1. Let $f(x, y) = 5 + 3x^2 + 3y^2 + 2y^3 + x^3$.

   (a) Final all critical points of f.

   (b) Use the second derivatives test to classify the critical points you found in (a) as a local maximum, local minimum, saddle point, or N/A if the test is not does not yield any information.

   (c) Determine whether or not $f(x, y)$ has a global maximum in the xy-plane. Justify your answer.

Solution:

(a) The partial derivatives of f are

$$\begin{aligned} f_x(x, y) &= 6x + 3x^2 = 3x(2 + x) \\ f_y(x, y) &= 6y + 6y^2 = 6y(1 + y) \end{aligned}$$

Setting each equation equal to zero yields four points at which both partials are zero. Hence f has four critical points: $(0, 0), (0, -1), (-2, 0), (-2, -1)$.

(b) First note that f is a polynomial in two variables, so that it has continuous second partial derivatives and the second derivatives test applies. The second partials of f are

$$\begin{aligned} f_{xx}(x, y) &= 6 + 6x \\ f_{xy}(x, y) &= 0 \\ f_{yy}(x, y) &= 6 + 12y \end{aligned}$$

In general, the quantity that we're interested in is

$$\begin{aligned} D(x, y) &= f_{xx}(x, y) f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (6 + 6x)(6 + 12y) \\ &= 36(1 + x)(1 + 2y) \end{aligned}$$

At each of the critical points, we see

$D(0, 0) = 36$ and $f_{xx}(0, 0) = 6$ so $(0, 0)$ is a **local minimum**
$D(0, -1) = -36$ so $(0, -1)$ is a **saddle point**
$D(-2, 0) = -36$ so $(-2, 0)$ is a **saddle point**
$D(-2, -1) = 36$ and $f_{xx}(-2, -1) = -6$ so $(0, 0)$ is a **local maximum**

(c) Consider the values of f along the x-axis, i.e. $f(x, 0) = 5 + 3x^2 + x^3$. By making x very big, the value of f can be made arbitrarily large, so f cannot have a maximum in the xy-plane.

2. Suppose that as you move away from the point $(2,0,1)$, the function $f(x,y,z)$ increases most rapidly in the direction of the vector $3\hat{\imath} - \hat{\jmath} + 5\hat{k}$ and the rate of increase of f in this direction is 7. What is $\nabla f(2,0,1)$?

   Solution: The direction of the gradient vector at any point is the direction of greatest increase of the function at that point, and its length is the the rate of that increase. Hence we can conclude that

$$\mathbf{u} = \frac{\langle 3,-1,5\rangle}{|\langle 3,-1,5\rangle|} = \frac{1}{\sqrt{35}}\langle 3,-1,5\rangle$$

   is a unit vector that points in the same direction as $\nabla f(2,0,1)$. Moreover $7 = |\nabla f(2,0,1)|$, so

$$\nabla f(2,0,1) = 7\mathbf{u} = \frac{7}{\sqrt{35}}\langle 3,-1,5\rangle$$

3. Consider the function $f(x, y) = x^3 + y^3 - 9xy + 1$

   (a) Find and classify the critical points on $f(x, y)$.

   (b) Find the absolute maximum and minimum values of $f(x, y)$ in the triangle with vertices $(0, 4)$, $(4, 0)$, and $(4, 4)$.

Solution:

(a) The partial derivatives of f are

$$
\begin{aligned}
f_x(x, y) &= 3x^2 - 9y \\
f_y(x, y) &= 3y^2 - 9x
\end{aligned}
$$

Setting the first partial equal to zero we have that $y = x^3/3$, and substituting this in the second equation, $x^4/3 - 9x = x(x^3/3 - 9) = 0$. The solutions to this equation are when $x = 0$ or $x = 3$. Hence the only critical points are $(0, 0)$ and $(3, 3)$. The second partial derivatives of f are

$$
\begin{aligned}
f_{xx}(x, y) &= 6x \\
f_{xy}(x, y) &= -9 \\
f_{yy}(x, y) &= 6y
\end{aligned}
$$

The quantity that we need to apply the second derivatives test is

$$D(x, y) = (6x)(6y) - (-9)^2 = 36xy - 81$$

and so

$D(0, 0) = -81$ so $(0, 0)$ is a **saddle point**

$D(3, 3) = 36 \cdot 9 - 81 > 0$ and $f_{xx}(3, 3) = 18 > 0$ so $(0, 0)$ is a **local minimum**

(b) We have already found the critical points in part (a), and only $(3, 3)$ lies in the triangle. The value of f there is $f(3, 3) = 2 \cdot 27 - 81 + 1 = -28$. So it remains to find the extreme points of f on the boundary:

Case 1: Along the hypotenuse, $y = 4 - x$,

$$f(x, 4 - x) = x^3 + (4 - x)^3 - 9x(4 - x) + 1 = x^3 + (4 - x)^3 + 36x + 9x^2 + 1$$

Setting the derivative of this function (with respect to x) to zero

$$
\begin{aligned}
0 &= 3x^2 - 3(4 - x)^2 + 36 + 18x = 3x^2 - 3(4 - 2x + x^2) + 36 + 18x \\
&= 24 + 24x
\end{aligned}
$$

shows that the only critical point is when $x = -1$. We are only interested in points where $0 \le x \le 4$, so we only need to check the endpoints:

$$\begin{aligned} f(0,4) &= 4^3 + 1 = 65 \\ f(4,0) &= 65 \end{aligned}$$

Case 2: Along the vertical edge of the triangle, we have

$$f(4,y) = 64 + y^3 - 36y + 1$$

Taking the derivative with respect to y and setting this equal to zero

$$0 = 3y^2 - 36$$

shows that the only critical point in the range $0 \le y \le 4$ occurs when $y = 2\sqrt{3}$. The value of f at this point and the corner that was not computed above is

$$\begin{aligned} &f(4, 2\sqrt{3}) = 64 + (2\sqrt{3})^3 - 36(2\sqrt{3}) + 1 = 65 + 8 \cdot 3\sqrt{3} - 72\sqrt{3} = 65 - 48\sqrt{3} \\ &f(4,4) = 2 \cdot 64 - 36 \cdot 4 + 1 = -15 \end{aligned}$$

Case 3: Along the horizontal edge of the triangle, we have

$$f(x,4) = x^3 + 64 - 36x + 1$$

Note that this is the same function of one variable as the one considered in case 2, so we know that the only critical point will be when $x = 2\sqrt{3}$. So the values of f there and at the last corner are

$$\begin{aligned} f(2\sqrt{3}, 4) &= 65 - 48\sqrt{3} \\ f(0,4) &= 65 \end{aligned}$$

Finally, to find the absolute maximum and minimum on the triangle, we take the maximum and minimum of the values found in the three cases and at the critical point $(3,3)$. Thus

$$\begin{aligned} &\text{absolute max on triangle is } 65 \\ &\text{absolute min on triangle is } -28 \end{aligned}$$

4. (a) Find a set of parametric equations for the line passing through the point $(2, 1, -1)$ and normal to the tangent plane of

$$4x + y^2 + z^3 = 8$$

Solution: This surface is a level surface of the function

$$F(x, y, z) = 4x + y^2 + z^3$$

Since the gradient F is normal to its level surfaces, we can use $\nabla F(2, 1, -1)$ as the direction of the line.

$$\begin{aligned} \nabla F(x, y, z) &= \langle 4, 2y, 3z^2 \rangle \\ \nabla F(2, 1, -1) &= \langle 4, 2, 3 \rangle \end{aligned}$$

Thus

$$\begin{aligned} x &= 2 + 4t \\ y &= 1 + 2t \\ z &= -1 + 3t \end{aligned}$$

(b) Suppose that $z = f(x, y)$, where $x = e^t$ and $y = t^2 + 3t + 2$. Given that $\partial z/\partial x = 2xy^2 - y$ and $\partial z/\partial y = 2x^2 y - x$, find dz/dt when $t = 0$.

Solution: The chain rule says that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy^2 - y)(e^t) + (2x^2 y - x)(2t + 3)$$

When $t = 0$, $x = 1$ and $y = 2$, so plugging in these values

$$\frac{dz}{dt} = (8 - 2)(1) + (4 - 1)(3) = 15$$

(c) Let $f(x, y) = (x - y)^3 + 2xy + x^2 - y$. Find the linear approximation $L(x, y)$ near the point $(1, 2)$.

Solution: The partial derivatives are

$$\begin{aligned} f_x(x, y) &= 3(x - y)^2 + 2y + 2x \\ f_y(x, y) &= -3(x - y)^2 + 2x - 1 \end{aligned}$$

and evaluated at $(1, 2)$

$$\begin{aligned} f_x(1, 2) &= 9 \\ f_y(1, 2) &= -2 \end{aligned}$$

Thus

$$L(x, y) = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) = 2 + 9(x - 1) - 2(y - 2)$$

5. Find the equation of the tangent plane to the surface $z = 4x^3y^2 + 2y$ at the point $(1, -2, 12)$.

Solution: Let $f(x, y) = 4x^3y^2 + 2y$. The partial derivatives are

$$\begin{aligned} f_x(x, y) &= 12x^2y^2 \qquad & f_x(1, -2) &= 48 \\ f_y(x, y) &= 8x^3y + 2 \qquad & f_y(1, -2) &= -14 \end{aligned}$$

Hence the tangent plane equation is

$$z = f(1, -2) + f_x(1, -2)(x - 1) + f_y(1, -2)(y + 2) = 12 + 48(x - 1) - 14(y + 2)$$

One could also realize the surface as a level surface of the function $F(x, y, z) = f(x, y) - z = 4x^3y^2 + 2y - z$. Then a normal vector to the tangent plane is given by the gradient of F:

$$\nabla F(1, -2, 12) = \langle f_x(1, -2), f_y(1, -2), -1\rangle = \langle 48, -14, -1\rangle$$

and the equation of the tangent plane is given by

$$0 = \nabla F(1, -2, 12) \cdot \langle x - 1, y + 2, z - 12\rangle = 48(x - 1) - 14(y + 2) - (z - 12)$$

6. Find all points at which the surface

$$x^2 + y^2 + 2x = z^2$$

has a vertical tangent plane.

Solution 1: Figure out what the surface looks like. First, we can complete the square to rewrite the equation as

$$(x+1)^2 + y^2 = z^2 + 1.$$

We get the intersection of the surface with the xz-plane by setting $y = 0$; this intersection is

$$(x+1)^2 = z^2 + 1,$$

a hyperbola with asymptotes $z = \pm(x+1)$.

Horizontal cross-sections, which we get by setting $z = k$, are circles

$$(x+1)^2 + y^2 = k^2 + 1,$$

in the plane $z = k$ with center $(x, y) = (-1, 0)$ and radius $\sqrt{k^2+1}$. If you sketch the surface, you will see that vertical tangent planes occur at points where the horizontal cross-section has the smallest possible radius. This occurs where

$$z = 0 \qquad (x+1)^2 + y^2 = 1.$$

Solution 2: The tangent plane is vertical when its normal vector is horizontal; that is, when the z-component of the normal vector is 0.

We can rewrite the equation of the surface as

$$x^2 + y^2 + 2x - z^2 = 0,$$

and view it as a level surface of the function

$$f(x, y, z) = x^2 + y^2 + 2x - z^2.$$

The normal vector to the surface (and, therefore, to the tangent plane) is the gradient of f,

$$\nabla f = \langle 2x + 2, 2y, -2z \rangle,$$

which has z-component equal to 0 when $z = 0$.

The points on the surface where $z = 0$ satisfy

$$x^2 + y^2 + 2x = 0 \qquad (x+1)^2 + y^2 = 1.$$

Both Solutions: These points form a circle in the plane $z = 0$ with radius 1 and center $(x, y) = (-1, 0)$.

7. Suppose that $f(x, y)$ is the distance between the origin $(0, 0)$ and the point (x, y). Use what you know about the significance of the gradient to find

$$\nabla f(3, 4)$$

without finding a formula for f or computing any derivatives.

Solution: The direction of ∇f is the direction in which f is increasing most rapidly. Distance from the origin increases most rapidly if you are moving directly away from the origin. At the point $(3, 4)$, the vector $\langle 3, 4\rangle$ points directly away from the origin.

The unit vector in this direction is $\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

The magnitude of ∇f is the directional derivative, or rate of change of f, in this direction. If you are moving directly away from the origin, your distance from the origin is increasing at a rate of one unit for each unit of distance you move. Therefore, $|\nabla f(3, 4)| = 1$.

Now we have the direction and magnitude, so we can say

$$\nabla f(3, 4) = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

8. Find a function parametrizing the curve γ given by the intersection of the plane with equation $z = x - y$ and the cylinder with equation $x^2 + y^2 = 1$.

   Find an equation for the line tangent to γ at the point where $(x, y) = (0, 1)$.

   Represent the arclength of γ as an integral. You do NOT have to evaluate this integral; just write it down.

   Solution: Since our curve lies on the cylinder $x^2 = y^2$, we can write $x = \cos t$ and $y = \sin t$ for $0 \le t \le 2\pi$. Then the equation of the plane gives z in terms of x and y, so $z = \cos t - \sin t$. Our parametrization is

$$\langle x, y, z\rangle = \vec{r}(t) = \langle \cos t,\, \sin t,\, \cos t - \sin t\rangle \qquad 0 \le t \le 2\pi.$$

   At the point where $(x, y) = (0, 1)$ we have $t = \dfrac{\pi}{2}$, so the point on γ is $\vec{r}\left(\frac{\pi}{2}\right) = \langle 0, 1, -1\rangle$ and a vector tangent to γ is given by $\vec{r}'\left(\dfrac{\pi}{2}\right)$.

$$\vec{r}'(t) = \langle -\sin t,\, \cos t,\, -\sin t - \cos t\rangle\,;$$

$$\vec{r}'\left(\frac{\pi}{2}\right) = \langle -1, 0, -1\rangle\,.$$

   The tangent line passes through the point $\vec{r}(t)$ in the direction of $\vec{r}'(t)$, so at $t = \dfrac{\pi}{2}$, it has equation

$$\langle x, y, z\rangle = \langle 0, 1, -1\rangle + t\,\langle -1, 0, -1\rangle\,.$$

   The arclength of γ is given by

$$\int_0^{2\pi} |\vec{r}'(t)|\; dt = \int_0^{2\pi} |\langle -\sin t,\, \cos t,\, -\sin t - \cos t\rangle|\; dt =$$

$$\int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + (-\sin t - \cos t)^2}\, dt =$$

$$\int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + (\sin^2 t + 2\sin t\cos t + \cos^2 t)}\, dt =$$

$$\int_0^{2\pi} \sqrt{2\sin t\cos t + 2}\, dt$$