Pattern matching in the cycle structures of permutations

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In this paper, we study matching conditions within the cycle structure of a permutation. Given a sequence $\sigma = \sigma_1 \ldots \sigma_n$ of distinct integers, let $\text{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\text{red}(\sigma) = 1\ 4\ 3\ 2$. Suppose that $\tau = \tau_1 \ldots \tau_j$ is a permutation in S_j and σ is a permutation in S_n with k cycles $C_1 \ldots C_k$. We shall always write cycles in the form $C_i = (c_{0,i}, \ldots, c_{p_i-1,i})$ where $c_{0,i}$ is the smallest element in C_i and p_i is the length of C_i and we arrange the cycles by decreasing smallest elements. That is, we arrange the cycles of σ so that $c_{0,1} > \cdots > c_{0,k}$. Then we say that σ has a *cycle τ-match* (c-τ-match) if there is an i such that $C_i = (c_{0,i}, \ldots, c_{p_i-1,i})$ where $p_i \geq j$ and an r such that $\text{red}(c_{r,i}c_{r+1,i} \ldots c_{r+j-1,i}) = \tau$ where we take indices of the form $r+s$ modulo p_i. Let c-τ-$\text{mch}(\sigma)$ be the number of cycle τ-matches in the permutation σ. For example, if $\tau = 2\ 1\ 3$ and $\sigma = (1,10,9)(2,3)(4,7,5,8,6)$, then 9 1 10 is a cycle τ-match in the first cycle and 7 5 8 and 6 4 7 are cycle τ-matches in the third cycle so that c-τ-$\text{mch}(\sigma) = 3$. Similarly, we say that τ *cycle occurs* in σ if there exists an i such that $C_i = (c_{0,i}, \ldots, c_{p_i-1,i})$ where $p_i \geq j$ and there is an r with $0 \leq r \leq p_i - 1$ and indices $0 \leq i_1 < \cdots < i_{j-1} \leq p_i - 1$ such that $\text{red}(c_{r,i}c_{r+i_1,i} \ldots c_{r+i_{j-1},i}) = \tau$ where the indices $r + i_s$ are taken mod p_i. We say that σ *cycle avoids* τ if there are no cycle occurrences of τ in σ. For example, if $\tau = 1\ 2\ 3$ and $\sigma = (1,10,9)(2,3)(4,8,5,7,6)$, then 4 5 7, 4 5 6, and 5 6 8 are cycle occurrences of τ in the third cycle. We can extend of the notion of cycle matches and cycle occurrences to sets of permutations in the obvious fashion. Given a set of permutations $\Upsilon \subseteq S_j$, we let $\mathcal{CAS}_{n,k}(\Upsilon)$ ($\mathcal{NCMS}_{n,k}(\Upsilon)$) denote the set of permutations $\sigma \in S_n$ such that σ has k-cycles and σ cycle avoids Υ (σ has no cycle Υ-matches). Similarly, we let $\mathcal{L}_m^{ca}(\Upsilon)$ ($\mathcal{L}_m^{ncm}(\Upsilon)$) be the set of m cycles γ in S_m such γ cycle avoids Υ (γ has no cycle Υ-matches).

Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we let $\text{des}(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\}|$. We say that σ_j is *left-to-right minima* of σ if $\sigma_j < \sigma_i$ for all $i < j$. We let $\text{lrmin}(\sigma)$ denote the number of left-to-right minma of σ. Given a cycle $C = (c_0, \ldots, c_{p-1})$ where c_0 is the smallest element in the cycle, we let $\text{cdes}(C) = 1 + \text{des}(c_0 \ldots c_{p-1})$. Thus $\text{cdes}(C)$ counts the number of descent pairs as we traverse once around the cycle because the extra factor of 1 counts the descent pair $c_{p-1} > c_0$. For example if $C = (1,5,3,7,2)$, then $\text{cdes}(C) = 3$ which counts the descent pairs 53, 72, and 21 as we traverse once around C. By convention, if $C = (c_0)$ is one-cycle, we let $\text{cdes}(C) = 1$. If σ is a permutation in S_n with k cycles $C_1 \ldots C_k$, then we define $\text{cdes}(\sigma) = \sum_{i=1}^{k} \text{cdes}(C_i)$. We let $\text{cyc}(\sigma)$ denote the number of cycles of σ.

The following theorem easily follows from the theory of exponential structures.

Theorem 1.

$$CA_{\Upsilon}(t,x,y) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} x^k \sum_{\sigma \in \mathcal{CAS}_{n,k}(\Upsilon)} y^{\text{cdes}(\sigma)} = e^{x \sum_{m \geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{ca}(\Upsilon)} y^{\text{cdes}(C)}}, \qquad (1)$$

and

$$NCM_{\Upsilon}(t,x,y) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} x^k \sum_{\sigma \in \mathcal{NCMS}_{n,k}(\Upsilon)} y^{\text{cdes}(\sigma)} = e^{x \sum_{m \geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{ncm}(\Upsilon)} y^{\text{cdes}(C)}}. \qquad (2)$$

It turns out that if $\tau \in S_j$ is a permutation that starts with 1, then we can reduce the problem of finding $NCM_{\tau}(t,x,y)$ to the usual problem of finding the generating function of permutations that have no τ-matches. For any permutation $\tau \in S_j$, let $\mathcal{NM}_n(\tau)$ be the set of $\sigma \in S_n$ such that σ has no τ-matches and

$$NM_\tau(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} x^{\text{lrmin}(\sigma)} y^{1+\text{des}(\sigma)}.$$

Then we can show that if τ starts with 1, then

$$NCM_\tau(t,x,y) = NM_\tau(t,x,y). \tag{3}$$

Using this fact, one can automatically refine a number of theorems on the literature on consecutive pattern avoidance. For example, Goulden and Jackson [1] proved a generating funtion for permutations that have no $12\ldots k$-matches which can be combined with Theorem 1 to prove the following refinement of their result.

Theorem 2. *If $\tau = 12\ldots k$ where $k \geq 2$, then*

$$NM_\tau(t,x,1) = \left(\frac{1}{\sum_{i\geq 0} \frac{t^{ki}}{(ki)!} - \frac{t^{ki+1}}{(ki+1)!}} \right)^x. \tag{4}$$

In fact using Theorem 1 and a theorem of Mendes and Remmel [2], we can show

Theorem 3. *For $k \geq 2$ and $\tau = 12\ldots k$,*

$$\begin{aligned} NCM_\tau(t,x,y) &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NCM}_n(\tau)} x^{cyc(\sigma)} y^{\text{cdes}(\sigma)} \qquad (5) \\ &= e^{x\ ln\left(\frac{1}{\sum_{n\geq 0} \frac{t^n}{n!} \sum_{i\geq 0} (-1)^i \mathcal{R}_{n-1,i,k-1} y^{n-i}}\right)} \end{aligned}$$

where $\mathcal{R}_{n,i,j}$ is the number of rearrangements of i zeroes and $n-i$ ones such that j zeroes never appear consecutively.

In the case where $\tau = 123$, we can give a more explicit formula for $NCM_{123}(t,x,y)$. That is, we can show

$$\begin{aligned} NCM_{123}(t,x,y) &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NCM}_n(123)} x^{cyc(\sigma)} y^{\text{cdes}(\sigma)} \qquad (6) \\ &= e^{x\ ln\left(\frac{1}{\sum_{n\geq 0} \frac{t^n}{n!} \left(\frac{y(1-t)}{1-yt+yt^2}|_{t^{n-1}}\right)}\right)} \\ &= \left(\frac{1}{\sum_{n\geq 0} \frac{t^n}{n!} \left(\frac{y(1-t)}{1-yt+yt^2}|_{t^{n-1}}\right)} \right)^x \end{aligned}$$

We prove similar results for several other types of permutations and sets of permutations.

This is joint work with Jeffrey Remmel (University of California, San Diego)[4].

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References

[1] I. Goulden and D. Jackson, *Combinatorial Enumeration*, John Wiley & Sons Inc. New York 1983.

[2] A. Mendes and J.B. Remmel, Permutations and words counted by consecutive patterns, Adv. Appl. Math, **37** 4, (2006) 443-480.