

ANOSOV AUTOMORPHISMS ON NILMANIFOLDS IN DIMENSIONS 9 AND 10.

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ABSTRACT. We study 9 and 10-dimensional Anosov Lie algebras, by using the properties of very special algebraic numbers. We classify k -step complex Anosov Lie algebras for $k > 2$ and in the two step case, we give an example in each possible type, or we give a non-existence result.

1. INTRODUCTION

A diffeomorphism f of a compact differentiable manifold M is called *Anosov* if the tangent bundle TM admits a continuous invariant splitting $TM = E^+ \oplus E^-$ such that df expands E^+ and contracts E^- exponentially. In [19], S. Smale raised the problem of classifying the compact nilmanifolds admitting Anosov diffeomorphisms, and up to now the only known examples of Anosov diffeomorphisms are on nilmanifolds or manifolds finitely covered by nilmanifolds, which are known as infranilmanifolds. It is even conjectured that any Anosov diffeomorphism is topologically conjugate to an Anosov automorphism of an infranilmanifold (see [18]). This conjecture is known to be true in some cases (see [7], [14]). This highlights the problem of classifying all compact nilmanifolds which admit an Anosov automorphism.

A rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ (i.e. with structure constants in \mathbb{Q}) of dimension n is said to be *Anosov* if it admits a *hyperbolic* automorphism A (i.e. all its eigenvalues have absolute value different from 1) which is *unimodular*, that is, $[A]_{\beta} \in GL_n(\mathbb{Z})$ for some basis β of $\mathfrak{n}^{\mathbb{Q}}$, where $[A]_{\beta}$ denotes the matrix of A with respect to β . We call a real or complex Lie algebra *Anosov* if it admits a rational form which is Anosov (see [10]).

Then, a simply connected n -dimensional nilpotent Lie group N , with its Lie algebra \mathfrak{n} , is a cover of a nilmanifold admitting an Anosov automorphism if and only if there exists a \mathbb{Z} -basis β (i.e. with integer structure constants) of \mathfrak{n} and a hyperbolic automorphism A of \mathfrak{n} such that $[A]_{\beta} \in GL(n, \mathbb{Z})$.

Hence as far as the question of classifying nilmanifolds admitting Anosov automorphisms is concerned, one is interested in real Anosov Lie algebras. The problem of deciding if a nilpotent Lie algebra \mathfrak{n} admits an Anosov automorphism is equivalent to see if \mathfrak{n} admits a hyperbolic automorphism A such that $[A]_{\beta} \in GL(n, \mathbb{Z})$, where β is a \mathbb{Z} -basis of \mathfrak{n} and n is the dimension of \mathfrak{n} .

The first example (due to Borel) of a non-toral nilmanifold admitting an Anosov automorphism was described by S. Smale ([19]). After that only relatively few

2000 *Mathematics Subject Classification.* Primary: 37D20; Secondary: 22E25, 20F34.

Key words and phrases. Anosov diffeomorphisms, nilmanifolds, nilpotent Lie algebras, hyperbolic automorphisms.

Supported by CONICET and grants from FONCyT and SeCyT. .

examples appeared in the literature. Recently, the families of nilmanifolds with Anosov automorphisms have been constructed showing that a complete classification does not seem to be possible (see [2], [3], [4], [5], [6], [10], [12], [15], [17]). In [13], all real and rational Anosov Lie algebras of dimension ≤ 8 are classified up to isomorphism and we note that curiously enough, there are quite few ones.

In this paper we will study Anosov complex Lie algebras. To classify Anosov Lie algebras over \mathbb{R} , one has to find all the others reals form of the given complex one and study which ones are Anosov. More specifically, we study Anosov Lie algebras of dimension 9 and 10 by using the properties of very special algebraic numbers, following the ideas introduced in [10] and used also in [12],[13] and [17]. This algebraic numbers are actually the eigenvalues of the corresponding Anosov automorphism, and they have very special properties that have been studied in [16]. Using all this and noting that the existence of such an algebra implies the existence of a lower dimensional one that can be found in the classification given in [13], we classify k -step complex Anosov Lie algebras for $k > 2$.

In the 2-step case, we consider every eventually possible type as in [13, Proposition 2.5], and in each one we either give a non existence result, or we give an example of Anosov Lie algebra showing that this case is actually possible. By using the results from [16], we prove that a 9-dimensional Anosov Lie algebra should be of type $(6, 3)$ or $(3, 3, 3)$ and moreover we show that there is only one complex Anosov Lie algebra of type $(3, 3, 3)$. We note that this is the first example of an Anosov Lie algebra of type $(3, 3, *)$ showing that the condition in [12, Proposition 2.3 (ii)] $n_3 \geq 3$ is in fact attained. In the $(6, 3)$ case, we give a necessary condition related to the Pfaffian form and from there we get a list of the possible Anosov Lie algebras of type $(6, 3)$, but for some of them we are not able to conclude whether they admit Anosov automorphism or not. In the last section we study 10-dimensional Anosov Lie algebras. We prove that the possible types of 2-step Anosov Lie algebras are $(8, 2)$, $(6, 4)$, $(4, 6)$ and $(5, 5)$. There are already known examples of type $(8, 2)$, $(6, 4)$ and $(4, 6)$, and we give a new example of an Anosov Lie algebra of type $(5, 5)$. For higher steps, we prove that there is only one Anosov Lie algebra in each of the types $(6, 2, 2)$, $(4, 2, 4)$, $(4, 2, 2, 2)$ and two of the type $(4, 4, 2)$.

We would like to thanks J. Lauret for his constant help and advice.

2. PRELIMINARIES

In this section, in order to introduce the framework in which we are going to work, we will begin by recalling some facts about the eigenvalues of an Anosov automorphism, proved in [12] that we are including for completeness.

Proposition 2.1. (see [12]) *Let \mathfrak{n} be a real r -step nilpotent Lie algebra. We define $C^i(\mathfrak{n})$ inductively by $C^i(\mathfrak{n}) = [\mathfrak{n}, C^{i-1}(\mathfrak{n})]$ for $i \geq 1$, where $C^0(\mathfrak{n}) = \mathfrak{n}$. If \mathfrak{n} is an Anosov Lie algebra then there exist a decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \dots \oplus \mathfrak{n}_r$ satisfying $C^i(\mathfrak{n}) = \mathfrak{n}_{i+1} \oplus \dots \oplus \mathfrak{n}_r$, $i = 0, \dots, r$, and a hyperbolic $A \in \text{Aut}(\mathfrak{n})$ such that*

- (i) $A\mathfrak{n}_i = \mathfrak{n}_i$ for all $i = 1, \dots, r$.
- (ii) A is semisimple (in particular A is diagonalizable over \mathbb{C}).
- (iii) For each i , there exists a basis β_i of \mathfrak{n}_i such that $[A_i]_{\beta_i} \in SL_{n_i}(\mathbb{Z})$, where $n_i = \dim \mathfrak{n}_i$ and $A_i = A|_{\mathfrak{n}_i}$.

If we have a decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \dots \oplus \mathfrak{n}_r$ as in the previous proposition, then we say that \mathfrak{n} is of type (n_1, n_2, \dots, n_r) where $n_i = \dim \mathfrak{n}_i$ for all $i = 1, \dots, r$.

It is not hard to see that the eigenvalues of an Anosov automorphism are necessarily algebraic units, that is, an eigenvalue of an Anosov automorphism satisfies a monic polynomial equation with integer coefficients and unit constant term. For an algebraic number $\lambda \in \mathbb{C}$, we denote by $\deg(\lambda)$ the degree of $m_\lambda(x)$, the irreducible monic polynomial over \mathbb{Q} annihilated by λ and by the *conjugated numbers* to λ we will denote the conjugated algebraic numbers of λ over \mathbb{Q} , that is, the other roots of $m_\lambda(x)$. We then have the following result (see [12]).

Lemma 2.2. *Let \mathfrak{n} be an Anosov real Lie algebra which is Anosov, and let A and $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \dots \oplus \mathfrak{n}_r$ be as in Proposition 2.1. If $A_i = A|_{\mathfrak{n}_i}$ then the eigenvalues $\lambda_1, \dots, \lambda_{n_i}$ of A_i , are algebraic units such that $1 < \deg(\lambda_j) \leq n_i$ for all $j = 1, \dots, n_i$ and we may assume that $\lambda_1 \dots \lambda_{n_i} = 1$.*

From this, for each A_i as in the lemma, we can define the *splitting* of A_i by the tuple $[k_1; \dots; k_m]$, $k_i \in \mathbb{N}$, such that there exists a reordering of the eigenvalues of A_i , $\{\lambda_1, \dots, \lambda_{n_i}\}$ (counted with their multiplicities) such that for all $l \geq 1$, $\deg \lambda_i = k_l$ if $i = \sum_{t=0}^{l-1} k_t + 1, \dots, \sum_{t=0}^{l-1} k_t + k_l$, where for consistency we define $k_0 = 0$. More specifically, the characteristic polynomial of A_i can be written as the product of m irreducible polynomials (over \mathbb{Z}) f_1, f_2, \dots, f_m such that $\deg f_i = k_i$ for all i , and the roots of f_l , $l \geq 1$, are

$$\lambda_{\sum_{t=0}^{l-1} k_t + 1}, \dots, \lambda_{\sum_{t=0}^{l-1} k_t + k_l}.$$

It is easy to see that $k_1 + \dots + k_m = n_i$ and since A_i is hyperbolic and unimodular, we also have that $k_j \neq 1$ for all j (see [12] Appendix). Also, since the order is not relevant, we can order it from greatest to lowest.

We note that each eigenvalue of $A - i$ is a product of certain eigenvalues of A_1 and eigenvalues of A are algebraic units. We are then interested in how the product of such special algebraic numbers can behave. It is proven in [16] that this behavior can not be so wild. In fact, one has the following lemma that can be deduced from [16].

Lemma 2.3. *Let A be an Anosov automorphism of a nilpotent Lie algebra \mathfrak{n} and let α and β be two eigenvalues of A . Then the following hold:*

1. *If $\deg(\alpha)$ and $\deg(\beta)$ are relatively prime then $\deg(\alpha\beta)$ cannot be a prime.*
2. *If $\alpha\beta$ is also an eigenvalue of A and $\deg(\alpha) \leq \deg(\beta)$ then*

$$g.c.d.(\deg(\beta), \deg(\alpha\beta)) \neq 1.$$

An *abelian factor* of \mathfrak{n} is an abelian ideal \mathfrak{a} for which there exists an ideal $\tilde{\mathfrak{n}}$ of \mathfrak{n} such that $\mathfrak{n} = \tilde{\mathfrak{n}} \oplus \mathfrak{a}$ (i.e. $[\tilde{\mathfrak{n}}, \mathfrak{a}] = 0$). Let $m(\mathfrak{n})$ denote the maximum dimension over all abelian factors of \mathfrak{n} . By [12, Theorem 3.6] if \mathfrak{n} is a rational Lie algebra with $m(\mathfrak{n}) = r$ and $\mathfrak{n} = \tilde{\mathfrak{n}} \oplus \mathbb{Q}^r$ is any decomposition in ideals, that is, \mathbb{Q}^r is a maximal abelian factor of \mathfrak{n} , then \mathfrak{n} is Anosov if and only if $\tilde{\mathfrak{n}}$ is Anosov and $r \geq 2$.

As we have said, we are going to start with an Anosov Lie algebra, \mathfrak{n} , and by using the properties we have mentioned of the eigenvalues of the Anosov automorphism it admits, we are going to deduce properties of \mathfrak{n} . Therefore, if \mathfrak{n} is an Anosov Lie algebra, with Anosov automorphism A , we are going to consider the complex Lie algebra $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n} \otimes \mathbb{C}$ which admits a decomposition $\mathfrak{n}_{\mathbb{C}} = (\mathfrak{n}_1)_{\mathbb{C}} \oplus \dots \oplus (\mathfrak{n}_r)_{\mathbb{C}}$, where

$(\mathfrak{n}_i)_{\mathbb{C}} = \mathfrak{n}_i \otimes \mathbb{C}$, as in Proposition 2.1, and a basis of eigenvectors of A_i for all i . Moreover, we also want to point out that if $\mathfrak{n}_{\mathbb{C}}$ has an abelian factor, then \mathfrak{n} must also carry an abelian factor. In fact, $\mathfrak{z}(\mathfrak{n}_{\mathbb{C}}) \cap [\mathfrak{n}_{\mathbb{C}}, \mathfrak{n}_{\mathbb{C}}] = (\mathfrak{z}(\mathfrak{n}) \cap [\mathfrak{n}, \mathfrak{n}])_{\mathbb{C}}$ and hence if $\mathfrak{z}(\mathfrak{n}_{\mathbb{C}}) \cap [\mathfrak{n}_{\mathbb{C}}, \mathfrak{n}_{\mathbb{C}}] \neq \mathfrak{z}(\mathfrak{n}_{\mathbb{C}})$ then $(\mathfrak{z}(\mathfrak{n}) \cap [\mathfrak{n}, \mathfrak{n}]) \neq \mathfrak{z}(\mathfrak{n})$.

Finally, we will state the following observation that can be easily proved by using Proposition 2.1.

Lemma 2.4. *Let \mathfrak{n} be an Anosov Lie algebra of type (n_1, \dots, n_r) and let $\mathfrak{n} = \mathfrak{n}_1 \oplus \dots \oplus \mathfrak{n}_r$ be the decomposition of \mathfrak{n} given in Proposition 2.1. Then $\tilde{\mathfrak{n}} = \mathfrak{n}/\mathfrak{n}_r$ and $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}]$ are both Anosov.*

In fact, if \mathfrak{n} is an Anosov Lie algebra, A and $\mathfrak{n} = \mathfrak{n}_1 \oplus \dots \oplus \mathfrak{n}_r$ are as in Proposition 2.1 then, since A is an automorphism, it preserves $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}_2 \oplus \dots \oplus \mathfrak{n}_r$. Also, since $A\mathfrak{n}_i = \mathfrak{n}_i$ for all $i = 1, \dots, r$, it is easy to see that it induces an automorphism of $\tilde{\mathfrak{n}} = \mathfrak{n}/\mathfrak{n}_r \simeq \mathfrak{n}_1 \oplus \dots \oplus \mathfrak{n}_{r-1}$. It is clear that, in both cases, these automorphisms are hyperbolic and to see that they also are unimodular, recall that for each i , there exists a basis β_i of \mathfrak{n}_i such that $[A_i]_{\beta_i} \in SL_{n_i}(\mathbb{Z})$, where $n_i = \dim \mathfrak{n}_i$ and $A_i = A|_{\mathfrak{n}_i}$.

Note that this argument is valid not only at real or complex level but also at the rational case. Concerning the type of $\tilde{\mathfrak{n}}$ or \mathfrak{n}' we note that if \mathfrak{n} is a three step nilpotent Anosov Lie algebra, then since $[\mathfrak{n}_2, \mathfrak{n}_2] = 0$ then $\tilde{\mathfrak{n}} = \mathfrak{n}/\mathfrak{n}_3 \simeq \mathfrak{n}_1 \oplus \mathfrak{n}_2$ and this decomposition gives the type of $\tilde{\mathfrak{n}}$. On the other hand, in this case, $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}]$ is an abelian Lie algebra.

3. DIMENSION 9

In this section we will study complex Anosov Lie algebras of dimension 9. In fact we will prove that any Anosov Lie algebra without an abelian factor has to be of type $(6, 3)$ or $(3, 3, 3)$. Moreover, there is only one complex Anosov Lie algebra (up to isomorphism) without an abelian factor of type $(3, 3, 3)$. In the case of $(6, 3)$ we give some new examples.

In the following, \mathfrak{n} , will be a complex Anosov Lie algebra with no abelian factor, and A, A_i and \mathfrak{n}_i will be as in Proposition 2.1. We will denote the eigenvalues of A_1 by λ'_i 's, eigenvalues of A_2 by μ'_j 's and eigenvalues of A_3 by ν'_k 's, and the corresponding eigenvectors by X'_i 's, Y'_j 's and Z'_k 's respectively.

For simplicity assume for a moment that \mathfrak{n} is a two step nilpotent Lie algebra. We may assume that for each Y_i , there exist X_j and X_k such that $Y_i = [X_j, X_k]$. On the other hand, if for each j, l , let $a_{j,l}^k \in \mathbb{C}$ be the scalars such that $[X_j, X_l] = \sum a_{j,l}^k Y_k$, since $\{Y_k\}$ are linearly independent, for each k we obtain

$$(1) \quad \lambda_j \lambda_l a_{j,l}^k = \mu_k a_{j,l}^k.$$

Hence, if $a_{j,l}^k \neq 0$, $\mu_k = \lambda_j \lambda_l$, and therefore, if $a_{j,l}^k \neq 0 \neq a_{j,l}^{k'}$, then $\mu_k = \mu_{k'}$. In particular, if $n_2 = 2$, since $\mu_1 \neq \mu_2$, for each j, l , there exists a unique k such that $[X_j, X_l] = a_k Y_k$. If it is so, by (1), $\lambda_j \lambda_l = \mu_k$. When $n_2 = 3$, the same property holds. Indeed, $\mu_i \neq \mu_j$ for all $i \neq j$ since $\deg \mu_i > 1$ for all i . If \mathfrak{n} is a 3-step Anosov Lie algebra and if $n_3 = 2$ or 3 , then for each Z_j we have that $Z_j = [X_i, Y_k]$ and therefore $\nu_j = \lambda_i \mu_k$ for some i, k (see [13]).

Using Proposition [12, Prop. 2.3] we can see that the possible types for a 9 dimensional Anosov Lie algebra are $(7, 2)$, $(6, 3)$, $(5, 4)$, $(4, 5)$, $(5, 2, 2)$, $(4, 3, 2)$, $(4, 2, 3)$ and $(3, 3, 3)$.

As a corollary of Lemma 2.4 and the fact that there is no non-toral 7 dimensional Anosov Lie algebra (see [13]), there are no Anosov Lie algebras of type $(5, 2, 2)$ and $(4, 3, 2)$.

We are now going to show that a nilpotent Lie algebra with no abelian factor of any of this types that actually admits an Anosov automorphism is of type $(6, 3)$ or $(3, 3, 3)$. For all the remaining cases but those, we will systematically use Lemma 2.3 to show that there must be an abelian factor. In these cases we will prove that $\mathfrak{n}_{\mathbb{C}}$ have an abelian factor, and therefore \mathfrak{n} must also carry one. Note that in this situation $\mathfrak{n} = \mathbb{C}^n \oplus \tilde{\mathfrak{n}}$ where $\tilde{\mathfrak{n}}$ is a lower dimensional Anosov Lie algebra with no abelian factor, and therefore should be one of those in the classification given in [13].

Case $(7, 2)$ Suppose that there exists an Anosov Lie algebra \mathfrak{n} of this type with no abelian factor, and let $\{\lambda_1, \dots, \lambda_7\}$ and $\{\mu_1, \mu_2\}$ denote the eigenvalues of A_1 and A_2 with eigenvectors $X_1, \dots, X_7, Y_1, Y_2$, respectively. Hence the splitting of A_1 can be $[7]$, $[5; 2]$, $[4; 3]$ or $[3; 2; 2]$. It is not hard to see that none of these cases are possible. In fact, if the splitting of A_1 is $[7]$ that is $\deg \lambda_i = 7$ for all $1 \leq i \leq 7$, then $\mu_k = \lambda_i \lambda_j$ for some $1 \leq i \neq j \leq 7$ with $\deg \mu_k = 2$, and therefore $(\deg \lambda_i \lambda_j, \deg \lambda_i) = 1$, a contradiction to Lemma 2.3. In fact, we note that if $\max(\deg \lambda_i, \deg \lambda_j)$ is odd, then the corresponding eigenvectors must satisfy $[X_i, X_j] = 0$ since $\lambda_i \lambda_j$ can never be an eigenvalue of A_2 . Therefore, the cases $[5; 2]$ and $[3; 2; 2]$ are not possible either, since $[X_i, X_j] = 0$ for all i, j and the algebra will be abelian.

Finally, since 2 is a prime number, Lemma 2.3 implies that if $\deg \lambda_i = 4$ and $\deg \lambda_j = 3$, (or equivalently 3 and 3) then $[X_i, X_j] = 0$ in the same way as before. Therefore case $[4; 3]$ we will have an abelian factor, namely the ideal generated by X_5, X_6 and X_7 .

Case $(5, 4)$ In this case we can argue in a similar way to the previous one, since the possible splitting of A_1 are $[5]$ and $[3; 2]$. In the first case, for each non zero bracket among the eigenvectors of A_1 , that is if $[X_i, X_j] \neq 0$ for some $1 \leq i, j \leq 5$, we then have that $\lambda_i \lambda_j = \mu_k$ has degree 4 or 2. It is easy to see that either way this contradicts Lemma 2.3 since $\text{g.c.d}(5, 4) = \text{g.c.d}(5, 2) = 1$. In the second one, by the same argument, we know that $[X_i, X_j] = 0$ for $i, j \in \{1, 2, 3\}$, and if $\mu_k = \lambda_i \lambda_j$ with $i \leq 3$ and $j = 4$ or 5 , then $\text{g.c.d}(\deg \lambda_i, \deg \mu_k) = 1$, contradicting again Lemma 2.3. Therefore, this case is not possible since $[X_i, X_j] = 0$ for all i, j .

Case $(4, 5)$ By using Lemma 2.3 and the previous techniques, we can see that in this case also we would have an abelian factor. We also note that this case is also considered in [6].

Case $(4, 2, 3)$ As before, we start by noting that the possible splitting of A_1 are $[4]$ or $[2; 2]$. Let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3\}$ denote the eigenvalues of A with corresponding eigenvectors $\{X_1, X_2, X_3, X_4, Y_1, Y_2, Z_1, Z_2, Z_3\}$. We note that since \mathfrak{n} has no abelian factor, each $\nu_k = \lambda_i \mu_j$ for some $1 \leq i, j \leq 3$ should have degree 3. We then have a contradiction to Lemma 2.3, since if we denote by d the degree of λ_i then either $d = 4$, or $d = 2$, d and 3 are coprime numbers in both cases.

Case (3, 3, 3) Let \mathfrak{n} be a nilpotent Lie algebra of type (3, 3, 3) that admits an Anosov automorphism A . Let $\{\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3\}$ denote the eigenvalues of A with corresponding eigenvectors $\{X_1, X_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3\}$ such that λ'_i 's, ν'_j 's and μ'_k 's are eigenvalues of A_1 , A_2 and A_3 respectively. It is easy to see that we can assume that

$$[X_1, X_2] = Y_1, \quad [X_2, X_3] = Y_2, \quad [X_1, X_3] = Y_3,$$

and since $\lambda_1 \lambda_2 \lambda_3 = \pm 1$ we also have that

$$[X_3, Y_1] = 0, \quad [X_1, Y_2] = 0, \quad [X_2, Y_3] = 0.$$

Let $\mathfrak{n}(a, b, c)$ denotes the Lie algebra given by

$$(2) \quad \begin{aligned} [X_1, X_2] &= Y_1, & [X_2, X_3] &= Y_2, & [X_1, X_3] &= Y_3, \\ [X_1, Y_1] &= a Z_1, & [X_2, Y_2] &= b Z_2, & [X_3, Y_3] &= c Z_3, \end{aligned}$$

for $a, b, c \in \mathbb{C}$. Moreover, since we are assuming that \mathfrak{n} has no abelian factor, we have that $abc \neq 0$ and in this case, by changing the basis in the center it is easy to see that $\mathfrak{n}(a, b, c)$ is isomorphic to $\mathfrak{n}(1, 1, 1)$. In the following we will show that $\mathfrak{n}(1, 1, 1)$ is the only Anosov Lie algebra with no abelian factor of type (3, 3, 3). To do this, we will first show that \mathfrak{n} is isomorphic to $\mathfrak{n}(1, 1, 1)$, and then we will give an hyperbolic automorphism A and a \mathbb{Z} -basis of $\mathfrak{n}(1, 1, 1)$ preserved by A such that the matrix of A in that basis has integer coefficients.

To see that \mathfrak{n} is isomorphic to $\mathfrak{n}(1, 1, 1)$, let's suppose on the contrary that

$$(3) \quad [X_i, Y_j] = Z_k, \quad \text{and} \quad [X_i, Y_l] = Z_r$$

for some i that we can assume to be 1. It is clear from (3) that $j, l \neq 2$ so we may assume that $j = 1$ and $l = 3$. Hence, if $k = r$, (3) implies that $\lambda_1^2 \lambda_2 = \lambda_1^2 \lambda_3$ and therefore we get to the contradiction $\lambda_2 = \lambda_3$. On the other hand, if $k \neq r$, with no loss of generality we can assume that

$$[X_1, Y_1] = a Z_1, \quad [X_1, Y_3] = b Z_2.$$

Now, as we have said before, one has that $Z_3 = [X_i, Y_j]$ for some i, j . It is not hard to see that the possibilities for (i, j) are (2, 1), (2, 2), (3, 2), and (3, 3) but all of them lead us to the same kind of contradiction by checking that the product of the eigenvalues in the center equals $\nu_1^2, \nu_1, \nu_2, \nu_2^2$ respectively. Indeed, if for example we consider case (2, 2), we obtain

$$[X_1, Y_1] = a Z_1, \quad [X_1, Y_3] = b Z_2, \quad [X_2, Y_2] = c Z_3,$$

and therefore $1 = \lambda_1^2 \lambda_2 \cdot \lambda_1^2 \lambda_3 \cdot \lambda_2^2 \lambda_3 = \lambda_1^2 \lambda_2 = \nu_1$, contradicting the fact that A_3 is hyperbolic.

Concerning the rational form which is Anosov, let us begin by noting that we do not have any restriction on the eigenvalues of A but the degree, so we can choose now very special ones. Consider the polynomial in $\mathbb{Z}[X]$ given by

$$(4) \quad f(X) = X^3 - 3X + 1.$$

Then

$$(5) \quad \lambda_1 = \xi + \xi^8, \quad \lambda_2 = \xi^2 + \xi^7, \quad \lambda_3 = \xi^4 + \xi^5,$$

are the roots of $f(X)$, where $\xi = e^{2i\pi/9}$ a ninth root of unity. In this case we have that the extension $\mathbb{Q}(\lambda_1)$ is a cyclic extension of degree 3 over \mathbb{Q} (see [1, p. 543]), and moreover straightforward calculation shows that

$$(6) \quad \lambda_1 = \lambda_3^2 - 2 \quad \lambda_2 = \lambda_1^2 - 2, \quad \lambda_3 = \lambda_2^2 - 2.$$

We consider now the automorphism A as above, corresponding to these λ_i 's, that is, defined by $A(X_i) = \lambda_i X_i$, $A(Y_i) = \mu_i Y_i$ and $A(Z_i) = \nu_i Z_i$ for $i = 1, 2, 3$. Recall that, from (2),

$$\mu_1 = \lambda_1 \lambda_2, \quad \mu_2 = \lambda_2 \lambda_3, \quad \mu_3 = \lambda_1 \lambda_3,$$

$$\nu_1 = \lambda_1 \mu_1, \quad \nu_2 = \lambda_2 \mu_2, \quad \nu_3 = \lambda_3 \mu_3.$$

Note that, due to our choice of λ_i , $\deg(\mu_i) = 3 = \deg(\nu_i)$ for all $1 \leq i \leq 3$. Concerning the new basis, for $i = 1, 2, 3$ let us denote by

$$\mathcal{X}_i = \sum_{j=1}^3 \lambda_j^{i-1} X_j,$$

$$\mathcal{Y}_i = \lambda_3^{1-i} (\lambda_2 - \lambda_1) Y_1 + \lambda_1^{1-i} (\lambda_3 - \lambda_2) Y_2 + \lambda_2^{1-i} (\lambda_3 - \lambda_1) Y_3,$$

$$\mathcal{Z}_i = (\lambda_1)^{i-1} (\lambda_2 - \lambda_1) Z_1 + (\lambda_2)^{i-1} (\lambda_3 - \lambda_2) Z_2 + (\lambda_3)^{i-1} (\lambda_3 - \lambda_1) Z_3.$$

Let $\beta_1 = \{\mathcal{X}_i, 1 \leq i \leq 3\}$, $\beta_2 = \{\mathcal{Y}_i, 1 \leq i \leq 3\}$ and $\beta_3 = \{\mathcal{Z}_i, 1 \leq i \leq 3\}$. It is easy to see that β_i 's are linearly independent sets. Moreover, we can see that $A(\mathcal{X}_i) = \mathcal{X}_{i+1}$, $A(\mathcal{Y}_i) = \mathcal{Y}_{i+1}$ for $i = 1, 2$, $A(\mathcal{X}_3) = 3\mathcal{X}_2 - \mathcal{X}_1$ and $A(\mathcal{Y}_3) = 3\mathcal{Y}_3 - \mathcal{Y}_1$ by using that $f(\lambda_i) = 0$ and therefore $\lambda_i^{-1} = 3 - \lambda_i^2$. Also, by using (6) one can see that $A(\mathcal{Z}_i)$ is an integer linear combination of the \mathcal{Z}_i 's. In fact, for example

$$\begin{aligned} A(\mathcal{Z}_1) &= \lambda_1^2 \lambda_2 (\lambda_2 - \lambda_1) Z_1 + \lambda_2^2 \lambda_3 (\lambda_3 - \lambda_2) Z_2 + \lambda_3^2 \lambda_1 (\lambda_3 - \lambda_1) Z_3 \\ &= \lambda_1^2 (\lambda_1^2 - 2) (\lambda_2 - \lambda_1) Z_1 + \lambda_2^2 (\lambda_2^2 - 2) (\lambda_3 - \lambda_2) Z_2 + \lambda_3^2 (\lambda_3^2 - 2) (\lambda_3 - \lambda_1) Z_3 \\ &= (\lambda_1^2 - \lambda_1) (\lambda_2 - \lambda_1) Z_1 + (\lambda_2^2 - \lambda_2) (\lambda_3 - \lambda_2) Z_2 + (\lambda_3^2 - \lambda_3) (\lambda_3 - \lambda_1) Z_3 \\ &= \mathcal{Z}_3 - \mathcal{Z}_2. \end{aligned}$$

Therefore, $\beta = \{\mathcal{X}_i, \mathcal{Y}_j, \mathcal{Z}_k : i, j, k = 1, 2, 3\}$ is a basis of $\mathfrak{n}(1, 1, 1)$ preserved by A and moreover $[A]_\beta \in GL(9, \mathbb{Z})$.

To finish, it remains to show that β is also a \mathbb{Z} -basis. We first note that since $\lambda_1 \lambda_2 \lambda_3 = 1$, $\lambda_1 + \lambda_2 \lambda_3 = 0$ and $\lambda_i^{-2} = 3\lambda_i^{-1} - \lambda_i$, we have that

$$(7) \quad [\mathcal{X}_1, \mathcal{X}_2] = \mathcal{Y}_1, \quad [\mathcal{X}_1, \mathcal{X}_3] = \mathcal{Y}_3 - 3\mathcal{Y}_2.$$

Moreover, by using (6) as above, we have

$$[\mathcal{X}_1, \mathcal{Y}_1] = \mathcal{Z}_1, \quad [\mathcal{X}_1, \mathcal{Y}_2] = \mathcal{Z}_2 - \mathcal{Z}_1, \quad [\mathcal{X}_1, \mathcal{Y}_3] = \mathcal{Z}_3 - 2\mathcal{Z}_2 + \mathcal{Z}_1.$$

Note that since A is an automorphism of \mathfrak{n} and by our construction of the basis, these are the only brackets one need to check. Hence β is a \mathbb{Z} -basis of $\mathfrak{n}(1, 1, 1)$ preserved by the hyperbolic automorphism A , such that $[A]_\beta \in GL(9, \mathbb{Z})$, and therefore $\mathfrak{n}(1, 1, 1)$ is an Anosov Lie algebra, as we wanted to show.

Remark 3.1. We will use the above proof as a model proof to show that a given Lie algebra is Anosov in some of the following cases where we will just exhibit the automorphism and the \mathbb{Z} -basis.

Case (6, 3) We first note that complex nilpotent Lie algebras of type (6, 3) have been studied in [8] sec 4. where they obtained a classification up to $SL(6) \times SL(3)$ action. Since we are going to use some of their results we will introduce now their notation.

Let $\{X_i : 1 \leq i \leq 9\}$ be the canonical basis of \mathbb{R}^9 and for any $t, s, r \in \mathbb{C}$ denote by $t U_1 + s U_2 + r U_3$ the 2-step nilpotent Lie algebra given by

$$(8) \quad \begin{aligned} [X_1, X_2] &= t X_7 & [X_3, X_4] &= t X_8 & [X_5, X_6] &= t X_9 \\ [X_5, X_4] &= s X_7 & [X_1, X_6] &= s X_8 & [X_3, X_2] &= s X_9 \\ [X_3, X_6] &= r X_7 & [X_5, X_2] &= r X_8 & [X_1, X_4] &= r X_9. \end{aligned}$$

where $U_i = (\mathbb{R}^9, [\cdot, \cdot]_i)$ is the 2-step nilpotent Lie algebra where $[\cdot, \cdot]_i$ is given by the i^{th} row of (8). It is easy to see that these are 2-step nilpotent Lie algebras of type (6, 3). The classification given in [8] splits the (6, 3) nilpotent Lie algebras in 7 families. The first 6, consists of Lie algebras with a semisimple part (closed $SL(6) \times SL(3)$ orbit) given in terms of $t U_1 + s U_2 + r U_3$, for some values of r, s, t plus a nilpotent part (non-closed $SL(6) \times SL(3)$ orbit) that are listed in a table. The last family consists of 76 nilpotent elements. Note that since we are interested in (6, 3) Lie algebras up to isomorphism, we need orbits for the action of $GL(6) \times GL(3)$, and therefore it is not hard to see that to this classification we only have to add the action of scalar multiples of the identity. Then, for example we have that $s U_2 + r U_3 \simeq s' U_2 + U_3$ for any $r \neq 0$ and the semisimple part of Family 4 (and 5) are all isomorphic.

Coming back to the Anosov problem, let \mathfrak{n} be an Anosov Lie algebra of type (6, 3) with no abelian factor, and let A, A_1, A_2 , be as in Proposition 2.1. Denote by $\lambda_1, \dots, \lambda_6$ the eigenvalues of A_1 , by μ_1, μ_2, μ_3 the eigenvalues of A_2 and the corresponding basis of eigenvectors by $\beta = \{X_1, \dots, X_6, Y_1, Y_2, Y_3\}$. Let f and g be the characteristic polynomials of A_1 and A_2 respectively. By using Lemma 2.3 it can be seen that that g is irreducible over \mathbb{Z} , and either f is irreducible or f has two irreducible factors of degree 3 over \mathbb{Z} . Note that this corresponds to the fact that the spitting of A_1 is [6] or [3; 3]. We are going to study now each one of these cases separately.

Case i: (f is irreducible over \mathbb{Z}). We note that, in particular, this implies that $\lambda_i \neq \lambda_j$ for all $i \neq j$. We begin by showing that we can assume that

$$(9) \quad [X_1, X_2] = Y_1, \quad [X_3, X_4] = Y_2, \quad [X_5, X_6] = Y_3.$$

In fact, if this is not the case we can reorder the basis so that

$$(10) \quad [X_1, X_2] = Y_1, \quad [X_1, X_3] = Y_2,$$

and in this situation we have to consider three possibilities for Y_3 ,

$$(11) \quad \text{(I)} [X_1, X_4] = Y_3, \quad \text{(II)} [X_2, X_j] = Y_3, \quad \text{(III)} [X_4, X_5] = Y_3.$$

Note that each one of this situations represents a few others that are totally equivalent to the considered one.

Case (III) is the simplest one because from (10) it follows at once that $\lambda_1^2 \lambda_2 \lambda_3 \lambda_4 \lambda_5 = 1$ and therefore we obtain the contradiction $\lambda_1 = \lambda_6$.

In case **(I)** we have that

$$(12) \quad \lambda_1^3 \lambda_2 \lambda_3 \lambda_4 = 1 \quad \text{or, equivalently} \quad \lambda_1^2 = \lambda_5 \lambda_6.$$

Let us consider then the possibilities for $[X_5, X_j] = Y_k$, $[X_6, X_l] = Y_r$. With no loss of generality we may assume that $[X_5, X_j] = Y_1$, and hence, $\lambda_5 \lambda_j = \lambda_1 \lambda_2$ or equivalently by (12), $\lambda_1 \lambda_j = \lambda_6 \lambda_2$. Therefore, by all this, it is easy to see that $j \neq 1, 2, 5, 6$ and hence $j = 3$ or 4 . Since both cases are entirely analogous, we will just consider $j = 4$. That is,

$$(13) \quad \begin{aligned} [X_1, X_2] &= Y_1, & [X_1, X_3] &= Y_2 & [X_1, X_4] &= Y_3 \\ [X_5, X_4] &= Y_1. \end{aligned}$$

From this, it is easy to see that $r \neq 1$, and if $r = 3$, then $l = 2$ since product of all λ_i 's is 1 and then by rearranging the basis we would have (9) as desired. To see that $r = 2$ is not possible either, suppose that $[X_6, X_l] = Y_2$. If this is the case, it is easy to see that $l \neq 1, 3, 6, 5, 2$, and so, we may assume that $[X_6, X_4] = Y_2$. Hence,

$$1 = \mu_1 \mu_2 \mu_3 = \lambda_1 \lambda_4^3 \lambda_5 \lambda_6.$$

By using (12), $(\lambda_1 \lambda_4)^3 = 1$ which implies that $\mu_3^3 = 1$, contradicting the fact that A_2 is hyperbolic.

Finally concerning case **(II)**, note first that we have to distinguish between $j = 3$ and $j \neq 3$. If $j \neq 3$, let us say $j = 4$ (or equivalently $j = 5$ or 6), we have

$$(14) \quad [X_1, X_2] = Y_1, \quad [X_1, X_3] = Y_2 \quad [X_2, X_4] = Y_3.$$

Therefore we obtain, equivalently,

$$(15) \quad \lambda_1^2 \lambda_2^2 \lambda_3 \lambda_4 = 1 \quad \text{or} \quad \lambda_5 \lambda_6 = \lambda_1 \lambda_2.$$

It is clear from this that if $[X_5, X_l] = a Y_1$ or $[X_6, X_k] = b Y_1$ then $l = 6$ and $k = 5$ and we would have (9) as desired. In a very similar way we did in the previous case, it is easy to see that this situation leads us to either case I or (9). Therefore, we can assume (9), that is

$$[X_1, X_2] = Y_1, \quad [X_3, X_4] = Y_2, \quad [X_5, X_6] = Y_3.$$

If there are no more non trivial Lie brackets but these, then $\mathfrak{n} \simeq \mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathfrak{h}_3 \simeq U_1$ which is known to be Anosov (see [13]). If there are more non trivial Lie brackets, without any loss of generality, we can assume that $[X_3, X_2] = c Y_3$. Moreover, using that there is no abelian factor, that the λ_i 's are all distinct and that $\lambda_1 \dots \lambda_6 = 1$, one can see that $\mathfrak{n}_{\mathbb{C}}$ is isomorphic to $\mathfrak{n}'(a, b, c)$, given by

$$(16) \quad \begin{aligned} [X_1, X_2] &= Y_1, & [X_3, X_4] &= Y_2 & [X_5, X_6] &= Y_3, \\ [X_5, X_4] &= a Y_1, & [X_1, X_6] &= b Y_2 & [X_3, X_2] &= c Y_3, \end{aligned}$$

for some $a, b, c \in \mathbb{C}$. By the same reasons given above, it is easy to see that we can not add more nontrivial Lie brackets.

To study how many non isomorphic Lie algebras we obtain from (16), we start by noting that if $abc = 0$, from [8] we get that we have only two non isomorphic Lie algebras:

$$\mathfrak{n}'(0, 0, c) \simeq \mathfrak{n}'(0, 0, 1) \simeq U_1 + \mathfrak{n}_{14}, \quad \text{and} \quad \mathfrak{n}'(a, b, 0) \simeq \mathfrak{n}'(1, 1, 0) \simeq U_1 + \mathfrak{n}_{11},$$

both in Family 6, where \mathfrak{n}_j denote the nilpotent part j given in Table 7.

On the other hand, if $abc \neq 0$, by changing the basis to

$$\beta' = \{X_1, aX_2, \frac{1}{a}X_3, \frac{1}{c}X_4, cX_5, X_6, aY_1, \frac{1}{ac}Y_2, cY_3\},$$

we obtain

$$(17) \quad \begin{aligned} [X_1, X_2] &= Y_1, & [X_3, X_4] &= Y_2, & [X_5, X_6] &= Y_3, \\ [X_5, X_4] &= Y_1, & [X_1, X_6] &= abc Y_2, & [X_3, X_2] &= Y_3, \end{aligned}$$

showing that $\mathfrak{n}'(a, b, c) \simeq \mathfrak{n}'(1, abc, 1)$, and then (if $abc \neq 0$)

$$\mathfrak{n}'(a, b, c) \simeq \mathfrak{n}'(s, s, s) \simeq U_1 + s U_2 \simeq s U_2 + U_3,$$

where $s^3 = abc$. Therefore, if $s^3 = s'^3$, $s U_2 + U_3 \simeq s' U_2 + U_3$. In the classification given in [8] one has the following non isomorphic Lie algebras

- $s U_2 + U_3$, corresponding to $s^3 \neq 0, \pm 1$ (Family 2)
- $U_2 + U_3$, corresponding to $s^3 = 1$ (Family 4)
- $-U_2 + U_3$, corresponding to $s^3 = -1$ (Family 5)
- $U_3 \simeq U_1$, corresponding to $s = 0$ (Family 6).

We are not able to decide if the algebras in Family 2 are Anosov or not, we note that since it gives an uncountable family of nonisomorphic Lie algebras, not all of them can be Anosov.

For $\mathfrak{s} = \mathbf{0}$, as we have already said $\mathfrak{n} \simeq \mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathfrak{h}_3$ is an Anosov Lie algebra.

If $\mathfrak{s}^3 = \mathbf{1}$, (assumed to be 1) we will show that $U_2 + U_3$ (see (8) with $t = 0, r = s = 1$), is an Anosov Lie algebra, by using the ideas of [17].

Let $\{\lambda_1, \lambda_2, \lambda_3\}$ be the roots of the polynomial in (4) used in the (3, 3, 3)-case. As we have noted there, λ_i are degree 3 algebraic units. Let μ be an algebraic unit of degree 2, such that

$|\mu^{\pm 1} \lambda_j| \neq 1$, for $j = 1, 2, 3$, and let A be the automorphism of \mathfrak{n} such that $[A]_\beta = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$, where

$$A_1 = \begin{bmatrix} \mu\lambda_3 & & & & & \\ & \mu^{-1}\lambda_3 & & & & \\ & & \mu\lambda_2 & & & \\ & & & \mu^{-1}\lambda_2 & & \\ & & & & \mu\lambda_1 & \\ & & & & & \mu^{-1}\lambda_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda_1\lambda_2 & & & \\ & \lambda_1\lambda_3 & & \\ & & \lambda_2\lambda_3 & \\ & & & \end{bmatrix}.$$

Due to our choice of μ and λ_i 's, it is clear that A is hyperbolic, and moreover, it is easy to see that A is also an (hyperbolic) automorphism for $U = s U_2 + U_3$, for any $s \in \mathbb{C}$. Straightforward calculation shows that if

$$\mathcal{X}_{(k,l)} = \lambda_3^l (\mu^k X_1 + \mu^{-k} X_2) + \lambda_2^l (\mu^k X_3 + \mu^{-k} X_4) + \lambda_1^l (\mu^k X_5 + \mu^{-k} X_6)$$

$$\mathcal{Y}_r = (\mu^{-1} - \mu)(\lambda_3^r Y_1 + \lambda_2^r Y_2 + \lambda_1^r Y_3),$$

then

$$\beta' = \{\mathcal{X}_{(k,l)}, \mathcal{Y}_r \mid k = 0, 1, \quad l = 0, 1, 2 \quad r = -1, 0, 1\},$$

is a \mathbb{Z} -basis of $\mathfrak{n} = U_2 + U_3$ preserved by A such that $[A]_{\beta'} \in GL(9, \mathbb{Z})$ (see [17] Example 2.5 and Remark 2.6).

For the case when $\mathfrak{s}^3 = -1$ let us change the basis of $-U_2 + U_3$ to

$$\beta = \{X_1 + X_2, X_3 + X_4, X_5 + X_6, X_1 - X_2, X_3 - X_4, X_5 - X_6, 2Y_3, 2Y_1, -2Y_2\}$$

and we then get

$$(18) \quad \begin{aligned} [X_1, X_2] &= Y_1, & [X_2, X_3] &= Y_2, & [X_1, X_3] &= Y_3, \\ [X_4, X_5] &= Y_1, & [X_5, X_6] &= Y_2, & [X_4, X_6] &= Y_3, \end{aligned}$$

which is an amalgamated sum of the type studied in [17]. Indeed, if we denote the algebra given by the first row by $\mathfrak{n}_1 = \mathfrak{v}_1 + \mathfrak{z}_1$ and by $\mathfrak{n}_2 = \mathfrak{v}_2 + \mathfrak{z}_2$ the one given by the second row, it is easy to see that both are isomorphic (Anosov) Lie algebras and moreover $\mathfrak{n} \simeq (\mathfrak{v}_1 + \mathfrak{v}_2) + \mathfrak{z}_1$. In this case by identifying both basis of the center, it is easy to see that this is an Anosov Lie algebra (see [17], Example 2.3).

(ii) f has two irreducible factors over \mathbb{Z} (each of degree 3). We can assume that both of them have constant term equal to 1 by considering the square of the automorphism if required. As usually, let $\{X_i : 1 \leq i \leq 6\}$ and Y_1, Y_2, Y_3 be the eigenvectors of A_1 and A_2 corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3$ respectively. It is clear that one may either have

$$(a) \quad [X_i, X_j] = Y_k \text{ for some } 1 \leq i, j \leq 3 \text{ (or equivalently } 4 \leq i, j \leq 6) \text{ or}$$

$$(b) \quad [X_i, X_j] = 0 \text{ for all } 1 \leq i, j \leq 3 \text{ and } [X_k, X_l] = 0 \text{ for all } 4 \leq k, l \leq 6.$$

In **a**), with no loss of generality we may assume $[X_1, X_2] = Y_1$. At the eigenvalue level, this means that $\nu_1 = \lambda_1 \lambda_2$ is a root of g . Since $\deg(\lambda_1 \lambda_2) = 3$ then $\lambda_1 \lambda_2 = \lambda_3^{-1}$, $\lambda_1 \lambda_3 = \lambda_2^{-1}$ and $\lambda_2 \lambda_3 = \lambda_1^{-1}$ are the roots of g . On the other hand, the absence of abelian factor implies that for every $k = 4, 5, 6$

$$[X_k, X_j] \neq 0 \quad \text{for } 1 \leq j \leq 3 \text{ or } 4 \leq j \leq 6.$$

It is not hard to see that from here we can either have

$$\{\mu_1, \mu_2, \mu_3\} = \{\lambda_1^{-2}, \lambda_2^{-2}, \lambda_3^{-2}\} \quad \text{or} \quad \{\mu_1, \mu_2, \mu_3\} = \{\lambda_1, \lambda_2, \lambda_3\}.$$

If $\mu_i = \lambda_j^{-2}$, we may rearrange the basis so that $\mu_i = \lambda_i^{-2}$, and hence it is easy to see that, after a new permutation if needed, the non trivial Lie brackets in \mathfrak{n} are given by

$$(19) \quad \begin{aligned} [X_1, X_2] &= Y_1, & [X_2, X_3] &= Y_2, & [X_1, X_3] &= Y_3, \\ [X_3, X_6] &= c Y_1, & [X_1, X_4] &= a Y_2, & [X_2, X_5] &= b Y_3 \end{aligned}$$

for some $a, b, c \in \mathbb{C}$. Due to our assumption of no abelian factor we have that $abc \neq 0$ and in this case it is easy to see that if we denote by $\tilde{\mathfrak{n}}(a, b, c)$ the Lie algebra defined above, we then have that $\tilde{\mathfrak{n}}(a, b, c) \simeq \tilde{\mathfrak{n}}(1, 1, 1) \simeq U_1 + \mathfrak{n}_{10}$ (is non-isomorphic to any of the ones previously studied). To see that this is an Anosov Lie algebra, we will use very similar arguments as in the case (3, 3, 3). Let λ'_i s denote the roots of the polynomial $X^3 - 3X + 1$ (see (4)) and let

$$\tilde{\beta} = \{\mathcal{X}_i, \mathcal{X}'_j, \mathcal{Y}_k \mid i, j, k = 1, 2, 3\}, \quad \text{where}$$

$$\mathcal{X}_i = \sum_{j=1}^3 \lambda_j^{i-1} X_j,$$

$$\mathcal{X}'_j = \lambda_3^{1-j} (\lambda_2 - \lambda_1) X_6 + \lambda_1^{1-j} (\lambda_3 - \lambda_2) X_4 + \lambda_2^{1-j} (\lambda_3 - \lambda_1) X_5,$$

$$\mathcal{Y}_k = \lambda_3^{1-k} (\lambda_2 - \lambda_1) Y_1 + \lambda_1^{1-k} (\lambda_3 - \lambda_2) Y_2 + \lambda_2^{1-k} (\lambda_3 - \lambda_1) Y_3.$$

Note that \mathcal{X}_i and \mathcal{Y}_k are defined in the same way as in case (3, 3, 3) since the brackets among the X_i and Y_j are the same as in that case. Then one only need to check for the \mathcal{X}'_j . Straightforward calculation shows that $\tilde{\beta}$ is a \mathbb{Z} -basis of $\tilde{\mathfrak{n}}(1, 1, 1)$ preserved by A and moreover the matrix $[A]_{\tilde{\beta}}$ is in $GL(9, \mathbb{Z})$, as we wanted to show (see [17]).

On the other hand, if the eigenvalues of A_1 are $\{\lambda_1, \lambda_2, \lambda_3, \lambda_1, \lambda_2, \lambda_3\}$, then the non zero Lie brackets on \mathfrak{n} are given by

$$\begin{aligned} [X_1, X_2] &= a Y_1, & [X_2, X_3] &= b Y_2 & [X_1, X_3] &= c Y_3, \\ [X_4, X_5] &= d Y_1, & [X_5, X_6] &= e Y_2 & [X_4, X_6] &= f Y_3, \\ [X_1, X_5] &= g Y_1, & [X_2, X_6] &= h Y_2 & [X_3, X_4] &= i Y_3, \\ [X_4, X_2] &= j Y_1, & [X_5, X_3] &= k Y_2 & [X_6, X_1] &= l Y_3, \end{aligned}$$

for some $a, b, c, d, e, f, g, h, i, j, k, l \in \mathbb{C}$.

Note that since the Pfaffian form of these Lie algebras (see [12]) is given by

$$(20) \quad Pf(x, y, z) = xyz(afk - ael + bdi - bgf + cje - cdh + lhg - ijk),$$

by calculating the pfaffian forms of the algebras listed in the classification given in [8] one can see that they should be isomorphic to one of the following:

- (1) $sU_2 + U_3$ $s \neq 0, \pm 1$ (Family 2)
- (2) $U_2 + U_3$ (Family 4)
- (3) $U_2 + U_3 + \mathfrak{n}_3$ (Family 4)
- (4) $-U_2 + U_3$ (Family 5)
- (5) $U_1 + \mathfrak{n}_i$ for $i = 6, 7, 10, 11, 12, 14$ (Family 6).

Note that cases (1),(2),(4) and the last one corresponding to $i = 10, 11, 14$ has been already considered. We can not decide for $i = 6, 7, 12$ but we will show that $\mathfrak{n} = U_2 + U_3 + \mathfrak{n}_3$ is an Anosov Lie algebra. To do this we first note that by rearranging the basis to

$$\beta = \{X_1, X_6, X_3, X_2, X_5, X_4, -Z_2, Z_1, Z_3\}$$

it is easy to see that this algebra corresponds to $\mathfrak{n}(1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 1)$. Hence it is clear that A is an automorphism of \mathfrak{n} and if λ_i $i = 1, 2, 3$ are (as above) the roots of the polynomial $X^3 - 3X + 1$, it is not hard to see that the \mathbb{Z} -basis that shows that this is an Anosov automorphism of \mathfrak{n} is given by $\beta' = \{\mathcal{X}_i, \mathcal{Y}_k : i, k = 1, 2, 3\}$ where

$$\mathcal{X}_i = \lambda_1^{i-1}(X_1 + X_4) + \lambda_2^{i-1}(X_2 + X_5) + \lambda_3^{i-1}(X_3 + X_6),$$

$$\mathcal{Y}_k = \lambda_3^{1-k}(\lambda_2 - \lambda_1)Y_1 + \lambda_1^{1-k}(\lambda_3 - \lambda_2)Y_2 + \lambda_2^{1-k}(\lambda_3 - \lambda_1)Y_3.$$

To conclude, let us study case (b) when $[X_i, X_j] = 0$ for $1 \leq i, j \leq 3$ and $[X_k, X_l] = 0$ for all $4 \leq k, l \leq 6$. It is clear that we may assume that $[X_1, X_4] = Y_1$, that is $\nu_1 = \lambda_1\mu_1$. We are now going to prove that we may assume that

$$[X_1, X_4] = Y_1, \quad [X_2, X_5] = Y_2 \quad [X_3, X_6] = Y_3,$$

and then the roots of g are given by $\nu_1 = \lambda_1\mu_1, \nu_2 = \lambda_2\mu_2, \nu_3 = \lambda_3\mu_3$, that is, they are products of different eigenvalues. In fact, if on the contrary we assume that

$$[X_1, X_4] = Y_1, \quad [X_1, X_5] = Y_2$$

it is easy to see that if $[X_i, X_j] = Y_3$, then $i \neq 1$, and $j \neq 3$. We may assume then that $[X_2, X_5] = Y_3$ and hence $\lambda_1^2 \lambda_2 \mu_1 \mu_2^2 = 1$ or equivalently, $\lambda_1 \mu_2 = \lambda_3 \mu_3$. From this, by considering all possibilities for $[X_i, X_6] = Y_k$ and $[X_3, X_j] = Y_l$ and rearranging the basis if required, it is not hard to check that we should have $[X_3, X_6] = Y_2$. Then by rearranging the basis if needed, we may assume that

$$[X_1, X_4] = Y_1, \quad [X_2, X_5] = Y_2 \quad [X_3, X_6] = Y_3,$$

as desired. According to this, \mathfrak{n} has at least these non trivial Lie brackets. If \mathfrak{n} has only this non trivial Lie brackets, then $\mathfrak{n} \simeq \mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathfrak{h}_3$ which is proven to be Anosov in [10].

If there are more non trivial Lie brackets, with no loss of generality, we may assume that $[X_3, X_5] = a Y_1$, and hence we have that

$$\nu_1 = \lambda_1 \mu_1 = \lambda_3 \mu_2.$$

Note that since $\lambda_i \neq \lambda_j$ and $\mu_i \neq \mu_j$, if $[X_i, X_j] = Y_k = [X_l, X_r]$ then $i \neq l$ and $j \neq r$. From this, it is not hard to see that the Lie bracket in \mathfrak{n} are given by

$$(21) \quad \begin{aligned} [X_1, X'_1] &= Y_1, & [X_2, X'_2] &= Y_2 & [X_3, X'_3] &= Y_3, \\ [X_3, X'_2] &= a Y_1, & [X_1, X'_3] &= b Y_2 & [X_2, X'_1] &= c Y_3, \end{aligned}$$

for some constants $a, b, c \in \mathbb{C}$. Note that by reordering the basis as

$$\beta' = \{X_1, X'_1, X_2, X'_2, X_3, X'_3, Y_1, Y_2, Y_3\}.$$

one can see that this algebra is isomorphic to $\mathfrak{n}(a, b, c)$ given in (16) that has already been studied.

Finally, if \mathfrak{n} has an abelian factor, that is $\mathfrak{n} = \tilde{\mathfrak{n}} \oplus \mathfrak{a}$, where \mathfrak{a} is an abelian ideal, according to [12, Theorem 3.6] one has that $\dim \mathfrak{a} \geq 2$. Moreover, $\tilde{\mathfrak{n}}$ is also an Anosov Lie algebra and $\dim \tilde{\mathfrak{n}} \leq 7$. Since there is no 7 dimensional ones we then have that \mathfrak{n} is one of the following ones (see [13]).

- \mathbb{R}^9
- $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^3$
- $\mathfrak{f}_3 \oplus \mathbb{R}^3$,

where \mathfrak{h}_3 is the 3-dimensional Heisenberg algebra, and \mathfrak{f}_3 is the free 2-step nilpotent Lie algebra on 3 generators.

Remark 3.2. From all of the above, we can assert that if \mathfrak{n} is a 9-dimensional Anosov Lie algebra then its Pfaffian form is projectively equivalent to xyz or 0. Moreover it is isomorphic to one of the following non isomorphic Lie algebras: $s U_2 + U_3$ $s \in \mathbb{C}$, $U_2 + U_3 + \mathfrak{n}_3$, $U_1 + \mathfrak{n}_i$ for $i = 6, 7, 10, 11, 12, 14$, \mathbb{R}^9 , $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^3$ or $\mathfrak{f}_3 \oplus \mathbb{R}^3$.

4. DIMENSION 10

We are now going to make some remarks about Anosov Lie algebras of dimension 10 with no abelian factor. We will give all possible types of Anosov Lie algebras of dimension 10 and give an example in each possible type.

We will begin by noting that restricted to the case of two steps Anosov Lie algebras of dimension 10, the possible cases are, (8, 2), (7, 3), (6, 4), (5, 5) and (4, 6). The (4, 6) is the free one which is known to be Anosov (see [3]). In the same way we discarded the case (7, 2) in the previous classification, one can see that there is

no $(7, 3)$ Anosov Lie algebra with no abelian factor. There is at least one $(8, 2)$ and one $(6, 4)$, namely $\mathfrak{n} \oplus \mathfrak{n}$ where \mathfrak{n} is the 5-dimensional 2-step nilpotent Lie algebra without an abelian factor of type $(4, 1)$ in the first case, and of type $(3, 2)$ in the second case (see [10]).

Concerning $(5, 5)$ case, we would like to note that our proof that such an algebra exists, depends on the existence of very special algebraic numbers that we are now going to exhibit. Let $f(X) = X^5 + X^4 - 4X^3 - 3X^2 + 3X + 1$. One can see that this is an irreducible polynomial with real roots $\lambda_1, \dots, \lambda_5$ such that $|\lambda_i| \neq 1$ and $|\lambda_i \lambda_j| \neq 1$ for all i, j . In fact, the roots are

$$\begin{aligned} \lambda_1 &= \zeta^5 + \zeta^6, & \lambda_2 &= \zeta^3 + \zeta^8, & \lambda_3 &= \zeta^4 + \zeta^7, \\ \lambda_4 &= \zeta^2 + \zeta^9, & \lambda_5 &= \zeta + \zeta^{10}, \end{aligned}$$

where ζ is the primitive 11th root of unity (see [1]). Straightforward calculation shows that

$$(22) \quad \lambda_{i+1}^2 = \lambda_i + 2, \quad i = 1, \dots, 4 \quad \text{and} \quad \lambda_1^2 = \lambda_5 + 2.$$

Hence, in particular, $\mathbb{Q}(\lambda_1)$ contains λ_5 and therefore the extension $\mathbb{Q}(\lambda_1)$ is cyclic Galois extension of degree 5 over \mathbb{Q} . Also, since $\deg(\lambda_1 \lambda_2)$ divides 5, we have that $\deg(\lambda_1 \lambda_2) = 5$ and $\lambda_1 \lambda_2, \lambda_2 \lambda_3, \lambda_3 \lambda_4, \lambda_4 \lambda_5, \lambda_5 \lambda_1$ are all conjugates over \mathbb{Q} (this can be seen by applying a cycle $(1\ 2\ 3\ 4\ 5)$ in the Galois group).

Now, let \mathfrak{n} be a $(5, 5)$ -type real Lie algebra with the basis $X_1, \dots, X_5, Y_1, \dots, Y_5$ and with the following non trivial brackets:

$$\begin{aligned} [X_1, X_2] &= Y_1, & [X_2, X_3] &= Y_2, & [X_3, X_4] &= Y_3, \\ [X_4, X_5] &= Y_4, & [X_5, X_1] &= Y_5. \end{aligned}$$

Let A denote the automorphism of \mathfrak{n} such that $A(X_i) = \lambda_i X_i$ for all i and $A(Y_j) = \lambda_j \lambda_{j+1} Y_j$ for all j . By our election of the λ 's it is clear that A is hyperbolic. We will now proceed in a similar way we did in the $(3, 3, 3)$ case to get a \mathbb{Z} -basis. In fact, let

$$\mathcal{X}_i = \sum_{j=1}^5 \lambda_j^{i-1} X_j, \quad i = 1, 2, \dots, 5,$$

then it is easy to see that for $i = 2, \dots, 5$

$$\begin{aligned} (23) \quad [\mathcal{X}_1, \mathcal{X}_i] &= (\lambda_2^{i-1} - \lambda_1^{i-1})Y_1 + \dots + (\lambda_1^{i-1} - \lambda_5^{i-1})Y_5 \\ &= (\lambda_2^{i-1} - (\lambda_2^2 - 2)^{i-1})Y_1 + \dots + (\lambda_1^{i-1} - (\lambda_1^2 - 2)^{i-1})Y_5, \end{aligned}$$

where we have used (22). Since $f(\lambda_i) = 0$ for $1 \leq i \leq 5$ it is not hard to see that (23) can be written as integer combinations of

$$(24) \quad \mathcal{Y}_i = \lambda_2^{i-1} Y_1 + \lambda_3^{i-1} Y_2 + \lambda_4^{i-1} Y_3 + \lambda_5^{i-1} Y_4 + \lambda_1^{i-1} Y_5 \quad i = 1, \dots, 5.$$

Hence, if $\beta = \{\mathcal{X}_i, \mathcal{Y}_j, 1 \leq i, j \leq 5\}$ it is easy to see that β is a basis of \mathfrak{n} and by the above observations we also have that it is a \mathbb{Z} -basis. To see that it is preserved by A note that

$$A(\mathcal{X}_i) = \mathcal{X}_{i+1}, \quad i = 1, \dots, 4 \quad \text{and} \quad A(\mathcal{X}_5) = -\mathcal{X}_5 + 4\mathcal{X}_4 + 3\mathcal{X}_3 - 3\mathcal{X}_2 - \mathcal{X}_1.$$

On the other hand, since $A(Y_i) = \lambda_i \lambda_{i+1} Y_i$ for $1 \leq i \leq 5$, by (22) we have that

$$A(Y_i) = (\lambda_{i+1}^2 - 2)\lambda_{i+1} Y_i \quad 1 \leq i \leq 4 \quad \text{and} \quad A(Y_5) = (\lambda_1^2 - 2)\lambda_1 Y_5,$$

and therefore

$$\begin{aligned} A(\mathcal{Y}_i) = & (\lambda_2^{i+2} - 2\lambda_2)Y_1 + (\lambda_3^{i+2} - 2\lambda_3)Y_2 + (\lambda_4^{i+2} - 2\lambda_4)Y_3 \\ & + (\lambda_5^{i+2} - 2\lambda_5)Y_4 + (\lambda_1^{i+2} - 2\lambda_1)Y_5, \quad \text{for } i = 1, \dots, 5. \end{aligned}$$

finally, using that $f(\lambda_i) = 0$, for all $1 \leq i \leq 5$ straightforward calculation shows that $A(\mathcal{Y}_i)$ is an integer combination of the \mathcal{Y}_i and therefore $[A]_\beta \in GL(10, \mathbb{Z})$. Hence \mathfrak{n} is an Anosov Lie algebra as we wanted to show.

we have recently become aware of [20] where this example of an Anosov Lie algebra is also obtained.

Concerning 3-steps, the the possible cases are, $(6, 2, 2)$, $(5, 3, 2)$, $(5, 2, 3)$, $(4, 4, 2)$, $(4, 3, 3)$, $(4, 2, 4)$ and $(3, 3, 4)$. Using Lemma 2.4 and the classification given in [13] it is easy to see that cases $(5, 2, 3)$ and $(4, 3, 3)$ are not possible, since there is no $(5, 2)$ or $(4, 3)$ type Anosov Lie algebra. We can also exclude case $(3, 3, 4)$ by using Lemma 2.3, in the same way as we did in dimension 9. To study the other cases, one should see in each case if it is possible or not to extend (in the sense of Lemma 2.4) the corresponding 2-step Anosov Lie algebra.

Let us begin with the case $(5, 3, 2)$. Here, according to [13] table 3, such Anosov Lie algebra should be an extension of $\mathfrak{f}_3 \oplus \mathbb{R}^2$, where \mathfrak{f}_3 is the free 2-step nilpotent Lie algebra on 3 generators. Also the corresponding hyperbolic automorphism satisfies that the splitting of A_1 is $[3; 2]$. Due to the fact that we are considering algebras with no abelian factor, it is easy to see that this situation leads us to a contradiction to the Lemma 2.3, since 2 and 3 are coprime numbers.

In the case of $(6, 2, 2)$, by Lemma 2.4 and [13, Table 3] an Anosov Lie algebra of this type has to be an extension of \mathfrak{g} or $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$, where \mathfrak{h}_3 is the Heisenberg Lie algebra and \mathfrak{g} is the 8-dimensional 2-step nilpotent Lie algebra defined by

$$(25) \quad [X_1, X_2] = Z_1, \quad [X_1, X_3] = Z_2, \quad [X_4, X_5] = Z_1, \quad [X_4, X_6] = Z_2.$$

It is not possible to have such a extension of \mathfrak{g} . In fact, if for example $[X_i, Z_1] \neq 0$, since $Z_1 = [X_1, X_2]$ by the Jacobi identity we must have that

$$[X_i, X_1] \neq 0, \quad \text{or} \quad [X_i, X_2] \neq 0.$$

Now, we also have that $Z_1 = [X_4, X_5]$ and hence

$$[X_i, X_4] \neq 0, \quad \text{or} \quad [X_i, X_5] \neq 0,$$

and it is clear that there is no such X_i satisfying both conditions.

On the other hand, if we denote by $\{X_1, X_2, Z_1, X_3, X_4, Z_2, X_5, X_6\}$ the canonical basis of $\mathfrak{n} = \mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$ corresponding to this decomposition, one can see that $\tilde{\mathfrak{n}}$ given by

$$(26) \quad \begin{aligned} [X_1, X_2] = Z_1, \quad [X_5, X_2] = W_1, \quad [X_1, Z_1] = W_1 \\ [X_3, X_4] = Z_2, \quad [X_6, X_4] = W_2, \quad [X_3, Z_2] = W_2 \end{aligned}$$

defines a extension of \mathfrak{n} of type $(6, 2, 2)$. Note that this algebra is isomorphic to $\mathfrak{n}' \oplus \mathfrak{n}'$ where \mathfrak{n}' is the Lie algebra given by the first line of (26), or equivalently by the second one. Therefore by [10] this is an Anosov Lie algebra of type $(6, 2, 2)$.

Moreover it is the only one, with no abelian factor. Indeed, to see this we first note that in order to get a $(6, 2, 2)$ extension (in the sense of the Lemma 2.4) of $\mathfrak{n} = \mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$ with no abelian factor we should add at least the following non trivial brackets

$$(27) \quad [X_5, X_i] = W_j, \quad [X_6, X_{i'}] = W_{j'}, \quad [X_k, Z_l] = W_r, \quad [X_{k'}, Z_{l'}] = W_{r'}$$

for some $i, i', k, k' = 1, \dots, 4$ $j, j', r, r' = 1, 2$. Now, using the Jacobi identity, the fact that \tilde{n} is a 3-step nilpotent and that any new bracket should be a multiple of some W_i we have that $[X_k, Z_l] = 0$ for $k = 1, 2$ $l = 2$ and $k = 3, 4$ $l = 1$. Also note that by Lemma 2.2 $\lambda_1 \lambda_2 \lambda_3 \lambda_4 = \lambda_5 \lambda_6 = 1$. Using all this, it is not hard to see that we may assume that

$$(28) \quad \begin{aligned} [X_1, X_2] &= Z_1, & [X_1, Z_1] &= W_1 \\ [X_3, X_4] &= Z_2, & [X_3, Z_2] &= aW_i \end{aligned}$$

for some $a \neq 0$. We have to distinguish between $i = 1$ and $i = 2$.

If $i = 2$, we get that $1 = \lambda_1^2 \lambda_2 \lambda_3^2 \lambda_4 = \lambda_1 \lambda_3$, and hence $\lambda_2 \lambda_4 = 1$. Therefore the matrix of A in this basis is $A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$, where the matrix A_i is given by

$$A_1 = \begin{bmatrix} \lambda & & & & & \\ & \lambda & & & & \\ & & \lambda^{-1} & & & \\ & & & \lambda^{-1} & & \\ & & & & \lambda_5 & \\ & & & & & \lambda_5^{-1} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda^2 & \\ & \lambda^{-2} \end{bmatrix} \quad A_3 = \begin{bmatrix} \lambda^3 & \\ & \lambda^{-3} \end{bmatrix}.$$

By considering the possibilities for $[X_i, X_5] = \alpha W_k$ and $[X_j, X_6] = \beta W_l$, it is easy to see that we may assume that $[X_1, X_5] = aW_1$ and $[X_6, X_3] = bW_2$ and from this we get that $\lambda_5 = \lambda^2$. Moreover, from this we can also see that $[X_i, X_5] = 0$ and $[X_j, X_6] = 0$ for $i = 3, 4$ and $j = 1, 2$. Hence

$$\mathfrak{n}' = \langle X_1, X_2, X_5, Z_1, W_1 \rangle \quad \text{and} \quad \mathfrak{n}'' = \langle X_3, X_4, X_5, Z_2, W_2 \rangle$$

are 3-step, 5-dimensional subalgebras of \tilde{n} with 2-dimensional derived algebra and $\tilde{n} = \mathfrak{n}' \oplus \mathfrak{n}''$. Recall that there is only two such nilpotent Lie algebras (see [11] Table 1), and only one with no abelian factor. Therefore since \tilde{n} has no abelian factor we have that \tilde{n} is given by (26) as we wanted to show.

We are now going to show that it i can not be equal to 1. Indeed, if $i = 1$, by considering the possibilities i, i', j and j' in (27), it is not hard to see that we may assume that \tilde{n} has at least the following non trivial Lie brackets

$$(29) \quad \begin{aligned} [X_1, X_2] &= Z_1, & [X_1, Z_1] &= W_1 & [X_5, X_2] &= W_2, \\ [X_3, X_4] &= Z_2, & [X_3, Z_2] &= aW_1 & [X_6, X_4] &= bW_2, \end{aligned}$$

for some $a, b \neq 0$. From this, straightforward calculations shows that A'_i 's are given by

$$A_1 = \begin{bmatrix} \lambda & & & & & \\ & \mu & & & & \\ & & \lambda^3 \mu^2 & & & \\ & & & \lambda^{-4} \mu^{-3} & & \\ & & & & (\lambda \mu)^2 & \\ & & & & & (\lambda \mu)^{-2} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda \mu & \\ & (\lambda \mu)^{-1} \end{bmatrix} \quad A_3 = \begin{bmatrix} \lambda^2 \mu & \\ & (\lambda^2 \mu)^{-1} \end{bmatrix},$$

Now since the degree of $\lambda \mu$ over \mathbb{Q} is 2, $\lambda, \mu, \lambda^{-4} \mu^{-3}$ and $\lambda^3 \mu^2$ are either two pairs of degree 2 conjugated numbers over \mathbb{Q} , or they are degree 4 conjugated numbers over \mathbb{Q} . It is easy to see that they can not be of degree 2 since if they were,

$\lambda^{-1} \in \{\mu, \lambda^{-4}\mu^{-3}, \lambda^3\mu^2\}$, and each of this situations leads us to a contradiction. In fact if for example we assume that $\lambda^{-1} = \lambda^{-4}\mu^{-3}$ then $(\lambda\mu)^{-3} = 1$ contradicting the fact that A_2 is hyperbolic.

Hence we may assume that $\lambda, \mu, \lambda^{-4}\mu^{-3}$ and $\lambda^3\mu^2$ are the roots of a monic polynomial of degree 4 with integer coefficients and constant term equal to ± 1 .

Since $\lambda\mu$ is a degree 2 algebraic integer, it is not hard to see that $[\mathbb{Q}(\lambda\mu) : \mathbb{Q}(\mu)] = 1$ (see [16, Lemma 2.5]) and from there one can see that $[\mathbb{Q}(\lambda, \mu) : \mathbb{Q}(\lambda\mu)] = 2$. Hence λ should be conjugated to any of $\mu, \lambda^{-4}\mu^{-3}$ or $\lambda^3\mu^2$ over $\mathbb{Q}(\lambda\mu)$. It is easy to see that any of this situations leads us to a contradiction. In fact if for example λ is conjugated to μ over $\mathbb{Q}(\lambda\mu)$ then $\lambda^3\mu^2$ should be conjugated to $\lambda^{-4}\mu^{-3}$ over $\mathbb{Q}(\lambda\mu)$. Hence there exists an automorphism σ of the field $\mathbb{Q}(\lambda, \mu)$ over $\mathbb{Q}(\lambda\mu)$ (that is $\sigma(x) = x$ for all $x \in \mathbb{Q}(\lambda\mu)$) such that

$$\sigma(\lambda^3\mu^2) = \lambda^{-4}\mu^{-3}, \quad \sigma(\lambda) = \mu, \quad \text{and} \quad \sigma(\lambda\mu) = \lambda\mu.$$

Therefore $\lambda^{-4}\mu^{-3} = \sigma(\lambda)\sigma((\lambda\mu)^2) = \mu \cdot (\lambda\mu)^2$, and then $(\lambda\mu)^6 = 1$ contradicting the fact that A_2 is hyperbolic.

Concerning case **(4, 4, 2)**, any Anosov Lie algebra of this type should be an extension of \mathfrak{h} , the 8-dimensional 2-step nilpotent Lie algebra defined by

$$(30) \quad [X_1, X_3] = Z_1, \quad [X_2, X_4] = Z_2, \quad [X_2, X_3] = Z_3, \quad [X_1, X_4] = Z_4.$$

We are now going to show that there are only two of them. To do this, we begin by noting that by Lemma 2.4 and [13] the restriction of A to \mathfrak{h} should be $A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$, where the matrix of A_i with respect to our basis is given by

$$A_1 = \begin{bmatrix} \lambda & & & \\ & \lambda^{-1} & & \\ & & \lambda^2 & \\ & & & \lambda^{-2} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda^3 & & & \\ & \lambda^{-3} & & \\ & & \lambda & \\ & & & \lambda^{-1} \end{bmatrix}, \quad A_3 = \begin{bmatrix} \mu & \\ & \mu^{-1} \end{bmatrix},$$

for λ a degree 2 algebraic number. Therefore, $A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix}$ where the eigenvalues of A_3 can be $\lambda^{\pm i}$ with $i = 2, 3, 4, 5$. In the last three cases, it is not hard to see that the corresponding Lie algebras are isomorphic. Let us denote by \mathfrak{n}_0 the algebra corresponding to $i = 2$ and by \mathfrak{n}_1 the one corresponding to $i = 4$. It is easy to see that $\mathfrak{n}_0 = \mathfrak{n}(a, b, c, d)$ is given by

$$(31) \quad \begin{aligned} [X_1, X_3] &= Z_1, & [X_2, X_4] &= Z_2, & [X_2, X_3] &= Z_3, & [X_1, X_4] &= Z_4 \\ [X_1, Z_3] &= aW_1, & [X_2, Z_1] &= bW_1, & [X_1, Z_2] &= cW_2, & [X_2, Z_4] &= dW_2, \end{aligned}$$

for some a, b, c, d such that $ab \neq 0$ and $cd \neq 0$. By the Jacobi identities, $[X_1, Z_3] = [X_2, Z_1]$ and therefore $a = b = 1$. It is easy to see, by reordering the basis, that $\mathfrak{n}(1, 1, c, 0)$ is isomorphic to $\mathfrak{n}(1, 1, 0, d)$. Then it suffices to see that $\mathfrak{n}(1, 1, 1, 0)$ is isomorphic to $\mathfrak{n}(1, 1, 1, 1)$. In this case the isomorphism is given by changing the basis to

$$\{X_1 + X_2, X_2, X_3, X_4, Z_1 + Z_3, Z_2 + Z_4, Z_3, Z_4, 2W_1, W_2\}.$$

Concerning $\mathfrak{n}_1 = \mathfrak{n}_1(a, b)$ it is easy to see that it is given by

$$(32) \quad \begin{aligned} [X_1, X_3] &= Z_1, & [X_2, X_4] &= Z_2, & [X_2, X_3] &= Z_3, & [X_1, X_4] &= Z_4 \\ [X_1, Z_1] &= aW_1, & [X_2, Z_2] &= bW_2. \end{aligned}$$

for some $a \neq 0 \neq b$ and in this case, it is easy to see by changing the Z 's by a multiple that this algebra is isomorphic to the one with $a = 1 = b$.

To see that \mathfrak{n}_0 is not isomorphic to \mathfrak{n}_1 we will make use of the following isomorphism invariant for 3-step nilpotent Lie algebras:

$$U(\mathfrak{n}) := \{X \in \mathfrak{n}/[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] : \dim \text{Im}(\text{ad } X) = 3\} \cup \{0\},$$

since it is clear that if $\varphi : \mathfrak{n} \mapsto \mathfrak{n}'$ is an isomorphism then $\varphi U(\mathfrak{n}) = U(\mathfrak{n}')$. It is easy to see that

$$(33) \quad U(\mathfrak{n}_0) = \langle X_1, X_2 \rangle_{\mathbb{C}},$$

but on the other hand, $U(\mathfrak{n}_1) = 0$, showing that they are not isomorphic as Lie algebras. Finally, one can check that

$$\beta = \{X_1 + X_2, \lambda X_1 + \lambda^{-1} X_2, X_3 + X_4, \lambda X_3 + \lambda^{-1} X_4, Z_1 + Z_2,$$

$$\lambda Z_1 + \lambda^{-1} Z_2, Z_3 + Z_4, \lambda Z_3 + \lambda^{-1} Z_4, W_1 + W_2, \lambda W_1 + \lambda^{-1} W_2\}.$$

is a \mathbb{Z} -basis of both extension (see [17]), and then we can conclude that they are both Anosov Lie algebra of type $(4, 4, 2)$. We would like to mention that this basis restricted to $\langle X_i, Z_j, 1 \leq i \leq 4, 1 \leq j \leq 4 \rangle$ generates the same lattice as in [13]. Indeed, if as in [13] $\lambda + \lambda^{-1} = 2a$ then $\lambda^{\pm 1} = a \pm (a^2 - 1)^{\frac{1}{2}}$ and therefore

$$\lambda X_1 + \lambda^{-1} X_2 = a(X_1 + X_2) + (a^2 - 1)^{\frac{1}{2}}(X_1 - X_2),$$

$$\lambda Z_1 + \lambda^{-1} Z_2 = a(Z_1 + Z_2) + (a^2 - 1)^{\frac{1}{2}}(Z_1 - Z_2).$$

To finish with the 3-step case we note that an Anosov Lie algebra of type $(4, \mathbf{2}, \mathbf{4})$ should be an extensions of $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ of degree 4. With the same ideas as before, it is not hard to see that there is only one possible such extension with no abelian factor, namely

$$(34) \quad \begin{aligned} [X_1, X_2] &= Z_1, & [X_1, Z_1] &= W_1, & [X_2, Z_1] &= W_2, \\ [X_3, X_4] &= Z_2, & [X_3, Z_2] &= W_3, & [X_4, Z_2] &= W_4. \end{aligned}$$

As in the previous cases, it is not hard to check that this algebra is isomorphic to $\mathfrak{n} \oplus \mathfrak{n}$ where \mathfrak{n} is the Lie algebra given by the first line of (34), or equivalently by the second one, and therefore is an Anosov Lie algebra by [10].

In the case of Lie algebras of more than 3-steps, again by Lemma 2.4, it should be an extension of a 3-step Anosov Lie algebra of dimension less than or equal to 8 and there is only one. In the notation of [13], $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ is the only 3-step Anosov Lie algebra of dimension less than or equal to 8. Explicitly, $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ is the 8 dimensional nilpotent Lie algebra of type $(4, 2, 2)$, given by

$$[X_1, X_2] = Z_1, \quad [X_1, Z_1] = W_1,$$

$$[X_3, X_4] = Z_2, \quad [X_3, Z_2] = W_2.$$

It is not hard to see that this algebra admits only one $(4, \mathbf{2}, \mathbf{2}, \mathbf{2})$ type extensions, explicitly given by

$$(35) \quad \begin{aligned} [X_1, X_2] &= Z_1, & [X_1, Z_1] &= W_1, & [X_1, W_1] &= U_1, \\ [X_3, X_4] &= Z_2, & [X_3, Z_2] &= W_2, & [X_3, W_2] &= U_2. \end{aligned}$$

Note that if $[X_2, W_1] = a U_1$ (or equivalently $[X_4, W_2] = b U_2$) then $a = 0$ ($b = 0$) since it would not satisfy Jacobi identities. Also, since (35) is isomorphic to $\mathfrak{n}_0 \oplus \mathfrak{n}_0$ where \mathfrak{n}_0 is given by the first line (or equivalently the second) by [10] it is an Anosov Lie algebra.

Note that the automorphism A is given by

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & A_4 \end{bmatrix} \quad \text{where} \quad A_1 = \begin{bmatrix} \lambda & & & \\ & \mu & & \\ & & \lambda^{-1} & \\ & & & \mu^{-1} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \lambda\mu & \\ & (\lambda\mu)^{-1} \end{bmatrix}, \quad A_3 = \begin{bmatrix} \lambda^2\mu & \\ & \lambda^{-2}\mu^{-1} \end{bmatrix}, \quad A_4 = \begin{bmatrix} \lambda^3\mu & \\ & \lambda^{-3}\mu^{-1} \end{bmatrix}.$$

We will finally note that if one consider as in Lemma 2.4 the Anosov Lie algebra $[\mathfrak{n}, \mathfrak{n}]$, we obtain in this case the abelian one.

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