Generating functions for Wilf equivalence under generalized factor order

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Kitaev, Liese, Remmel, and Sagan recently defined generalized factor order on words comprised of letters from a partially ordered set $(P, \leq_P)$ by setting $u \leq_P w$ if there is a subword v of w of the same length as u such that the i-th character of v is greater than or equal to the i-th character of u for all i. This subword v is called an embedding of u into w. Generalized factor order is related to generalized subword order, in which the characters of v are not required to be adjacent [2]. For the case where P is the positive integers with the usual ordering, they defined the weight of a word $w = w_1 \dots w_n$ to be $\mathrm{wt}(w) = x^{\sum_{i=1}^{n} w_i} t^n$, and the corresponding weight generating function

$$F(u; t, x) = \sum_{w \geq_P u} \mathrm{wt}(w).$$

They then defined two words u and v to be Wilf equivalent, denoted $u \backsim v$, if and only if $F(u; t, x) = F(v; t, x)$. They also defined the related generating function $S(u; t, x) = \sum_{w \in \mathcal{S}(u)} \mathrm{wt}(w)$ where $\mathcal{S}(u)$ is the set of all words w such that the only embedding of u into w is a suffix of w, and showed that $u \backsim v$ if and only if $S(u; t, x) = S(v; t, x)$. We continue this study by giving an explicit formula for $S(u; t, x)$ if u factors into a weakly increasing word followed by a weakly decreasing word.

Kitaev, Liese, Remmel and Sagan [1] gave two examples of classes of words u such that $S(u; t, x)$ has a simple form. That is, they proved that if $u = 1\,2\,3 \dots n-1\; n$ or $u = 1^k b^\ell$ for some $k \geq 0$, $\ell \geq 1$, and $b \geq 2$, then $S(u; t, x) = \frac{x^s t^r}{P(u; t, x)}$ for some polynomial $P(u; t, x)$, and produced an explicit expression for $P(u; t, x)$ in each case.

We shall show that there is a much richer class of of words u such that $S(u; t, x)$ has this same form. Specifically, for any word u, let u_{inc} be the longest weakly increasing prefix of u. If $u = u_{inc} v$ and v is weakly decreasing, then we shall say that u has an *increasing/decreasing factorization* and denote v as u_{dec}. Thus if $u = u_1 u_2 \dots u_n$ has an increasing/decreasing factorization, then either $u_1 \leq \cdots \leq u_n$, in which case $u_{inc} = u$ and u_{dec} is the empty string ε, or there is a $k < n$ such that $u_1 \leq \cdots \leq u_k > u_{k+1} \geq \cdots \geq u_n$, in which case $u_{inc} = u_1 \dots u_k$ and $u_{dec} = u_{k+1} \dots u_n$. For the theorem that follows, we define

$$D^{(i)}(u) = \{n - i + j : 1 \leq j \leq i \text{ and } u_j > u_{n-i+j}\}$$

and $d_i(u) = \sum_{n-i+j \in D^{(i)}(u)} (u_j - u_{n-i+j})$. Our main result is the following theorem.

Theorem 1. *Let $u = u_1 u_2 \dots u_n \in \mathcal{P}^*$ have an increasing/decreasing factorization. For $1 \leq i \leq n-1$, let $s_i = u_{i+1} u_{i+2} \dots u_n$ and $d_i = d_i(u)$. Also let $s_n = \varepsilon$ and $d_n = 0$. Then*

$$S(u; t, x) = \frac{t^n x^{\Sigma(u)}}{t^n x^{\Sigma(u)} + (1 - x - tx) \sum_{i=1}^{n} t^{n-i} x^{d_i + \Sigma(s_i)} (1-x)^{i-1}}.$$

We can use this formula as an aid to classify Wilf equivalence in a variety of cases, specifically we can classify all equivalences for all words of length 3. In fact, it turns out that the coefficients of related generating functions are well-known sequences in several special cases. Finally, we discuss a conjecture that if $u \backsim v$ then u and v must be rearrangements, and the stronger conjecture that there also must be a weight-preserving bijection $f : \mathcal{S}(u) \to \mathcal{S}(v)$ such that $f(u)$ is a rearrangement of u for all u.

Much of the work in [1] demonstrates and verifies Wilf equivalence when the standard order on the positive integers is used. However, different posets have also yielded non-trivial Wilf equivalences that are worth mention. We will also discuss various equivalences when using two specific partial orders on $\mathbb{P}^*$: the mod k partial order, defined by setting $m \leq_k n$ if $m \leq n$ and $m = n$ mod k (see [3]), and the fence partial order defined by the cover relations $2i - 1 < 2i$ and $2i + 1 < 2i$ for all positive integers i.

This is joint work with Thomas Langley and Jeffrey Remmel.

References

[1] Sergey Kitaev, Jeffrey Liese, Jeffrey Remmel, Bruce E. Sagan, Rationality, Irrationality, and Wilf equivalence in generalized factor order. *Electron. J. Combin.* **16** (2009).

[2] Bruce E. Sagan and Vincent Vatter, The Möbius function of a composition poset. *J. Algebraic Combin.* **24** (2006), 111–136.

[3] Thomas Langley, Jeffery Liese, Jeffrey Remmel, Wilf equivalence for generalized factor orders modulo k, preprint.