus (u^1 \cup \cdots \cup u^{i-1})$, such that the length of each u^i is given by λ_i.*

The proof is based on the *inversion poset* of σ and the fact that when σ is separable its inversion poset has no subposet isomorphic to $\begin{array}{ccc} * & & * \\ | & / & | \\ * & & * \end{array}$. We present one corollary and one application.

Corollary 2. *If a word w contains a separable permutation σ as a pattern, then* $\mathrm{sh}(w) \supseteq \mathrm{sh}(\sigma)$.

The necessity of having σ be separable is illustrated by the pair $\sigma = 3142$ (of shape $(2, 2)$) and $w = 41352$ (of shape $(3, 1, 1)$).

Our application involves *shortest containing supersequences*. These have important applications in such areas as statistics and computational biology. A word w is a *supersequence* of a permutation σ if σ is a subsequence. For any set of permutations $B \subseteq S_n$, w is a *supersequence* of B if w is a supersequence of each element of B. Let $\mathrm{scs}_n(B)$ denote the minimum length of a supersequence of the set B. Koutas and Hu [**?**] construct explicit supersequences to show that $\mathrm{scs}_n(S_n) \leq n^2 - 2n + 4$. On the other hand, Kleitman and Kwiatkowski [**?**] have shown that $\mathrm{scs}_n(S_n) \geq n^2 - Cn^{7/4+\varepsilon}$ where $\varepsilon > 0$ and C depends on ε. We consider the problem of finding explicit sets B for which we can compute a *lower* bound for $\mathrm{scs}_n(B)$ in a simple manner.

So, let $\mu(n)$ be the Ferrers diagram obtained by taking the union of all Ferrers diagrams of size n. The size of $\mu(n)$ is $\sum_{i=1}^{n} d(i)$ where $d(i)$ denotes the number of divisors of i. Asymptotically, $|\mu(n)| \sim n(\ln n + 2\gamma + \cdots)$. The number of corners in $\mu(n)$ is counted by $\lfloor \sqrt{4n+1} \rfloor - 1$. If we associate a separable permutation to each corner of $\mu(n)$, then Corollary 2 implies that there is a set B of $\lfloor \sqrt{4n+1} \rfloor - 1$ permutations for which $\mathrm{scs}_n(B) \geq n(\ln n + 2\gamma + \cdots)$.

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References

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[3] P. J. Koutas and T. C. Hu. Shortest string containing all permutations. *Discrete Math.*, 11:125–132, 1975.