A SHORT NOTE ON 2-STEP NILPOTENT LIE ALGEBRAS ASSOCIATED WITH GRAPHS

MEERA G. MAINKAR

Abstract. In this note we consider the 2-step nilpotent Lie algebras associated with graphs. We prove that the 2-step nilpotent Lie algebras $\mathfrak{n}$ and $\mathfrak{n}'$ associated with graphs (S, E) and (S', E') respectively are isomorphic if and only if (S, E) and (S', E') are isomorphic.

1. Introduction

We consider the 2-step nilpotent Lie algebras associated with finite graphs. This class of 2-step nilpotent Lie algebras was introduced in [1]. The group of Lie automorphisms of such a 2-step nilpotent Lie algebra was determined in terms of the graph in [1]. Also, Anosov and ergodic automorphisms on corresponding nilmanifolds were studied. Recently this class was studied in [2], [3], [5] and [4] from different points of views. These 2-step nilpotent Lie algebras have nice combinatorial structure. In [4], explicit examples and non-examples of Einstein solvmanifolds were constructed using graphs. A combinatorial construction of the first and second cohomology groups for the 2-step nilpotent Lie algebras associated with graphs was given in [5], and was used to construct symplectic and contact nilmanifolds.

In this note, we will prove that there are only finitely many non-isomorphic 2-step nilpotent Lie algebras associated with graphs in a given dimension. In fact, we prove that the 2-step nilpotent Lie algebras $\mathfrak{n}$ and $\mathfrak{n}'$ associated with graphs (S, E) and (S', E') respectively are isomorphic if and only if (S, E) and (S', E') are isomorphic. This result has already been used in [4] and [5]. In [4], the authors provide a method to construct Einstein solvmanifolds by using 2-step nilpotent Lie algebras associated with graphs. Using our main theorem, these solvmanifolds are isometric if and only if the graphs are isomorphic, which gives them examples of nonisometric Einstein solvmanifolds. In [5], the authors construct symplectic 2-step nilpotent Lie algebras associated with graphs. It is easy to see that there are only five non-isomorphic 2-step nilpotent Lie algebras

Mathematics Subject Classification. Primary: 22E25

Key words and phrases. nilpotent Lie algebras.

of dimension six associated with graphs (see Remark 1 in [5]) by using our main theorem. The authors further prove that all of them are symplectic.

Acknowledgements: I am thankful to Prof. S. G. Dani and Prof. J. Lauret for their help.

2. Preliminaries

In this section we recall the construction of a 2-step nilpotent Lie algebra associated with a finite graph (see [1]). Let (S, E) be a finite simple graph, where S is the set of vertices and E is the set of edges. Let V be a real vector space with S as a basis and let W be a subspace of $\wedge^2 V$ spanned by $\{\alpha \wedge \beta : \alpha, \beta \in S, \alpha\beta \in E\}$. Let $\mathfrak{n} = V \oplus W$. We define Lie bracket $[,]$ on $\mathfrak{n}$ by specifying that $[\alpha, \beta] = \alpha \wedge \beta$ if α and β are vertices joined by an edge and 0 otherwise, and $[x, w] = 0$ for all $x \in \mathfrak{n}$ and $w \in W$. This determines a unique Lie algebra structure on $\mathfrak{n}$. We note that $[\mathfrak{n}, \mathfrak{n}] = W$ and W is contained in the center on $\mathfrak{n}$, and hence $\mathfrak{n}$ is a 2-step nilpotent Lie algebra. We call $\mathfrak{n}$ *a 2-step nilpotent Lie algebra associated with* (S, E). We also note that the dimension of $\mathfrak{n}$ is $|S| + |E|$ and dimension of $[\mathfrak{n}, \mathfrak{n}]$ is $|E|$.

3. Main Theorem

In this section, we prove our main theorem. It is easy to see that the isomorphic graphs correspond to isomorphic 2-step nilpotent Lie algebras. To prove the converse part, first we prove the following:

Theorem 3.1. *Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra associated with graphs (S, E) and (S', E'). Then the graphs (S, E) and (S', E') are isomorphic.*

Proof. By construction, we note that $|S| = |S'|$ and $|E| = |E'|$. Let V be vector space with a basis S and W be the subspace of $\wedge^2 V$ spanned by $\{\alpha \wedge \beta : \alpha, \beta \in S, \alpha\beta \in E\}$. Now since $[v + w, v' + w'] = 0$ if and only if $[v, v'] = 0$ for all $v, v' \in V$ and $w, w' \in W$, we may assume that S' is a basis of V.

We consider the complexification of $\mathfrak{n}$ and of all automorphisms if necessary. Let D (respectively D') denote the subgroup of $GL(V^{\mathbb{C}})$ consisting of all elements which can be represented as diagonal matrices w.r.t. S (respectively S'). Let G be the subgroup of $GL(V^{\mathbb{C}})$ consisting of restrictions of Lie automorphisms τ of $\mathfrak{n}^{\mathbb{C}}$ such that $\tau(V^{\mathbb{C}}) = V^{\mathbb{C}}$. In particular, G contains linear automorphisms of $V^{\mathbb{C}}$ which can be extended to Lie automorphisms of $\mathfrak{n}^{\mathbb{C}}$. We note that G is an algebraic group and it contains both D and D'. Moreover, D and D' are maximal tori in G^0, the connected component of identity of G. Hence there exists $g \in G^0$ such that $D = gD'g^{-1}$ (see [6]). Let $d \in D$ be such that $d(\alpha) = d_\alpha \alpha$ for all $\alpha \in S$, where d_α's are all nonzero real and $d_\alpha d_\beta \neq d\gamma d_\delta$ unless $\{\alpha, \beta\} = \{\gamma, \delta\}$. There exists $d' \in D'$ such that $d = gd'g^{-1}$. We note that the entries of d' are d_α's for all $\alpha \in S$. Hence there exists a bijection σ of S onto S' such that $d'(\sigma(\alpha)) = d_\alpha \sigma(\alpha)$ for all $\alpha \in S$.

We will prove that $d_\alpha d_\beta$ is an eigenvalue of an extended automorphism d of $\mathfrak{n}$ if and only if it is an eigenvalue of an extended automorphism d' of $\mathfrak{n}$. Now $d_\alpha d_\beta$ is an eigenvalue of d if and only if $[\alpha, \beta] \neq 0$ (by our choice of d). We note that

$$[g^{-1}(d(\alpha)), g^{-1}(d(\beta))] = d_\alpha d_\beta [g^{-1}(\alpha), g^{-1}(\beta)].$$

On the other hand, we have

$$[g^{-1}(d(\alpha)), g^{-1}(d(\beta))] = d'[g^{-1}(\alpha), g^{-1}(\beta)] \text{ since } g^{-1}d = d'g^{-1}.$$

Since $g \in G$, $[g^{-1}(\alpha), g^{-1}(\beta)] \neq 0$ if and only if $[\alpha, \beta] \neq 0$. Hence $d_\alpha d_\beta$ is an eigenvalue of d if and only if it is an eigenvalue of d'. Thus $\alpha\beta \in E$ if and only if $\sigma(\alpha)\sigma(\beta) \in E'$, and (S, E) is isomorphic to (S', E'). □

Theorem 3.2. *Let $\mathfrak{n}$ and $\mathfrak{n}'$ be the 2-step nilpotent Lie algebras associated with graphs (S, E) and (S', E') respectively. Then $\mathfrak{n}$ and $\mathfrak{n}'$ are isomorphic if and only if (S, E) and (S', E') are isomorphic.*

Proof. If the graphs (S, E) and (S', E') are isomorphic, then it can be seen that $\mathfrak{n}$ and $\mathfrak{n}'$ are isomorphic by construction (see §2). Conversely let τ be an isomorphism of $\mathfrak{n}$ onto $\mathfrak{n}'$. Let V (respectively V') be a vector space with S (respectively S') as a basis. Now let $S'' = \{\tau(\alpha) : \alpha \in S\}$ and let E'' be the set of unordered pairs $\tau(\alpha)\tau(\beta)$ such that $\alpha\beta \in E$. Then $\mathfrak{n}'$ is a 2-step nilpotent Lie algebra associated with the graph (S'', E''). Now by using Theorem 3.1, we conclude that (S', E') is isomorphic to (S'', E''), which is isomorphic to (S, E) by our construction. □

Remark 3.3. In particular, there are only finitely many non-isomorphic 2-step nilpotent Lie algebras associated with graphs in a given dimension.

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Department of Mathematics, Dartmouth College, Hanover, NH 03755, USA

E-mail address: meera.g.mainkar@dartmouth.edu