Another look at bijections for pattern-avoiding permutations

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Bijections between $S_n(321)$ and $S_n(132)$ which preserve certain permutation statistics have received considerable attention in the literature. Robertson defined, through an iterative process, a remarkable natural bijection between these two sets. For any $\sigma \in S_n(321)$ this bijection repeatedly removes the 132-pattern $\sigma_x\sigma_y\sigma_z$ that is smallest in the lexicographic ordering of positions (x, y, z) and replaces it with the 321-pattern $\sigma_y\sigma_z\sigma_x$ until $\sigma \in S_n(132)$. At the 2005 Integers conference he conjectured that this bijection, Γ, is fixed point preserving. Bloom and Saracino affirmed this conjecture and also showed that Γ preserves the number of excedances. It was also shown that Γ can also be defined such that at each iteration one removes the 132-pattern that is smallest either in terms of positions (x, y, z) or in terms of values $(\sigma_x, \sigma_y, \sigma_z)$. The proof depended on first showing that $\Gamma(\sigma^{-1}) = (\Gamma(\sigma))^{-1}$ for all $\sigma \in S_n(321)$. Due to the iterative nature of Γ the proof of this fact was quite technical.

Here we will give a pictorial non-iterative definition of Γ. We accomplish this through a new combinatorial object, which we call a permutation template, that generalizes the notion of a permutation diagram. This new definition results in clearer and *independent* proofs that Γ commutes with inverses and preserves fixed points and excedances. Additionally, permutation templates provide the necessary structure to better understand why Γ may be defined in terms of any combination of position or value based 132-pattern replacements.